Qualifying Exam

LOGIC

August 26, 1982

Do FOUR of the following problems, at most two elementary.

I. Elementary Questions

1. Classify each of the following classes (with regard to first-order logic) as
   a) finitely axiomatizable (give the axioms)
   b) axiomatizable but not finitely axiomatizable (give the axioms and a proof that no finite set of axioms suffices).
   c) not axiomatizable (give proof).

   i) The class of finite linearly ordered sets, \( (A, <) \)
   ii) The class of infinite linearly ordered sets, \( (A, <) \)
   iii) The class of densely linearly ordered sets, \( (A, <) \)
   iv) The class of well-ordered sets, \( (A, <) \).

2. Prove the following or give a counter-example. Let \( S \) be a decidable (i.e., recursive) set of sentences in propositional logic, using proposition letters \( P_n \ (n \in \omega) \). Then \( \{ \varphi : S \vdash \varphi \} \) is decidable.

3. Give a counterexample to the following statement:

   If \( \alpha, \gamma \) are ordinals and \( A \subseteq \gamma, A \cong \alpha \) and \( (\gamma - A) \cong \alpha \),
   then \( \gamma = \alpha + \alpha \) or \( \gamma = \alpha \).
II. Model Theory.

1. Which of the following properties of complete theories and/or models are preserved under the operation of taking reducts (explain):

   i) $\aleph_0$-categoricity;
   iv) being prime;

   ii) $\aleph_1$-categoricity;
   v) being saturated;

   iii) $\omega$-stability;
   vi) having only finitely many countable models.

2. Assume $T$ is a complete theory, $\{\Gamma_\eta | \eta \in 2^{<\omega}\}$ and $\{A_\eta | \eta \in 2^{<\omega}\}$ are types and models of $T$ respectively, satisfying:

   i) if $\eta \subseteq \xi$, then $A_\xi$ realizes $\Gamma_\eta$ and $\Gamma_\eta \subseteq \Gamma_\xi$; and

   ii) if $\eta$ and $\xi$ are incompatible, then $A_\xi$ omits $\Gamma_\eta$.

   Prove then that $T$ has $2^\aleph_0$ pairwise non-isomorphic countable models.

3. Suppose $M \prec M'$, $a, b \in |M|$, $c \in |M'|$, $p, q$ types over $M$, $\varphi(x, a)$ a formula with parameters from $|M| \cup \{a\}$ satisfy:

   i) $a$ realizes $p$ and $b$ realizes $q$ in $M'$;

   ii) $p(\bar{x}) \cup q(\bar{y})$ is a complete type; and

   iii) $\varphi(x, a)$ is a complete formula for the type over $|M| \cup \{a\}$ realized by $c$.

   Prove then that $\varphi(x, a)$ is also a complete formula for the type that $c$ realizes over $|M| \cup \{a, b\}$. [Hint: Otherwise there is $c' \in M'' \succ M'$ realizing the same type as $c$ over $|M| \cup \{a\}$, but not over $|M| \cup \{a, b\}$. Proceed from there!].
III. Recursion Theory.

1. Prove that there is no recursive \( f \) satisfying:

   i) \( W_f(n) \) codes a well order, \( n < \omega \);

   ii) if \( W_n \) is a well order, then \( W_n = W_f(n) \), \( n < \omega \).

   \[ A \subseteq \omega \text{ codes a well order if } \langle \text{ran } A \cup \text{dom } A, \{ (x,y) \mid (x,y) \in A \} \rangle \]
   is a well order, where \( \text{ran } A = \{ y \mid \exists x ((x,y) \in A) \} \) and
   \( \text{dom } A = \{ x \mid \exists y ((x,y) \in A) \} \).

   [Hint: One way to do this uses the recursion theorem].

2. Suppose \( C \) is a non-recursive \( \Delta^0_2 \) set. Prove that there is a
   simple set \( A \) such that \( C \not\leq_T A \). \[ \underline{Hint:} \text{ Use the limit lemma to}
   represent } C \text{ as } \lim_{s} C_s, \text{ where } \{C_s\}_{s<\omega} \text{ is a recursive}
   sequence.

3. Suppose \( A \) is a countable, saturated structure realizing recursive types
   \( \Gamma_1, \Gamma_2 \). Prove that there is a decidable structure realizing \( \Gamma_1 \) and \( \Gamma_2 \).

4. Suppose \( \{ A_s \}_{s<\omega} \) is an effective enumeration of \( A \), where each \( A_s \)
   is recursive, and \( \forall e ( \exists \leq_s (e)^s + (e)^A(e)^+ ) \).

   Prove \( A \) is low, i.e. \( A' = 0' \).
IV. Set Theory.

1. Let \((A, <)\) be a linear order such that \(\forall X \subseteq A \ (X \approx A \text{ or } A - X \approx A)\).
   Show \(A\) is a well order or an inverse well order.

2. Call \(F\) a mad \(f\) (maximal almost disjoint family of subsets of \(\omega_1\)) iff
   \[
   \begin{align*}
   &i) \ F \subseteq P(\omega_1) \\
   &\forall x \in F \ (|x| = \omega_1) \\
   &\forall x, y \in F \ (x \neq y \rightarrow |x \cap y| < \omega_1) \\
   &\text{iv)} \ F \text{ is maximal with respect to (i) - (iii).}
   \end{align*}
   \]
   Assume: \(M\) is a countable transitive model of ZFC, \(F \in M\), and \((F \text{ is a mad } f)^M\).
   Show that \(F\) remains a mad \(f\) in any ccc forcing extension of \(M\).

3. If \(f, g \in \omega^\omega\), we say \(f <^* g\) iff \(\{n : g(n) < f(n)\}\) is finite.
   Let \(\alpha\) be the least ordinal which is not the type of a chain in the partial order, \((\omega^\omega, <^*)\).
   Prove that \(\alpha\) is a regular cardinal.

4. Assume that there is an ordinal \(\alpha\) such that \(R(\alpha) \models \text{ZFC}\), and let \(\alpha\) be the smallest such.
   Prove that \(\alpha\) is a strong limit cardinal and that \(\text{cf}(\alpha) = \omega\).