QUALIFYING EXAM IN LOGIC

JANUARY 20, 1983

INSTRUCTIONS: Do four questions; at most two elementary.

NOTATIONS:
1. \( \{\varphi_e \mid e < \omega\} \) is a standard enumeration of all partial recursive functions.

2. \( \mathcal{W}_e = \text{df dom } \varphi_e \)

3. \( \mathcal{W}_{e,s} = \text{df } \{x \leq s \mid \varphi_e(x) \text{ converges in } \leq s \text{ steps}\} \)

4. \( D_y = \{x_1 < x_2 < \cdots < x_n\} \) where \( y = 2^{x_1} + 2^{x_2} + \cdots + 2^{x_n} \).

POLICY ON MISPRINTS

The Doctoral Exam Committee tries to proofread the exams as carefully as possible. Nevertheless, the exam may contain misprints. If you are convinced a problem has been stated incorrectly, mention this to the proctor and indicate your interpretation in your solution. In such cases do not interpret the problem in such a way that it becomes trivial.
I. Elementary Questions

1. Find a set $S$ of sentences in an uncountable language such that
   a) for all $n \in \omega$, $S$ has a finite model of size $\geq n$.
   b) $S$ has no countably infinite models.

2. Let $L$ be a fixed finite language. If $S, T$ are consistent sets of sentences of $L$, say $S \Rightarrow T$ iff $S \models \varphi$ for all $\varphi \in T$.
   Show:
   a) If $T$ is r.e. and consistent, there is a complete $\Pi_1^0$ (co-r.e.) $S$ with $S \Rightarrow T$.
   b) There is an r.e. consistent $T$ with no complete r.e. $S$ such that $S \Rightarrow T$.

3. Prove the following or give a counterexample: Let $S_n, T_n$ ($n \in \omega$) be sets of sentences. Assume that for all $n$, $S_n \subseteq S_{n+1}$, $T_n \subseteq T_{n+1}$, and there is a model, $\mathcal{U}_n$ such that $\mathcal{U}_n \models T_n$ and $\mathcal{U}_n \not\models S_n$.
   Then there is a $L$ such that $L \models \bigcup_{n \in \omega} T_n$ and $L \not\models \bigcup_{n \in \omega} S_n$. 
II. Recursion Theory

1. Prove or disprove:

There exists a complete $\Delta_2$ set; i.e. $\exists A \in \Delta_2 \forall B \in \Delta_2 \exists f$ recursive $\forall n \in B \iff f(n) \in A$.

2. Let $\{U_{e,s} \mid e,s < \omega\}$ satisfy:

i) $U_{e,s} \subseteq U_{e,s+1}$; and

ii) $\exists f$ recursive $\forall e,s [U_{e,s} = D_{f(e,s)}]$.

Let $U_e = \bigcup_{s<\omega} U_{e,s}$.

Prove there is a recursive $g$ such that for all $e,s$:

1) $W_g(e) = U_e$; and

2) $W_{g(e),s+1} \subseteq U_{e,s}$.

Hint: Recursion Theorem.

3. Suppose a recursive $f$ satisfies

$\forall n,m \ [W_{f(n)} \neq \{0,1,2,\ldots,m-1\}]$.

Prove that there is a recursive $g$ satisfying:

1) $\forall i \exists j [W_i = W_{f(j)} \text{ or } W_i = W_{g(j)}]$;

2) $\forall i,j [W_{f(i)} \neq W_{g(j)}]$; and

3) $\forall i \neq j [W_{g(i)} \neq W_{g(j)}]$.

Hint: Construct $\{A_i\}_{i<\omega}$ such that there is a recursive $g$, $A_i = W_{g(i)}$ for all $i$. 
III. Model Theory

1. A model $A$ is almost-$\omega$-homogeneous iff there is an $\bar{a} \in |A|^<\omega$ such that $(A,\bar{a})$ is $\omega$-homogeneous. Prove that if every countable model of a theory $T$ is almost-$\omega$-homogeneous, then every model of $T$ is almost-$\omega$-homogeneous.

2. If $\Gamma,\Sigma$ are complete types of a theory $T$, say that $\Gamma$ forces $\Sigma$ iff whenever $\Gamma$ is realized in a model, then $\Sigma$ is realized too. Suppose $T$ is a complete theory with the property that if $\Gamma$ forces $\Sigma$ then either

i) $\Sigma$ is principal; or

ii) $\Sigma(x_{i_1},\ldots,x_{i_m}) \subseteq \Gamma(x_1,\ldots,x_n)$ for some $1 \leq i_1 \leq i_2 \leq \cdots \leq i_m \leq n$.

Suppose $\Gamma(x)$ is a complete 1-type. Prove that there is a homogeneous model of $T$ omitting $\Gamma(x)$.

3. Assume that $R,S \subseteq \omega \times \omega$ are recursive, $(\omega,R) \equiv (\omega,S)$, and $(\omega,R)$ is homogeneous. Prove that there is an $f \in \Lambda^0_3$ such that $f : (\omega,R) \equiv (\omega,S)$.

4. Let $U$ be a uniform ultrafilter on $\omega_1$. If $K$ is a family of structures for $L$, let $K^*$ be the set of all structures of the form $\prod \mathcal{U}_\alpha/U$, where each $\mathcal{U}_\alpha \in K$. Let $S$ be a set of sentences of $L$ with $|S| = \omega_1$. Assume that for each finite $F \subseteq S$, there is an $\mathcal{U} \in K$ with $\mathcal{U} \models F$. Show that there is an $\mathcal{U} \in (K^*)^*$ with $\mathcal{U} \models S$. 
IV. Set Theory

1. Let $\mathcal{M}$ be a countable transitive model for ZFC + GCH. Assume that $\mathcal{P} \in \mathcal{M}$, $\mathcal{P}$ is a partial order, and $(\mathcal{P} \text{ is c.c.c.})^\mathcal{M}$. Let $G$ be $\mathcal{P}$-generic over $\mathcal{M}$. Show that the following holds in $\mathcal{M}[G]$:

$$\forall F \subset \omega_1 (|F| \geq \omega_3 \rightarrow \exists f, g \in F (f \neq g \land \{\alpha < \omega_1 : f(\alpha) = g(\alpha)\} = \omega_1)).$$

2. If $X$ is a set of real numbers, call $X$ a Bernstein set iff $|X| \geq \omega_1$ and $X$ has no perfect subsets. Suppose that $x_\alpha$, for $\alpha < \omega_1$, are distinct real numbers. Show that there is a closed unbounded $C \subset \omega_1$ such that $\{x_\alpha : \alpha \in C\}$ is a Bernstein set.

3. Assume $\text{MA} + \neg \text{CH}$. For each limit $\gamma \in \omega_1$, let $A_\gamma$ be a cofinal $\omega$-sequence. Show that there are $B_n \subset \omega_1$ for $n \in \omega$ such that each $B_n \cap A_\gamma$ is finite and $\bigcup_{n \in \omega} B_n = \omega_1$. 