Logic Ph D Qualifying Exam

January, 1972

Instructions.

Majors and Minors: Do five problems, at most three from one section.

Third area: Do four problems.

A. Model Theory.

A1. The diagram of a model $\mathfrak{A}$ is the set $D(\mathfrak{A})$ of all atomic and negated atomic sentences true in $(\mathfrak{A}, a)_{a \in A}$. A model $\mathfrak{A}$ of a theory T is said to complete T if $T \cup D(\mathfrak{A})$ is complete. Prove that if $\mathfrak{A}$ completes T then every elementary submodel $\mathfrak{B}$ of $\mathfrak{A}$ completes T.

A2. Let $\mathfrak{A} = \langle A, <, \ldots \rangle$ be a model where $<$ is a linear ordering with no last element. Prove that there is an elementary extension $\mathfrak{B}$ of $\mathfrak{A}$ such that every subset of B of power $\leq \aleph_{47}$ has an upper bound.

A3. Let $L(R_0, R_1, \ldots)$ be the language formed by adding countably many relation symbols $R_0, R_1, \ldots$ to the countable language L. Let T be a complete theory in $L(R_0, R_1, \ldots)$ and T_n the set of all consequences of T in $L(R_0, \ldots, R_n)$. Let $\Sigma(x)$ be a set of formulas of L. Suppose each T_n has a model which omits $\Sigma(x)$. Prove that T has a model which omits $\Sigma(x)$.

A. Model Theory (Continued).

A4. Let $\mathfrak{A} = \langle \omega, + \rangle$ be the standard model of additive number theory and let D be a non-principal ultrafilter over ω. Prove that in the ultrapower $\Pi_D \mathfrak{A}$ there is an element $a \neq 0$ such that for all positive $n < \omega$, there is a b with $\underbrace{b + \ldots + b}_{n} = a$.

B. Set Theory.

B1. Prove without using Gödel's Theorem that ZF is not finitely axiomatizable.

B2. If $2^{\aleph_0} = \aleph_2$ prove $\aleph_3^{\aleph_0} = \aleph_3$. You may use the following facts only: $\left(2^{\aleph_0}\right)^{\aleph_0} = 2^{\aleph_0}$, $\aleph_3 \cdot \aleph_3 = \aleph_3$, $\aleph_3$ is regular.

B3. Let M be a transitive model of ZF + "every uncountable cardinal is singular". (A cardinal is an initial ordinal.) Show that no transitive set N with $M \subseteq N$, $M \cap \mathrm{Ord} = N \cap \mathrm{Ord}$, satisfies ZF + AC.

B4. Show that for every infinite ordinal α, there is a countable transitive set A with $\langle R_\alpha, \varepsilon \rangle \equiv \langle A, \varepsilon \rangle$.

C. Recursion Theory.

C1. Call a formula $\varphi(x)$ strongly finite if in every model M of Peano arithmetic, only a finite number of $m \in M$ satisfy φ. Prove that the set of Gödel numbers of strongly finite formulas is r.e.