Qualifying Examination in Logic

January 1980

Instructions: Do 5 problems, not more than 3 from one part.
Model Theory

1. Let $L(R_0, R_1, \ldots)$ be the language formed by adding countably many relation symbols $R_0, R_1, R_2, \ldots$ to the countable language $L$. Let $T$ be a complete theory in $L(R_0, R_1, \ldots)$ and $T_n$ the set of all consequences of $T$ in $L(R_0, \ldots, R_n)$. Let $\Sigma(x)$ be a set of formulas of $L$. Suppose each $T_n$ has a model which omits $\Sigma(x)$. Prove that $T$ has a model which omits $\Sigma(x)$.

2. Let $(X, <)$ be an infinite set of indiscernibles in a model $A$ with built-in Skolem functions. Show that for each $Y \subseteq X$, $A$ has an elementary submodel $B$ such that $B \cap x = y$.

3. Give an example of a model $A$ for a countable language such that $A$ has power $\omega_1$ but every proper elementary submodel of $A$ is countable.

4. Let $D$ be an ultrafilter over $I$. Suppose that for each $i \in I$, the model $A$ is elementarily embeddable in the model $B_i$. Prove that $A$ is elementarily embeddable in the ultrapower $\pi_D B_i$.

Set Theory

1. Prove that if $\alpha$ and $\beta$ are limit ordinals, $\alpha < \beta$, and $(R(\alpha), \varepsilon)$ is an elementary submodel of $(R(\beta), \varepsilon)$, then $(R(\alpha), \varepsilon)$ is a model of ZFC.

2. (a) Show $ZF \vdash \forall x (P(x) \neq x)$

(b) Show that if $ZF$ is consistent then so is $ZF^+ + \exists x (P(x) \vDash x)$, where $ZF^+$ is $ZF$ without the axiom of regularity.

int: Try to find a model with an $x, y$ such that $x = \{y\}$, $y = \{x, 0\}$ (so that $y = P(x)$).
3. Assume that ZF is consistent. Show that there is a finite subtheory \( T \) of ZF such that in ZF it cannot be proved that \( T \cup \text{"there is an uncountable inaccessible cardinal"} \) is consistent.

4. Let \( M \) be a transitive model of ZF + "every uncountable cardinal is singular". Show that no transitive set \( N \) with \( M \in N \), \( M \cap \text{Ord} = N \cap \text{Ord} \), satisfies ZFC.

C. Recursion Theory

1. Let \( T \) be a recursively axiomatized theory in a countable language with finite and infinite models such that \( T \) is \( \omega_1 \)-categorical. Prove that \( T \) has a decidable model.

2. Show that there is a sequence \( f_{\alpha}, \alpha < \omega_1 \) of functions mapping \( \omega \) into \( \omega \) such that whenever \( \alpha < \beta < \omega_1 \), \( f_{\alpha} \) is recursive in \( f_{\beta} \) but \( f_{\beta} \) is not recursive in \( f_{\alpha} \).

3. Show that there is an \( e \) such that \( d_{e} \) is the characteristic function of the set \( \{0, 1, \ldots, e\} \), where \( \{d_{i} \mid i < \omega\} \) is an effective enumeration of all partial recursive functions.

4. Call a formula \( \varphi(x) \) strongly finite if in every model \( M \) of Peano arithmetic, only a finite number of \( m \in M \) satisfy \( \varphi \). Prove that the set of Gödel numbers of strongly finite formulas is r.e. but not recursive.