Qualifying Exam
Logic
Aug 27 1987

Instructions: Do any four problems, but at most two elementary. Please use a separate packet of paper for each problem since not all of your answers will be graded by the same person. If you think a problem has been stated incorrectly, mention this to the proctor and indicate your interpretation in your solution. In such cases do not interpret the problem in such a way that it becomes trivial.
DEFINITIONS

1. \( \omega = N \) = the set of natural numbers.

2. \( A \preceq B \) means \( A \) is elementarily embeddable in \( B \).

3. \( C^{<\omega} \) are all finite sequences from \( C \).

4. \( \{W_e(n) \mid e^{<\omega}\} \) is the standard enumeration of all r.e. subsets of \( \omega^n \). \( W_e = W_e^{(1)} \).

5. \( K = \{e \mid e \in W_e\} \).

6. \( Tr \subseteq \omega^{<\omega} \) is a tree iff \( \forall \alpha, \beta \in \omega^{<\omega} : \alpha \subseteq \beta \) and \( \beta \in Tr \), then \( \alpha \in Tr \).

7. \( A \preceq B \) means \( A \) is an elementary submodel of \( B \).

8. \( A \preceq_T B \) means \( A \) is Turing reducible to \( B \).
Elementary

1. Given a countable set of students and a countable set of classes. Suppose each student wants one of a finite set of classes, and each class has a finite enrollment limit. Use the compactness theorem to prove that if each finite set of students can be accommodated, then the whole set can.

2. Let $T$ and $U$ be first order theories in a language $L$. Suppose that for each finite subset $T_0 \subseteq T$ and $U_0 \subseteq U$ there are models $\mathcal{A}_0 \models T_0$ and $\mathcal{B}_0 \models U_0$ such that $\mathcal{B}_0$ is a submodel of $\mathcal{A}_0$. Prove that there are models $\mathcal{A} \models T$ and $\mathcal{B} \models U$ such that $\mathcal{B}$ is a submodel of $\mathcal{A}$.

3. Given a partial order $< A, <^* >$ with no infinite decreasing sequences. Prove that there is a well order $< A, < >$ such that $<^* \subseteq <$.

4. Let $\kappa$ be an uncountable cardinal of countable cofinality. Show there exists $< f_\alpha : \omega \times \kappa \mid \alpha < \kappa^+ >$ such that for all $\alpha \neq \beta$ and for all but finitely many $n$, $f_\alpha(n) \neq f_\beta(n)$.
Recursion Theory

1. Prove that there is no recursive g such that for all e < ω:
   1) \( W_g(e) \) is finite; and
   2) if \( W_e \) is finite, then \( W_e \supseteq W_g(e) \).

2. Given an infinite r.e. set A, construct a low simple set S containing the complement of A.

3. Prove
   \[ \forall A \subseteq N \exists B, C \subseteq N \exists e \left[ A = \phi_e^B = \phi_e^C \text{ and } B \not{\equiv}T C \right]. \]

4. Prove that there exist a minimal triple of Turing degrees such that no two of the degrees form a minimal pair.

    \[ [ A, B, C \text{ are a minimal triple } \iff \text{ they are non-recursive and } \forall D (D \equiv_T A, B, C \rightarrow D \text{ recursive}) ] \]
Model Theory

1. Let $\mathcal{A}$ be an infinite model of a countable language. Prove that for each $b \in \mathcal{A}$, $\text{Th}(\mathcal{A})$ is $\omega$-categorical iff $\text{Th}(\mathcal{A}, b)$ is $\omega$-categorical.

2. Let $\mathcal{A}$ be a model with the property that each subset $U$ of $A$ is a relation of $\mathcal{A}$ and each function $f : A \to A$ is a function of $\mathcal{A}$. Suppose $\mathcal{A} < \mathcal{B}$ and there is an element $b \in \mathcal{B}$ such that $\mathcal{B}$ has no proper submodels containing $b$. Prove that there is an ultrafilter $D$ over $A$ such that $\mathcal{B} \cong \prod_D \mathcal{A}$.

3. Let $T$ be a complete theory with infinite models in a countable language and let $\kappa$ be an infinite cardinal. Prove that $T$ has models $\mathcal{A}$ and $\mathcal{B}$ of power $\kappa$ where $\mathcal{B}$ is a proper submodel of $\mathcal{A}$ and there is an automorphism $f$ of $\mathcal{A}$ such that
   
   $\mathcal{B} < f(\mathcal{B}) < f(f(\mathcal{B})) < f(f(f(\mathcal{B}))) < \ldots$

   and

   $\mathcal{A} = \mathcal{B} \cup f(\mathcal{B}) \cup f(f(\mathcal{B})) \cup f(f(f(\mathcal{B}))) \cup \ldots$. 

   Hint: use indiscernibles.

4. Prove that for any consistent complete theory $T$ there is a model $\mathcal{A} \models T$ such that

   $\forall a, b \in \mathcal{A}$ [ $a, b$ realize the same $\lambda$-type in $\mathcal{A}$ ] iff

   $\exists \theta(x, y) [ \mathcal{A} \models \theta(a, b) \text{ and } \forall \sigma(x) [ T \vdash \forall x, y[\theta(x, y) \to [\sigma(x) \leftrightarrow \sigma(y)]]]]$, 

   where '$\theta$' and '$\sigma$' range over formulas of $L(T)$.
Set Theory

1. Assume CH and let Lim be the set of limit ordinals less than $\omega_1$. Show that there exists $\langle A_\alpha \mid \alpha \in \text{Lim} \rangle$ such that for every $\alpha \in \text{Lim}$, $A_\alpha \subseteq \alpha$ and for $\alpha < \beta$, $A_\alpha \cap A_\beta$ is finite, but there does not exist $X \in [\text{Lim}]^{\omega_1}$ and $\alpha < \omega_1$ such that for every $\gamma \not\in X$, $A_\gamma \cap A_\beta \subseteq \alpha$.

2. Assume there exists an uncountable transitive model of ZFC. Show there exists an uncountable transitive model of ZFC+$\forall \lambda \exists \theta$. Hint: Consider forcing with $P = (2^{<\kappa})^{L_\alpha}$ for appropriate $\alpha$, $\kappa$.

3. Let $P=\text{FIN}(\omega_2)$ be the partial order of functions with finite domain contained in $\omega_2$ and range $\{0,1\}$. Show that in the generic extension obtained by forcing with $P$ that there does not exist a linear order of cardinality $\omega_1$ such that every other linear order of cardinality $\omega_1$ can be embedded.