LOGIC QUALIFYING EXAM, AUGUST 29, 1991

INSTRUCTIONS: Do any four problems. Use a separate packet of paper for each problem, since not all of your answers will be graded by the same person. If you think a problem has been stated incorrectly, mention this to the proctor and indicate your interpretation in your solution. In such cases, do not interpret the problem in such a way that it becomes trivial.

NOTATION: ω and $\mathbf{Q}$ are the sets of natural numbers and rationals. W_e is the domain of the recursive function with Godel number e. $X <_T Y$ means that X is Turing reducible to Y and Y is not Turing reducible to X. R_α is the set of all sets of rank less than α. ZFC is Zermelo-Fraenkel set theory with choice. cub means closed unbounded.

ELEMENTARY PROBLEMS

E1. Prove that if κ and λ are cardinals with $\omega \leq \lambda \leq \kappa$, then $(\kappa^+)^\lambda \leq \max(\kappa^\lambda, \kappa^+)$.

E2. Let T be the theory of equivalence relations such that each equivalence class is infinite. Prove that T is not finitely axiomatizable.

RECURSION THEORY

R1. Let A be an r.e. set. Show that there exists an $e \in \omega$ such that $W_e = \{x : e + x \in A\}$.

R2. Prove that there are sets $X_q \subseteq \omega$, $q \in \mathbf{Q}$, such that whenever $q < r$, $X_q <_T X_r$.

MODEL THEORY

M1. Suppose $\kappa \geq \omega$ is a cardinal, $\mathfrak{A}_\alpha$, $\alpha < \kappa$, is an elementary chain, and for each $\alpha < \kappa$, $\mathfrak{A}_\alpha$ is elementarily embeddable in $\mathfrak{B}_\alpha$. Prove that there is an ultrafilter D over κ such that $\bigcup_\alpha \mathfrak{A}$ is elementarily embeddable in the ultraproduct $\Pi_D \mathfrak{B}_\alpha$.

M2. Let $\mathfrak{A}$ be a countably infinite recursively saturated model. Prove that $\mathfrak{A}$ has a nontrivial automorphism.

SET THEORY

S1. Let κ be an uncountable inaccessible cardinal. Prove that the set

$$\{\alpha < \kappa : \langle R_\alpha, \in \rangle \models \text{ZFC}\}$$

has a cub subset but is not a cub set.

S2. If $f, g \in {}^{\omega_1}\omega_1$, let $f < g$ mean $(\forall \alpha < \omega_1)(f(\alpha) < g(\alpha))$. Prove that it is consistent with ZFC to have $2^\omega = 2^{\omega_1} = 2^{\omega_2} = \omega_3$, together with a family $\mathcal{F} \subseteq {}^{\omega_1}\omega_1$ of size ω_2 such that for each $f \in {}^{\omega_1}\omega_1$ there exists $g \in \mathcal{F}$ with $f < g$.