QUALIFYING EXAM IN LOGIC
January, 1990

INSTRUCTIONS: Do any four problems. Use a separate packet of paper for each problem, since not all of your answers will be graded by the same person. If you think a problem has been stated incorrectly, mention this to the proctor and indicate your interpretation in your solution. In such cases, do not interpret the problem in such a way that it becomes trivial.

NOTATION: ω is the set of natural numbers. ℝ is the set of real numbers. X ≤_T Y means that X is Turing reducible to Y. We is the domain of the partial recursive function ϕ_e coded by e, and We,t is the set of x ∈ We such that the computation code of ϕ_e with input x is < t.

The universe of a model U is denoted by A. ≡ means elementarily equivalent, and < means elementary submodel. A model U is ω-homogeneous iff for all finite tuples a, b in A such that (U, a) ≡ (U, b) and all c ∈ A there exists d ∈ A with (U, a, c) ≡ (U, b, d).

A set X ⊆ ω₁ is stationary iff it meets every closed unbounded set Y ⊆ ω₁.
ELEMENTARY PROBLEMS

E1. a) Give an example of a first-order theory $T$ with exactly $\aleph_0$ non-isomorphic models.
   b) Show that there is no first-order theory $T$ in a finite language with exactly $\aleph_0$ non-isomorphic models.
   Caution: Note that we are considering all models of $T$, possibly of differing cardinalities.

E2. Prove that for every infinite cardinal $\kappa$, there exists a set, $\mathcal{F}$, of subsets of $\kappa$ such that $|\mathcal{F}| = 2^\kappa$ and for all $A, B \in \mathcal{F}$, if $A \neq B$ then $A$ is not a subset of $B$.

RECURSION THEORY

R1. Prove that there is a non-recursive set $X \subseteq \omega$ such that for all infinite $Y \subseteq X$, $X$ is recursive in $Y$.

R2. Suppose that $f$ is a total one-to-one recursive function. Prove that there exists $e$ such that $W_e = \text{range}(f)$ and

$$\forall n \exists t W_{e,t} = \{f(0), \ldots, f(n-1)\}.$$ 

MODEL THEORY

M1. Let $\mathcal{A}$ be an uncountable $\omega$-homogeneous model for a countable language. Prove that for every countable elementary submodel, $\mathcal{B} \prec \mathcal{A}$, there is a countable $\omega$-homogeneous $\mathcal{C}$ with $\mathcal{B} \prec \mathcal{C} \prec \mathcal{A}$.

M2. Let $T$ be a complete first-order theory and let $\Gamma(x)$ be a set of formulas with $x$ free. For each model, $\mathcal{A} \models T$, let $\Gamma(\mathcal{A})$ be the set of all $b \in A$ such that $b$ realizes $\Gamma(x)$ in $\mathcal{A}$. Suppose that $\Gamma(\mathcal{A})$ is finite for all $\mathcal{A} \models T$. Prove that $\Gamma(\mathcal{A})$ has the same cardinality for all $\mathcal{A} \models T$.

SET THEORY

S1. Prove that there exists a set $\{X_\alpha : \alpha < 2^{\omega_1}\}$ of subsets of $\omega_1$ such that whenever $\alpha \neq \beta$, the symmetric difference, $X_\alpha \Delta X_\beta$, is stationary in $\omega_1$.

S2. Prove that it is consistent with ZFC to have a family of functions, $f_\alpha \in \omega^\omega$, for $\alpha < \omega_2$, such that for each $g \in \omega^\omega$, $\{\alpha : f_\alpha \leq^* g\}$ is countable. Here, $f \leq^* g$ means that $f(n) \leq g(n)$ for all but at most finitely many $n$.
   Hint. Use Cohen real forcing.