Qualifying Exam
Logic
August 28, 1986

Instructions: Do any **four** problems, but at most **two** elementary. Please use a separate packet of paper for each problem since not all of your answers will be graded by the same person. If you think a problem has been stated incorrectly, mention this to the proctor and indicate your interpretation in your solution. In such cases do not interpret the problem in such a way that it becomes trivial.

1 Elementary

1. Suppose T is a consistent first order theory such that for any sentence θ in the language of T, if $T \cup \{\theta\}$ is consistent, then $T \cup \{\theta\}$ is not complete, i.e. no finite extension of T is complete, for example Peano arithmetic. Show there exist a family $\{\theta_n \ : \ n \in \omega\}$ of sentences in the language of T which are completely independent of T. (i.e. for any $X \subset \omega$ the theory:
$$T \cup (\bigcup \{\theta_n \ : \ n \in X\}) \cup (\bigcup \{\neg\theta_n \ : \ n \notin X\})$$
is consistent.)

2. Let κ be an infinite cardinal and let λ be the least cardinal such that $\kappa^\lambda > \kappa$. Show that λ is regular.

3. Assume Con(ZF). Find a consistent theory $T \supseteq ZF$ and a first order sentence θ (in any language) such that
$$T \vdash \theta \text{ has a finite model}$$
but θ does not have a finite model.

2 Recursion theory

1. Prove or disprove:

   For every total recursive function f, there exists an $n < \omega$ satisfying

   $$W_{f(n)} = \{n, f(n)\}$$

2. Prove that if $A \leq_m 0^{(n)}$ where $1 \leq n < \omega$, then $A \leq_1 0^{(n)}$.

3. Prove that there are sets $A, B \subset \omega$ satisfying:

   (a) $0 <_T A <_T B$;

   (b) $\neg \exists C \subset \omega \ \ [A <_T C <_T B]$

3 Model theory

1. Let T be the theory with binary relation $<$ and unary operation f and axioms stating that:

   (a) $<$ is a strict linear ordering;

   (b) $<$ is dense and has no greatest or least element;

   (c) f is an automorphism of the ordering $<$; and

   (d) $\forall x \; x < f(x)$.

   Show that T is complete.

2. Let A be a model for a countable language with E and other relations such that E is an equivalence relation with each class countably infinite.

   (a) Prove that A has a countable elementary substructure B such that any element of A which is E-equivalent to an element of B is an element of B.

   (b) Prove that A has an elementary extension in which each E-equivalence class has cardinality the continuum. (Warning: A may be larger than the continuum.)

3. Suppose that T is a countable theory with an infinite model, but no countable saturated model. Show that T has uncountably many pairwise nonisomorphic models in every infinite power.

4 Set Theory

1. Say that a set A is quasi-finite iff for every $B \subset A$ either B is finite or $A \setminus B$ is finite. Show that Con(ZFC) implies Con(ZF+there exists an infinite quasi-finite set).

2. Suppose that M is a standard model of ZFC, P is a partial order in M such that $(P \text{ is countable})^M$, and G is P-generic over M. Show that for every $X \in M[G] \cap [\omega]^\omega$ there exists $Y \in M \cap [\omega]^\omega$ such that both $X \cap Y$ and $X \setminus Y$ are infinite.

3. Show that $\diamondsuit_{\omega_1}$ implies that there exists a Souslin tree $(T, \leq)$ which is rigid, i.e. every automorphism is the identity, but it is weakly homogeneous, i.e. for every $\alpha < \beta < \omega_1$ and $s, t \in T_\alpha$ there is an automorphism of $(T_{<\beta}, \leq)$ taking s to t, where:

$$T_\alpha = \{s \in T \quad : \quad \{t \in T \quad : \quad t \lhd s\} \text{ has order type } \alpha\}$$

   and

$$T_{<\beta} = \bigcup_{\alpha<\beta} T_\alpha$$