A. Elementary problems

1. State the Completeness Theorem for first-order logic and give a sketch of its proof.

2. Let $T$ be a theory in a countable language with at least one function symbol, and assume that $T$ has an infinite model. Show that $T$ has a countable model which is not finitely generated.

3. Show $|\gamma|^{\omega} = (\omega^\omega)^\omega$ (cardinal exponentiation).

4. Assume ZFC is consistent. Show that there is a recursive extension $T$ of ZFC such that $T$ is consistent and $T \vdash \neg \text{Con}(T)$.

5. For ordinals $\alpha, \beta$, define $\alpha \neq \beta = \sup \{ \gamma : 3A \subseteq \gamma : \text{type}(A) = \alpha \land \text{type}(\gamma - A) = \beta \}$.
Compute $(\omega^3 + \omega) \neq (\omega^2 + 1)$, and prove your answer is correct.

B. Model Theory

1. Let $\kappa$ be an infinite model and let $\kappa \geq |\kappa|$ be such that $\kappa^\omega = \kappa$. Show that there is a $\kappa > \kappa$ such that $|\kappa| = \kappa$.

   Note. There is no restriction on the cardinality of the language.

2. Let $T$ be the complete theory of the model

$$\langle \mathbb{Q}, <, 1, \frac{1}{2}, \frac{1}{3}, \ldots, -1, -\frac{1}{2}, -\frac{1}{3}, \ldots \rangle$$

where $\mathbb{Q}$ is the set of rationals. Determine how many non-isomorphic countable models $T$ has, and identify the prime and countable saturated models. Note. The language has one binary relation plus a constant symbol for each of: $1, \frac{1}{2}, \frac{1}{3}, \ldots, -1, -\frac{1}{2}, -\frac{1}{3}, \ldots$
3. Show that there is no set of sentences $T$, in the language of group theory, such that the models of $T$ are precisely the free groups with $\geq 2$ generators.

**C. Recursion Theory**

1. Define $a \in b$ iff $a < W^1_0$. Show that there is a recursive $S \subseteq \omega$ and a recursive total order $<_q$ of $S$ isomorphic to the rationals such that for $a, b \in S$, $a <_q b$ iff $a < b$.

2. Show that there is a recursive total order whose well-founded initial segment has type $\omega^*_\alpha$.

3. Show that there is no r.e. set $A \subseteq \omega$ such that for all $\epsilon \in \omega$, if $\phi^1_\epsilon$ is total then $\phi^1_\epsilon$ is 1-1 iff $\epsilon < A$.

**D. Set Theory**

1. For $x, y \in P(\omega)$, define $x \leq^L y$ iff $x \subseteq L[y]$. Show that it is consistent with $\text{ZFC} + \text{GCH}$ that there is an $\mathcal{U} \subseteq P(\omega)$ such that $|\mathcal{U}| = \omega_1$ and the elements of $\mathcal{U}$ are pairwise incomparable under $\leq^L$.

2. Assume $\text{MA} + \text{GCH}$. Let $X_\alpha \subseteq [0, 1]$ for $\alpha < \omega_1$, and assume each $X_\alpha$ is Lebesgue measurable and has positive measure. Show that for some $\mathcal{A} \subseteq \omega_1$, $|\mathcal{A}| = \omega_1$ and $\cap_{\alpha \in \mathcal{A}} X_\alpha$ has positive measure.

3. Assume $\exists \mathcal{A} \subseteq \omega_1 (V = L[\mathcal{A}])$. Prove $\text{CH}$. 