QUALIFYING EXAM IN LOGIC

January, 1990

INSTRUCTIONS: Do any four problems. Use a separate packet of paper for each problem, since not all of your answers will be graded by the same person. If you think a problem has been stated incorrectly, mention this to the proctor and indicate your interpretation in your solution. In such cases, do not interpret the problem in such a way that it becomes trivial.

NOTATION: ω is the set of natural numbers. $\mathbb{R}$ is the set of real numbers. $X \leq_T Y$ means that X is Turing reducible to Y. W_e is the domain of the partial recursive function ϕ_e coded by e, and $W_{e,t}$ is the set of $x \in W_e$ such that the computation code of ϕ_e with input x is $< t$.

The universe of a model $\mathfrak{A}$ is denoted by A. $\equiv$ means elementarily equivalent, and $\prec$ means elementary submodel. A model $\mathfrak{A}$ is ω-homogeneous iff for all finite tuples $\vec{a}, \vec{b}$ in A such that $(\mathfrak{A}, \vec{a}) \equiv (\mathfrak{A}, \vec{b})$ and all $c \in A$ there exists $d \in A$ with $(\mathfrak{A}, \vec{a}, c) \equiv (\mathfrak{A}, \vec{b}, d)$.

A set $X \subseteq \omega_1$ is stationary iff it meets every closed unbounded set $Y \subseteq \omega_1$.

ELEMENTARY PROBLEMS

E1. a) Give an example of a first-order theory T with exactly $\aleph_0$ non-isomorphic models.

b) Show that there is no first-order theory T in a finite language with exactly $\aleph_0$ non-isomorphic models.

Caution: Note that we are considering all models of T, possibly of differing cardinalities.

E2. Prove that for every infinite cardinal κ, there exists a set, $\mathcal{F}$, of subsets of κ such that $|\mathcal{F}| = 2^\kappa$ and for all $A, B \in \mathcal{F}$, if $A \neq B$ then A is not a subset of B.

RECURSION THEORY

R1. Prove that there is a non-recursive set $X \subseteq \omega$ such that for all infinite $Y \subseteq X$, X is recursive in Y.

R2. Suppose that f is a total one-to-one recursive function. Prove that there exists e such that $W_e = range(f)$ and

$$\forall n \exists t W_{e,t} = \{f(0), \ldots, f(n-1)\} \quad .$$

MODEL THEORY

M1. Let $\mathcal{A}$ be an uncountable ω-homogeneous model for a countable language. Prove that for every countable elementary submodel, $\mathcal{B} \prec \mathcal{A}$, there is a countable ω-homogeneous $\mathcal{C}$ with $\mathcal{B} \prec \mathcal{C} \prec \mathcal{A}$.

M2. Let T be a complete first-order theory and let $\Gamma(x)$ be a set of formulas with x free. For each model, $\mathcal{A} \models T$, let $\Gamma(\mathcal{A})$ be the set of all $b \in A$ such that b realizes $\Gamma(x)$ in $\mathcal{A}$. Suppose that $\Gamma(\mathcal{A})$ is finite for all $\mathcal{A} \models T$. Prove that $\Gamma(\mathcal{A})$ has the same cardinality for all $\mathcal{A} \models T$.

SET THEORY

S1. Prove that there exists a set $\{X_\alpha : \alpha < 2^{\omega_1}\}$ of subsets of ω_1 such that whenever $\alpha \neq \beta$, the symmetric difference, $X_\alpha \triangle X_\beta$, is stationary in ω_1.

S2. Prove that it is consistent with ZFC to have a family of functions, $f_\alpha \in \omega^\omega$, for $\alpha < \omega_2$, such that for each $g \in \omega^\omega$, $\{\alpha : f_\alpha \leq^* g\}$ is countable. Here, $f \leq^* g$ means that $f(n) \leq g(n)$ for all but at most finitely many n. *Hint.* Use Cohen real forcing.