INSTRUCTIONS: Do five problems, at most two from part A.
Do not use the continuum hypothesis.

Glossary:
\[ |x| = \text{cardinality of } x. \]
\[ \operatorname{cf}(\lambda) = \text{cofinality of } \lambda. \]
\[ R(\alpha) = \{x : x \text{ has rank } < \alpha\}. \]
\[ U \text{ is } \omega \text{-homogeneous if whenever } (U, a_1, \ldots, a_n) \equiv (U, b_1, \ldots, b_n), \]
we have \[ \forall c \exists d (U, a_1, \ldots, a_n, c) \equiv (U, b_1, \ldots, b_n, d). \]
\[ \mathbb{Q} = \text{set of rational numbers}. \]
\[ \Diamond(S) \text{ means that } S \subseteq \omega_1 \text{ and there is a family of sets } A_\alpha \subseteq \alpha, \alpha \in S, \]
such that for all \[ A \subseteq \omega_1, \]
\[ \{\alpha \in S : A \cap \alpha = A_\alpha\} \]
is stationary in \[ \omega_1. \]
\[ \Diamond \text{ means } \Diamond(\omega_1). \]
\[ \text{ZF is Zermelo-Fraenkel set theory.} \]
A. Elementary Problems

A1. Let \( \mathbb{C} = \langle \mathbb{C}, +, \cdot, 0, 1 \rangle \) be the field of complex numbers and let \( R \subseteq \mathbb{C} \) be the set of real numbers. Show that \( R \) is not definable in \( \mathbb{C} \).

A2. Let \( \text{HF} = \langle \text{HF}, \in \rangle \) be the structure of hereditarily finite sets. State and prove a version of Tarski's Theorem on Truth which applies to \( \text{HF} \).

A3. Let \( T \) be a universal theory. Assume \( T \models \forall x \exists y P(x, y) \). Show that there are terms \( t_1(x), \ldots, t_n(x) \) such that \( T \models \forall x \bigvee_{m=1}^{n} P(x, t_m(x)) \).

A4. Let \( \kappa \) be a cardinal and let

\[
\lambda = \sup \{ 2^\alpha : \alpha < \kappa \}.
\]

Show that \( \text{cf}(\lambda) = \text{cf}(\kappa) \) or \( \text{cf}(\lambda) > \kappa \).

A5. Find the mistake in the following proof.

a) For each finite \( S \subseteq \text{ZF} \), \( \text{ZF} \models (\exists \alpha)((\alpha, \in) \text{ is a model of } S) \).

b) \( \text{ZF} \models (\text{If every finite } S \subseteq \text{ZF} \text{ has a model, then } \text{ZF} \text{ has a model}) \).

c) By a) and b) \( \text{ZF} \models (\text{ZF has a model}) \).

d) By Gödel's second theorem and c), \( \text{ZF} \) is inconsistent.
B. Model Theory

B1. Let $T$ be the theory with the axioms
\[
\begin{align*}
\forall y \exists x & \ y = F(x) \\
\forall x \forall y & \ F(x) = F(y) \rightarrow x = y
\end{align*}
\]

Show that every complete extension of $T$ is $\omega$-stable and has Morley rank at most two.

B2. Let $\mathcal{U} = \langle A, <, \cdots \rangle$ be an $\omega$-homogeneous model for a countable language such that $<$ well orders $A$. Prove that $A$ has cardinality at most $2^\omega$.

In problems B3-B5, let $T$ be a countable complete theory whose models are infinite.

B3. Prove that $T$ has a family of countable models $\mathcal{U}_S$, $S \subseteq \omega$, such that if $S$ is a proper subset of $T$ then $\mathcal{U}_S$ is a proper elementary submodel of $\mathcal{U}_T$. Hint: Use indiscernibles.

B4. Show that $T$ has an $\omega$-homogeneous model of power $\omega_1$ with only countably many types. Hint: Similar to Vaught's two-cardinal argument.

B5. (Shelah). Let $S$ be a set of fewer than $2^\omega$ types $\Gamma(x)$ which are maximal consistent with $T$ and locally omitted by $T$. Prove that $T$ has a model which simultaneously omits each $\Gamma(x) \in S$.

Hint: Represent the Henkin construction by a binary tree.
C. Recursion Theory

C1. Find a function $d : \omega \times \omega \rightarrow \mathbb{Q}$ such that:
   a) $d$ is recursive.
   b) $d$ is a metric.
   c) the set \{n $\in$ $\omega$ : n is isolated in the space $(\omega, d)$\} is not recursive.

C2. Show that there is a set of Turing degrees \{d$_q$ : q $\in$ $\mathbb{Q}$\} such that q < r implies $0 < d_q < d_r < 0'$.

C3. Let $f : \omega \rightarrow \omega$ be recursive. Show that there is a function $g : \omega \rightarrow \omega$ such that $f$ is primitive recursive in $g$ and $g$ has a primitive recursive graph.

C4. Let $C(X)$ be a $\Sigma_1^1$ predicate with no $\Delta_1^1$ solutions. Prove that the set of solutions of $C(X)$ has cardinality $2^\omega$.

The following problems are based on the topics course in admissible sets.

C5. Let $\alpha$ be the first admissible ordinal $> \omega_1$. Show that $L_\alpha$ has property Beta. Conclude that there is a $\Delta_2^1$ ordinal $\alpha$ which is admissible but not recursively inaccessible such that $L_\alpha$ has property Beta. (You may assume any theorem proved in Barwise's book.)
C6. Let \( \mathfrak{m} = (M, <, p, R, \cdots, R_y) \) be a structure where \( < \) well-orders \( M \) and \( p \) is a pairing function. Using the relation between \( \text{HYP}_\mathfrak{m} \) and inductions on \( \mathfrak{m} \) prove the following uniformization theorem:

For every inductive relation \( R \) there is an inductive relation \( S \subseteq R \) such that \( \text{dom}(R) = \text{dom}(S) \) and, for all \( x \in \text{dom}(R) \)

\[ \exists ! y \; S(x, y) \]
D. Set Theory

D1. Prove that $\text{ZF} \vdash \text{Con}(\text{ZFL-P})$, where ZFL-P is ZF with the axiom of constructibility but not the power set axiom.

D2. Show that forcing with $\mathbb{P}$ collapses $\aleph_\omega$ to $\omega$, where $\mathbb{P}$ is the set of all partial functions $p : \aleph_\omega \to 2$ with $|\text{domain } p| < \aleph_\omega$.

D3. For $\alpha$ a limit ordinal less than $\omega_1$, let $C_\alpha$ be a cofinal $\omega$-sequence in $\alpha$. Show that there is an uncountable set $X \subset \omega_1$ of limit ordinals such that

$$\forall \alpha, \beta \in X (\alpha < \beta \rightarrow \alpha \notin C_\beta)$$

D4. Assume $\diamondsuit$. Show that there is an $S \subset \omega_1$ such that $\diamondsuit(S)$ and $\diamondsuit(\omega_1 \setminus S)$. Hint: Consider the ideal,

$$I = \{S \subset \omega_1 : \text{not } \diamondsuit(S)\}$$

D5. Let $M$ be a countable, transitive model of ZFC, let $M[G]$ be a $\mathbb{P}$-generic extension of $M$ where $\mathbb{P}$ is c.c.c. in $M$. Let $X, Y \in M$. Show that for every $F \in M[G]$ with $F : X \to Y$ there is a $f \in M$ such that for every $x \in X$, we have $f(x) \subset Y$, $(|f(x)| \leq \omega)^M$ and $F(x) \in f(x)$. 