Logic Ph D Qualifying Exam

September, 1971

Instructions:
Majors and Minors: Do five problems.
Third area: Do four problems.

1. **Model Theory. (Do at most three)**

Let $D$ be an ultrafilter over $I$. Suppose that for each $i \in I$, the model $M_i$ is elementarily embeddable in the model $L_i^0$. Prove that $\mathcal{M}$ is elementarily embeddable in the ultrafilter $\prod_D L_i^0$.

2. Let $\langle X, < \rangle$ be an infinite set of indiscernibles in a model $\mathcal{M}$. Let $\langle Y, < \rangle$ be any linearly ordered set. Prove that there is a model $\mathcal{L} \equiv \mathcal{M}$ such that $Y \subseteq B$ and for all increasing $n$-tuples $x_1 < \ldots < x_n$ from $X$ and $y_1 < \ldots < y_n$ from $Y$,

$$ (\mathcal{L}, x_1 \ldots x_n) \equiv (\mathcal{L}, y_1 \ldots y_n) $$

3. Prove that every sentence which is preserved under submodels is logically equivalent to a universal sentence.

4. Show that the theory of an equivalence relation with infinitely many equivalence classes, each of which is infinite, is model-complete.
B. **Set Theory.** (Do at most two)

**Notation:** $R(\alpha)$ denotes the set of all sets of rank $< \alpha$, that is,

$$R(0) = \emptyset, \quad R(\alpha) = \bigcup_{\beta < \alpha} S(R(\beta)).$$

Let ZFC be Zermelo-Fraenkel set theory with the axiom of choice.

5. What, if anything, is wrong with the following argument?

Let $ZFC_n$ be the first $n$ axioms of ZFC.

a) By the reflection principle, for all $n < \omega$.

$$ZFC \vdash (\exists \alpha) R(\alpha) \text{ is a model of } ZFC_n.$$

b) For all $n < \omega$, $ZFC \vdash (ZFC_n \text{ has a model}).$

c) $ZFC \vdash \text{(if every finite subset of } ZFC \text{ has a model, then } ZFC$ has a model).

d) $ZFC \vdash (ZFC \text{ has a model}).$

e) $ZFC \vdash (ZFC \text{ is consistent}).$

f) By Gödel's incompleteness theorem, ZFC is inconsistent.

6. Prove in ZFC that every well-founded model $\langle A, E \rangle$ of the axiom of extensionality is isomorphic to a model $\langle B, \varepsilon \rangle$ for some transitive $B$.

7. Prove that if $\alpha$ and $\beta$ are limit ordinals, $\alpha < \beta$, and $\langle R(\alpha), \varepsilon \rangle$ is an elementary submodel of $\langle R(\beta), \varepsilon \rangle$, then $\langle R(\alpha), \varepsilon \rangle$ is a model of ZFC.
C. Recursion Theory (Continued).

C2. Show that the set defined in C1 is not recursive.

C3. Let \( p(x, y) \) be an r.e. predicate such that

\[ \forall x \exists y \ p(x, y). \]

Show that there is a total recursive function \( f \) such that

\[ \forall x \ p(x, f(x)). \]

C4. Let \( f \) be a total function on the natural numbers. Show that if \( f \)

is \( \Pi_1^1 \) then \( f \) is hyperarithmetic.

D. Special Topics.

D1. For each formula \( \phi(x) \) of set theory, prove in ZF that

\[ \{x \in \omega : \phi^L(x) \} \in L. \]

D2. Show that \( \mathbb{Z}(p^\omega) \oplus \mathbb{Q} \equiv \mathbb{Z}(p^\omega) \). Here \( \mathbb{Q} \) is the additive group of the rationals and \( \mathbb{Z}(p^\omega) \) is the divisible hull of \( \mathbb{Z}(p) \), the integers mod \( p \).