Instructions: Do any four problems, but at most two elementary. Please use a separate packet of paper for each problem since not all of your answers will be graded by the same person. If you think a problem has been stated incorrectly, mention this to the proctor and indicate your interpretation in your solution. In such cases do not interpret the problem in such a way that it becomes trivial.
1. A set of sentences $\Sigma$ is independent iff

$$\forall \sigma \in \Sigma \left[ \Sigma - \{\sigma\} \not\vdash \sigma \right]$$

Prove that if $\Gamma, \Sigma$ are sets of formula satisfying:

(a) $\Sigma \cap \Gamma = \emptyset$;
(b) $|\Gamma| \leq |\Sigma|$;
(c) $\forall \sigma \in \Sigma \left[ (\Sigma \cup \Gamma) - \{\sigma\} \not\vdash \sigma \right]$

Then $\Sigma \cup \Gamma$ can be axiomatized by a set of independent sentences.

2. Find $A \subset B \subset C \subset D$ countable structures in the same language such that: $A$ is isomorphic to $C$, $B$ is elementarily equivalent to $D$ but not isomorphic to it, and the theory of $A$ is $\aleph_0$-categorical.

3. Show that the set of validities in the first order theory of pure equality is recursive.

4. Let $(P, <)$ be an infinite partial order. Show that $P$ contains an infinite subset of order type $\omega$ or $\omega^*$ (i.e. converse $\omega$), or an infinite set of pairwise incomparable elements.
Recursion Theory

5. Prove that if f,g are total recursive functions and A is a simple set, then
   \( \exists n \in A \) satisfying
   \[ W_{f(n)} \cup W_{g(n)} = W_n \]

6. Assume A,B r.e. satisfying
   (a) \( A \subset B \);
   (b) \( \forall n \ [B^{[n]} - A^{[n]} \text{ infinite}] \).

   Prove \( \exists C, D \) r.e. satisfying
   (a) \( A \subset C, D \subset B \);
   (b) \( C|T D \).

7. Prove that there are no A,B r.e. recursively inseparable sets and simple
   set S such that \( B \leq_m S \).

8. Prove there exists recursively incomparable maximal sets.
9. Find $T_i, \Gamma_i, L_i \ i < 2$ such that:

(a) $T_i$ is complete theory in $L_i, i < 2$;
(b) $\Gamma_i$ complete non-principal type of $T_i, i < 2$;
(c) $T_0 \cup T_1$ is a consistent theory in $L_0 \cup L_1$; and
(d) there exists a $L_0 \cup L_1$ formula $\theta(\vec{x})$ which is consistent with $T_0 \cup T_1$
    and for every formula $\psi(\vec{x}) \in (\Gamma_1(\vec{x}) \cup \Gamma_2(\vec{x}))$

$$T_0 \cup T_1 \vdash \theta(\vec{x}) \rightarrow \psi(\vec{x})$$

10. Assume $T$ is a complete consistent theory such that no complete consistent expansion of $T$ by finitely many constants has a complete principal type. Prove that every model of $T$ has a proper elementary substructure.

11. $L(T) = \{<, c_i : i < \omega\}$. $T$ is a complete consistent theory which says that $<$ is a dense linear order without endpoints and the $c_i$’s are distinct constants. What are the possible cardinalities of the class of countable isomorphism types of models of $T$?

12. Let $T$ be the theory with countably many unary relation symbols $\{P_n : n \in \omega\}$ and all axioms of the form:

$$\exists \bar{x}(\land_{n \in A} P_n \land \land_{n \in B} \neg P_n)$$

where $A$ and $B$ are disjoint finite subsets of $\omega$. Show that $T$ is a complete theory.
13. Suppose $A_\alpha$ for $\alpha < \omega_1$ are countable and for all $\alpha < \omega_1$:

$$A_\alpha \cap (\bigcup_{\beta < \alpha} A_\beta)$$

is finite.

Show there exists $X \in [\omega_1]^{\omega_1}$ and a set $Z$ such that for every distinct $\alpha, \beta \in X$, $A_\alpha \cap A_\beta = Z$, i.e. an uncountable $\Delta$-system.

14. Assume $MA + \neg CH$ and suppose $a_\alpha \subset \omega$ for $\alpha < \omega_1$. Show there exists $X \in [\omega]^{\omega}$ such that for every $\alpha < \omega_1$

$$\text{either } X \subset^* a_\alpha \text{ or } X \cap a_\alpha =^* \emptyset$$

where $^*$ means modulo finite.

15. Show there exists an almost disjoint family $F$ of countably infinite subsets of the real line $\mathbb{R}$ such that for every uncountable $X \subset \mathbb{R}$ there exists a $Y \in F$ such that $Y \subset X$.

16. Assume $MA + \neg CH$ and suppose that $P$ is a poset with the ccc. and $\tau$ is a term in the forcing language of $P$ such that

$$1 \vdash \tau \subset \omega_1 \text{ is stationary}$$

Show that for some $P$-filter $G$ the set

$$\{ \alpha < \omega_1 : \exists p \in G \; p \vdash \alpha \in \tau \}$$

is stationary in $\omega_1$. 
