QUALIFYING EXAM IN LOGIC
January, 1991

INSTRUCTIONS: Do any four problems. Use a separate packet of paper for each problem, since not all of your answers will be graded by the same person. You should not hand in more than four problems; if you do more, only the first four will be graded.

If you think a problem has been stated incorrectly, mention this to the proctor and indicate your interpretation in your solution. In such cases, do not interpret the problem in such a way that it becomes trivial.

NOTATION: $\omega$ is the set of natural numbers. All languages are understood to be languages in first order predicate logic. The universe of a model $\mathfrak{U}$ is denoted by $A$. A set $X \subseteq Y$ is cofinite in $Y$ if $Y \setminus X$ is finite. The cofinality of a linearly ordered set $(S, \leq)$ is the least cardinal $\kappa$ such that for some $T \subseteq S$ of size $\kappa$, $(\forall s \in S)(\exists t \in T)s \leq t$. The cofinality of $(S, \leq)$ is defined similarly but with $s \geq t$. If $A, B \subseteq \omega$, $A \equiv_T B$ means that $A$ is Turing equivalent to $B$, and $A <_T B$ means that $A$ is Turing reducible to $B$ and not $A \equiv_T B$. $W_X$ is the domain of the partial recursive function with Gödel number $x$. $A^{(n)}$ is the $n$th jump of $A$. $A \oplus B$ is the disjoint union $\{2^x : x \in A\} \cup \{3^y : y \in B\}$. ZFC is Zermelo-Fraenkel set theory with choice. MA is Martin's Axiom, CH is the continuum hypothesis, and GCH is the generalized continuum hypothesis.
ELEMEN~TARY PROBLEMS

E1. Let T be a theory in a finite language which has no in~finite models. Show that T is decidable.

E2. Let U be elementarily equivalent to the model (ω,0,s) where s is the successor function. Show that for every formula φ(x) in the language of U, the set \{a ∈ A : U ⊨ φ[a]\} is either finite or cofinite in A.

MODEL THEORY

M1. Let T be a theory with infinite models in a countable language. Prove that T has a countable model U which has 2^ω distinct elementary submodels.

M2. Let κ and λ be infinite regular cardinals. Prove that the standard model of arithmetic has an elementary extension U = ⟨A, +, ×, U, ≤⟩ such that ⟨A − ω, ≤⟩ has cointiality κ and cofinality λ.

RECURSION THEORY

R1. Show that there is no partial recursive function φ(x) on ω such that for all x, W_x ≠ ∅ implies that φ(x) is defined and equals min{y : y ∈ W_x}.

R2. Show that there are r.e. sets A and B such that for all n, A^{(n)} <_T φ^{(n+1)}, B^{(n)} <_T φ^{(n+1)}, and A^{(n)} ∨ B^{(n)} ≡_T φ^{(n+1)}.

Hint: Use the Sacks Splitting Theorem, the Robinson Jump Interpolation Theorem, and the Recursion Theorem.

SET THEORY

S1. Let M be a countable transitive model of ZFC + GCH. Show that there is a forcing extension of M satisfying ZFC + GCH together with the statement that not every subset of ω_1 is constructible from a subset of ω.

S2. Assume MA and ¬CH. Let X be a set of real numbers of size κ_1. For each x ∈ X, let S_x be an ω-sequence of elements of X which converges to x. Prove that there is an uncountable Y ⊆ X such that Y ∩ S_x is finite for all x ∈ X.
SET THEORY

**S1.** Let $M$ be a countable transitive model of $ZFC + GCH$. Show that there is a forcing extension of $M$ satisfying $ZFC + GCH$ together with the statement that not every subset of $\omega_1$ is constructible from a subset of $\omega$.

**Solution:** In $M$, let $P$ be finite partial functions from $\omega_1$ to 2. Since (in $M$) $P$ is ccc and has size $\omega_1$, $M[G]$ still satisfies $GCH$. Every element of $M[G\cap P]$ is of the form $\tau G$ for some $P$ name $\tau$. In particular, $G \notin M[G\cap P]$ for any $\alpha < \omega_1$. In $M[G]$, if $x \subset \omega$, then $x \in M[G\cap P]$ for some $\alpha < \omega_1$, so $G \notin M[x]$, so $G \notin L[x]$. Thus, if $A = \{\alpha \in \omega_1 : G(\alpha) = 1\}$, then $A$ is a subset of $\omega_1$ not constructible from any subset of $\omega$.

**S2.** Assume MA and \neg CH. Let $X$ be a set of real numbers of size $\aleph_1$. For each $x \in X$, let $S_x$ be a simple sequence in $X$ which converges to $x$. Prove that there is an uncountable $Y \subset X$ such that $Y \cap S_x$ is finite for all $x \in X$.

**Solution:** Let $S = \{S_x : x \in S\}$. Let $P$ be the set of all pairs, $p = (a_p, F_p)$ such that $a_p$ is a finite subset of $X$ and $F_p$ is a finite subset of $S$. Say $q \leq p$ iff $a_q \supseteq a_p$, $F_q \supseteq F_p$, and

$$\forall S \in F_p \forall x \in (a_q \setminus a_p)(x \notin S) \ .$$

Dense sets: Say $X = \{X_\alpha : \alpha \in \omega_1\}$, where each $X_\alpha$ is countable. If $G$ meets $\{p : a_p \setminus X_\alpha \neq \emptyset\}$ for each $\alpha$ and $\{p : S \in F_p\}$ for each $S \in S$, then $Y = \bigcap_{\alpha \in \omega_1} F_\alpha$ satisfies the requirements of the problem.

ccc: Suppose $A$ is an uncountable antichain in $P$. By the standard $\Delta$-system and thinning arguments, we may assume that the $a_p$ for $p \in A$ form a $\Delta$-system, and then that the root is empty. We may then assume that $A = \{p_\xi : \xi < \omega_1\}$, where $a_\alpha p_\xi = \{x^1, \ldots, x^\eta_\xi\}$. Let $T_\xi = \bigcap_{p_\xi} F_{p_\xi}$. Thinning again, we may assume $\alpha > \xi$ implies $x^i_\alpha \notin T_\xi$. Now, fix $\alpha < \omega_1$ such that whenever $I_1 \ldots I_n$ are rational intervals, if there exists a $\xi$ such that each $x^i_\xi \in I_i$ ($i = 1 \ldots n$), then there is such a $\xi$ less than $\alpha$. Since $T_\alpha$ has countable closure, we may find a $\xi$ such that each $x^i_\xi \notin T_\alpha$, and then fix rational neighborhoods $I_i$ of each $x^i_\xi$ missing $T_\alpha$. Now, by our assumption on $\alpha$, we can choose $\xi < \alpha$ - but then $p_\xi$ and $p_\alpha$ are compatible.