A. Elementary Problems.

1. Let $T$ be a finitely axiomatizable theory with only a countable number of complete extensions. Prove that one of these complete extensions is finitely axiomatizable.

2. Let $T$ be a model-complete theory. Prove that $T$ is complete if and only if for every two models $M, N$ of $T$ there is a model $D$ of $T$ such that both $M, N$ are isomorphic to submodels of $D$.

3. Prove in ZFC that if $0 < \lambda \leq \kappa$ then
   
   \[(\kappa^+)^\lambda = \max(\kappa^\lambda, \kappa^+)\]

4. Prove that for every set $X$ and ordinal $\alpha$ there is a function $f$ with domain $\alpha$ such that $f(0) = X$ and for all $\beta < \gamma < \alpha$, $f(\beta) \subseteq f(\gamma)$.

5. Let $\mathcal{A} = (\mathbb{A}, <)$, $\mathcal{B} = (\mathbb{B}, <')$ be infinite linear orderings. Prove that $\mathcal{B}$ is isomorphic to a submodel of some elementary extension of $\mathcal{A}$.
B. Model Theory

Always assume the language is countable.

1. Let $T$ be a complete theory which has $2^{|T|}$ complete types. Show that for each infinite cardinal $\kappa \leq 2^{|T|}$ there are models of $T$ of arbitrarily large cardinality which realize exactly $\kappa$ complete types.

2. Let $\kappa$ be an inaccessible cardinal and let $\mathfrak{M}$ be a saturated model of power $\kappa$. Prove that $\mathfrak{M}$ is the union of a proper elementary chain $\mathfrak{M}_\beta$, $\beta < \kappa$, of models isomorphic to $\mathfrak{M}$.

3. Let $\mathcal{D}$ be an ultrafilter and let $\mathfrak{M} \times \mathfrak{N}$ be the direct product of $\mathfrak{M}$ and $\mathfrak{N}$. Prove that $\mathcal{D}(\mathfrak{M} \times \mathfrak{N}) = \mathcal{D}\mathfrak{M} \times \mathcal{D}\mathfrak{N}$.

4. Prove that the complete theory of $(\mathbb{Z}, <)$ has exactly 2 non-isomorphic countably homogeneous models.
G. Recursion Theory.

1. Let \( \prec \) be a recursive well ordering of type \( \omega \). Show that there is a recursive well ordering of type \( \omega^\sigma \) (ordinal exponentiation).

2. Let \( d \) be a Turing degree with the property that
\[
\exists x \in d \, \forall y \leq_T d \left[ y = x \right]
\]
Prove that \( 0' \not\leq d \).

3. Prove that there is a recursive linear ordering \( \prec \) such that \( \prec \) is not a well ordering but such that \( \prec \) has no arithmetic descending sequences.

4. Suppose that for every set \( A \) of natural numbers there is a (unique) set \( B \) such that,
\[
\forall n [K_B(n) = \varphi_e^A(n)]
\]
Write \( \Upsilon(A) = B \) if the above holds. Prove that there is a recursive function \( \Upsilon \) and a recursive relation \( R \) such that for all \( A \),
\[
\Upsilon(A) = \{ n \mid R(\varphi_e^A(n)) \}
\]
D. Set Theory

1. Prove that if \( \langle R(a), a \rangle \) is a model of \( \mathcal{N} \), then \( a \) is a cardinal. Show that if there is such an \( a \) then the least such \( a \) has cofinality \( \omega \).

2. Assume that \( \text{ZFC} \) has a standard model. Let \( \mathcal{L}_n \) be the set of all sentences \( \varphi \) such that in every standard model \( M \) of \( \text{ZFC} \),

\[
M \models \left( \text{\( \mathcal{L}_n \) is a model of \( \varphi \)}\right)
\]

Prove that \( \mathcal{T}_1 \subseteq \mathcal{T}_2 \subseteq \ldots \subseteq \mathcal{T}_n \subseteq \ldots \).

3. For \( x, y \leq \omega \) define \( x \leq_L y \) iff \( x \in I[y] \). Assume that \( \forall x, y \left( x \leq_L y \text{ or } y \leq_L x \right) \). Prove that \( 2^{\aleph_0} \leq \aleph_2 \).

4. Let \( \kappa \) be inaccessible. Let

\[
A = \{ \alpha < \kappa \mid R(\alpha) \models \text{ZFC} \}
\]

Prove that \( A \) is not closed but that it contains a set \( B \) which is closed and unbounded in \( \kappa \).