Qualifying Examination in Logic

January, 1979

A. Elementary Problems.

1. Let L be a countable language without function symbols. State and prove the usual Compactness Theorem as it applies to L .

2. Work in ZF without the axiom of choice. Show that (a) iff (b) .

(a) Every structure $\mathfrak{A} = \langle A, R \rangle$, R binary, has a countable elementary substructure.

(b) The axiom of dependent choice; that is: If $R \subseteq X \times X$, where X is non-empty, and if for every $a \in X$ there is a $b \in X$ such that $\langle a, b \rangle \in R$ then there is a function for the natural numbers such that, for every n ,

$$\langle f(n), f(n+1) \rangle \in R \quad .$$

[Hint for (b) $\Longrightarrow$ (a) . Choose X cleverly.]

3. Consider weak second order logic, logic where we allow extra quantifiers $\forall X_i$, $\exists X_i$ where X_i ranges over arbitrary finite subsets of the domain. Give a proof or counter example to each of the following:

a) If φ has an infinite model then it has an uncountable model.

b) If φ has an infinite model then it has a countable model.

4. Let $L = \{+, -, 0\}$ be the language of abelian groups. Prove that the class of divisible abelian groups is axiomatizable but is not finitely axiomatizable. (G is divisible if for each $x \in G$ and each natural number n there is a $y \in G$ such that $n \cdot y = x$.)

B. Model Theory.

1. Prove that every model of ZF has an elementary extension which is ω_1-saturated but not ω_2-saturated.

2. Let T be a κ-stable complete theory in a countable language. Prove that T has a model M of power κ in which every model of T of power κ is elementarily embeddable. (i.e. M is κ^+-universal).

3. Let N be an elementary extension of M and let M_1 be an extension of M. Prove that N has an extension N_1 elementarily equivalent to M_1.

4. Let M be the model

$$\langle \mathbb{R} \times \mathbb{R}, \quad E_1, E_2 \rangle$$

where $(x_1, x_2) E_i \; (y_1, y_2)$ iff x_i and y_i have the same integral part. Find the Morley rank of the theory of M.

C. Recursion Theory.

1. State the normal form theorem for a function $\varphi(a)$ recursive in a function $\alpha(x)$. Sketch the method of proof (a and x range over the natural numbers).

2. State and prove the theorem that all arithmetical predicates P(a) (a ranging on the natural numbers) fall into a hierarchy called the "arithmetical hierarchy."

3. Define the system $\underline{0}$ of notations for constructive ordinals, and show how they can be used to define an extension of the arithmetic hierarchy, called the hyperarithmetic hierarchy.