Qualifying Examination in Logic

January, 1970

Instructions:

Majors and Minors: Do 3 problems.

Third Area: Do 4 problems.

A. Model Theory.

A1. Let D be an ultrafilter over a set I such that $\Pi_D\,\omega$ is countable. Prove that $\Pi_D \langle \omega, < \rangle$ is isomorphic to $\langle \omega, < \rangle$.

A2. Let $T_0 \subset T_1 \subset T_2 \subset \ldots$ be a sequence of sets of sentences such that for each n there exists a model of T_n which is not a model of T_{n+1}. Prove that $\bigcup_{n<\omega} T_n$ is not finitely axiomatizable.

A3. Let $\mathfrak{A} = \langle A, U, \ldots \rangle$ be a model for a countable language such that A has power $\aleph_2$ and U has power $\aleph_0$. Prove that there are two models $\mathfrak{B}, \mathfrak{C}$ of power $\aleph_1$ such that $\mathfrak{B}$ and $\mathfrak{C}$ are elementarily equivalent to $\mathfrak{A}$ but are not isomorphic to each other.

A4. Let $\mathfrak{A}$ be a model for a countable language. Show that $\mathfrak{A}$ can be made into a model $\mathfrak{A}^*$, by adding countably many functions and keeping the same elements, such that every submodel of $\mathfrak{A}^*$ is an elementary submodel of $\mathfrak{A}^*$.

B. Set Theory

B1. Let κ be any infinite cardinal and let $2^{\kappa} = \aleph_{\alpha}$. Show that for all $n < \omega$, $(\aleph_{\alpha+n})^{\kappa} = \aleph_{\alpha+n}$.

B2. Prove that $\prod_{n<\omega} \aleph_n > \aleph_\omega$.

B3. Let M be a transitive model of ZF. Prove that the axiom of constructibility holds in M if and only if there is no transitive submodel $N \subsetneq M$ of ZF such that all the ordinals of M belong to N.

B4. Let M be a transitive model of ZFC + GCH and let $\kappa > \omega$ be a regular cardinal in M. Show that there is a transitive model $N \supseteq M$ of ZFC with the same ordinals and cardinals as M such that $N \models 2^{\omega} = \kappa$.

C. Recursion Theory

C1. Assume that S is a non-empty recursive subset of $\omega \times \omega$. Prove that the set

$$\{m : (\exists n) < m, n > \in S\}$$

is the range of a recursive function.

C2. Prove that the set

$\{n \in \omega :$ n is the Gödel number of a sentence which is true in the standard model of arithmetic$\}$

is not recursively enumerable.

C3. Let T be a set of sentences of predicate logic. Assume that there is no recursive set $S \subset \omega$ such that whenever $T \vdash \varphi$, the Godel number of φ is in S and the Godel number of $\neg\varphi$ is in $\omega - S$. Prove that for every sentence ψ consistent with T, $T \cup \{\psi\}$ is undecidable.

C4. Let P be the theory of Peano arithmetic (as in Kleene's book). Let $pr(x)$ be the usual formula representing "x is the Godel number of a sentence provable from P". Let con be the usual sentence expressing "P is consistent". Suppose φ is a sentence with Godel number e such that $P \vdash \varphi \leftrightarrow \neg pr(e)$. Show that $P \vdash \varphi \leftrightarrow con$.