A. Elementary

1. In ordinal arithmetic, multiply (putting your answer in canonical form, base ω):

$$(\omega^3 + \omega^2 \cdot 5 + 7) \cdot (\omega^2 \cdot 2 + 4) \ .$$

2. In cardinal arithmetic, compute (using the Axiom of Choice but not G(H):

(i) $(\beth_{\omega_1})^{\aleph_0}$

(ii) $(\beth_{\omega_1})^{\aleph_1}$

(iii) $(\beth_{\omega_1})^{\beth_\omega}$

3. [Work in ZFC.] Assume there is an inaccessible cardinal κ. Let α be the least ordinal such that $R(\alpha) \models \mathrm{ZFC}$. Prove that α is a cardinal and $cf(\alpha) = \omega$.

4. Outline a straight-forward proof that if ZFC has a standard model, then ZFC is not decidable. [Of course, this is an easy version of Gödel's Second Theorem.]

5. Prove that there is a ring R elementarily equivalent to the ring of integers, but where R is not Noetherian.

6. An orderable group is a group $\langle G, \cdot \rangle$ for which there exists a linear ordering $\leq$ such that:

$$\forall x, y, z\,[x \leq y \rightarrow (z \cdot x \leq z \cdot y \wedge x \cdot z \leq y \cdot z)] \ .$$

Show that there exists a set of universal axioms in the language of groups whose

B. Recursion Theory

The problems in this part are open ended. You are not expected to provide all details.

1. State the normal form theorem for a partial recursive function $\varphi(x_1, \ldots, x_n)$ (= Turing computable partial function) and indicate the method of proof. How is the statement altered for a function $\varphi(x_1, \ldots, x_n)$ partial recursive in a 1-place function $\alpha(x)$? What new idea is needed in the proof?

2. State and prove the recursion theorem.

3. Define a universal system of ordinal notations, give an example of one, and outline the proof that it is universal.

4. Describe the arithmetical hierarchy, prove that each of the classes of relations in it contains relations not in any of the classes except ones higher in the hierarchy, and sketch the proof of Post's theorem.

C. Model Theory

1. Let T be a complete theory. Suppose T has models $\mathfrak{A}$ and $\mathfrak{B}$ such that $\mathfrak{A} \cong \mathfrak{A} \times \mathfrak{A} \times \mathfrak{B}$ and $\mathfrak{B} \cong \mathfrak{B} \times \mathfrak{B} \times \mathfrak{B} \times \mathfrak{B}$. Prove that T has a model $\mathfrak{C}$ such that $\mathfrak{C} \cong \mathfrak{C} \times \mathfrak{C}$.

2. Prove that every model of Peano arithmetic has an elementary extension which is not ω_1-saturated.

3. Let T be a complete theory in a uncountably language with uncounted many types $\Sigma(x)$. Show that for each cardinal $\kappa > \omega$, T has a model of power κ in which exactly ω_1 types $\Sigma(x)$ are realized.

4. Let D be a countably incomplete ultrafilter. Show that the ultrapower $\Pi_D(\omega, <)$ has a decreasing sequence of length ω_1.

this is garbled
perhaps it means "ctble languge".
or $|L_T| = \omega_1$
Could we indisc. in this case.

D. Set Theory

1. Let T be a recursive complete theory in ordinary first-order logic. Show (T is ω-stable) $\Longleftrightarrow$ (T is ω-stable)^{L}.

2. Show that it is not provable in ZFC that

$$\exists\alpha\beta[(L(\alpha), \epsilon) \prec (L(\beta), \epsilon) \text{ and } \alpha < \beta \text{ and } \alpha \text{ is a cardinal}]$$

(assuming ZFC is consistent).

3. Let $f : \omega \to 2$ be Cohen generic over L. Show $f\restriction\{2n : n \in \omega\} \notin L[f\restriction\{2n+1 : n \in \omega\}]$.

4. Assume MA + $\neg$CH. Let $F \subset P(\omega)$, such that $|F| = \omega_1$ and all finite intersections from F are infinite. Show that for some infinite $x \subset \omega$, $|\{y \in F : x \subseteq y\}| = \omega_1$.