QUALIFYING EXAM IN LOGIC
August, 1990

INSTRUCTIONS: Do any four problems. Use a separate packet of paper for each problem, since not all of your answers will be graded by the same person. If you think a problem has been stated incorrectly, mention this to the proctor and indicate your interpretation in your solution. In such cases, do not interpret the problem in such a way that it becomes trivial.

NOTATION: \( \omega \) is the set of natural numbers. \( \mathbb{R} \) is the set of real numbers and \( \mathbb{Q} \) is the set of rational numbers. The universe of a model \( \mathcal{U} \) is denoted by \( A \). If \( \mathcal{U} \) is a model and \( X \subseteq A \), \( \mathcal{U}_X \) is the expansion of \( \mathcal{U} \) formed by adding a constant for each \( x \in X \). If \( A, B \subseteq \omega \), \( A \equiv_T B \) means that \( A \) is Turing equivalent to \( B \). \( A' \) is the jump of \( A \), and \( A \oplus B = \{2^a3^b : a \in A \text{ and } b \in B\} \). \( \text{ZF} \) is Zermelo-Fraenkel set theory, and \( \text{PA} \) is Peano arithmetic. \( \kappa < \kappa = \bigcup \{\kappa^\alpha : \alpha < \kappa\} \). CCC denotes the countable chain condition.
ELEMENTARY PROBLEMS

E1. Show that in the theory ZF–∞ consisting of all axioms of ZF except the axiom of infinity, the consistency of PA is not provable.

E2. Prove that the complete theory of the model \((\mathbb{R}, \mathbb{Q}, \leq)\) is decidable.

MODEL THEORY

M1. Prove that there exists a saturated dense linear order of cardinality \(\kappa\) if and only if \(\kappa = \kappa^\lt\).

M2. Let \(T\) be an \(\forall 3\) theory in a countable language which has infinite models. Prove that \(T\) has a model \(M\) of power \(2^\omega\) such that whenever \(M \subseteq B \models T\), every countable set of existential formulas with constants from \(A\) which is satisfiable in \(B_A\) is satisfiable in \(M_A\).

RECURSION THEORY

R1. Prove or disprove: If \(A\) and \(B\) are r.e., then \(A' \oplus B' \equiv_T (A \oplus B)'\).

R2. Let \(A\) be hypersimple and define

\[ B = \langle (m, n) : m \leq n \text{ or } m \in A \rangle. \]

Prove that \(B\) is hypersimple but not hyperhypersimple.

SET THEORY

S1. If \(P\) and \(Q\) are partial orderings in a countable model \(M\) of ZFC such that \(P\) is countably closed and \(Q\) is CCC in \(M\), then \(Q\) is CCC in the generic extension of \(M\) over \(P\).

S2. Prove that there is a subset \(S \subseteq \mathbb{R}\) of power \(2^\omega\) such that it and its complement meet uncountable Borel subset of \(\mathbb{R}\).