The combinatorial equivalence of a computability theoretic question

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Oberwolfach 2021 Workshop on Computability Theory

April 28, 2021

Introduction

- We prove that a question of Miller and Solomon——whether every coloring $c : d^{<\omega} \to k$ admits a c-computable variable word infinite solution, is equivalent to a combinatorial question.
- The combinatorial question asked whether there is a sequence of positive integers so that each of its initial segment satisfies a Ramsey type property.
- Moreover, the negation of the combinatorial question is a generalization of Hales-Jewett theorem.

We thank Denis Hirschfeldt, Benoit Monin and Ludovic Patey for helpful discussion.

VWI problem

We adopt the problem-instance-solution framework.

Definition 1 (Variable word)

- An *n-variable word over d* is a sequence v (finite or infinite) of $\{0, \cdots, d-1\} \cup \{x_0, x_1, \cdots\}$ where there are n many variables in v.

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- Given an $\vec{a} \in d^m$, an n-variable word v, suppose $x_{m_0}, x_{m_1}, \cdots, x_{m_{n-1}}$ occur in v with $m_{\hat{n}-1} < m_{\hat{n}}$ for all $\hat{n} < n$. We write $v(\vec{a})$ for the $\{0, \cdots, d-1\}$-string obtained by substitute $x_{m_{\hat{n}}}$ with $\vec{a}(\hat{n})$ in v for all $\hat{n} < m$ and then truncating the result just before the first occurrence of $x_{m_{\hat{n}+1}}$.

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- We write $P_{x_m}(v)$ for the set of positions of x_m in v, namely $\{t : v(t) = x_m\}$; the *first occurrence* of a variable x_m in v refers to the integer $\min P_{x_m}(v)$.

VWI problem

Example 2

Infinite variable word v on $\{0,1\}$:

$$
\begin{aligned}
& 011 \quad x_0x_0\ 011 \quad x_1 \quad x_0x_0 \quad x_1x_1 00 \quad x_2x_2 \cdots \qquad (1.1)\\
\vec{a} = 10, v(\vec{a}) =& 011 \quad 11\ 011 \quad 0 \quad 11 \quad 0000.\\
P_{x_0}(v) =& \{3,4 \quad ,9,10,\cdots\}.
\end{aligned}
$$

VWI problem

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Infinite variable word v on $\{0,1\}$:

$$\begin{aligned} &011 \quad x_0x_0\ 011 \quad x_1 \quad x_0x_0 \quad x_1x_100 \quad x_2x_2\cdots \quad (1.1)\\ \vec{a} = 10, v(\vec{a}) =&011 \quad 11\ 011 \quad 0 \quad 11 \quad 0000.\\ P_{x_0}(v) =&\{3,4\ ,9,10,\cdots\}.\end{aligned}$$

Definition 3

- Problem: $\mathsf{VWI}(d,k)$.
- Instance: $c : d^{<\omega} \to k$.
- Solution: an ω-variable word v such that $\{v(\vec{a}) : \vec{a} \in d^{<\omega}\}$ is monochromatic.

VWI vs RCA

Joe Miller and Solomon proposed the following question in [Miller and Solomon, 2004].

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Or in terms of reverse mathematics:

Question 5

Is $\mathsf{VWI}(d, k)$ provable in RCA?

Other versions of variable word problem

Definition 6 (VW, OVW)

If we require the occurrence of x_i being finite for all i then the problem is called VW.
If we require all the occurrence of x_i comes before any occurrence of x_{i+1} then it is called OVW (ordered variable word).

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The problem is proposed by [Carlson and Simpson, 1984] and studied in [Miller and Solomon, 2004], [Liu et al., 2019]. Clearly,

Theorem 7

$\mathsf{VWI}(d,k) \leq \mathsf{VW}(d,k) \leq \mathsf{OVW}(d,k)$.
$\mathsf{VWI}(d,k) \Leftrightarrow \mathsf{VWI}(d,k+1), \mathsf{VW}(d,k) \Leftrightarrow \mathsf{VW}(d,k+1), \mathsf{OVW}(d,k) \Leftrightarrow \mathsf{OVW}(d,k+1)$.

The complexity of $\mathsf{OVW}, \mathsf{VW}$

Theorem 8 ([Miller and Solomon, 2004])

There exists a computable instance of $\mathsf{OVW}(2,2)$ *that does not admit* Δ^0_2 *solution. Thus* $\mathsf{RCA}_0 + \mathsf{WKL}$ *does not prove* $\mathsf{VW}(2,2)$.

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The following result answers a question of [Miller and Solomon, 2004] and [Montalbán, 2011].

Theorem 9 (Monin, Patey, L)

- *For every computable* $\mathsf{OVW}(2,2)$*-instance* c*, every* $\emptyset'$*-PA degree compute a solution to* c.
- *There exists a computable* $\mathsf{OVW}(2,2)$*-instance such that every solution is* $\emptyset'$*-DNC degree.*

Corollary 10 (Monin, Patey, L)

ACA *proves* $\mathsf{OVW}(2,2)$.

Question 11 ([Miller and Solomon, 2004])

Does $\mathsf{OVW}(d,k)$ or $\mathsf{VW}(d,k)$ implies ACA_0 for some l?

A combinatorial equivalence of "VWI$(2,2)$ vs RCA"

Definition 12 ($ENSH^d_k$)

- Let $n_0, n_1, \cdots, n_{r-1}$ be a sequence of positive integers, let $N_0 = \{0, \cdots, n_0 - 1\}$, $N_1 = \{n_0, \cdots, n_0 + n_1 - 1\}, \cdots$, $N_{r-1} = \{n_0 + \cdots + n_{r-2}, \cdots, n_0 + \cdots + n_{r-1} - 1\}$, and $N = \cup_{s \leq r-1} N_s$; let $f : d^N \to k$. We say $n_0 \cdots n_r$ is *sectionally-homogeneous* for f if there exists an $s \leq r-1$, an n_s-variable word v over d of length N such that the first occurrence of variables in v consist of N_s, i.e.,

$$\{\min P_{x_m}(v) : m \in \omega\} = N_s,$$

  and v is monochromatic for f.
- We write $ENSH^d_k(n_0 \cdots n_{r-1})$ iff there exists a coloring $f : d^N \to k$ such that $n_0 \cdots n_{r-1}$ is *not* sectionally-homogeneous for f. In that case we say f witnesses $ENSH^d_k(n_0 \cdots n_{r-1})$.

A combinatorial equivalence of "VWI(2, 2) vs RCA"

Let $ENSH_k^d$ denote the set of infinite sequence of integers $n_0 n_1 \cdots$ such that $ENSH_k^d(n_0 \cdots n_r)$ holds for all $r \in \omega$.

Theorem 13 ([Liu, 2020])

The following are equivalent:

- *There exists a* $\mathsf{VWI}(d, k)$*-instance* c *that does not admit* c*-computable solution.*
- *There exists an* $X \in ENSH_k^d$.

Intuition on $ENSH_k^d(n_0 \cdots n_{r-1})$

Proposition 14

If $\vec{n}$ is a subsequence of $\vec{\hat{n}}$ or $\vec{n} \geq \vec{\hat{n}}$, then $ENSH_k^d(\vec{\hat{n}})$ implies $ENSH_k^d(\vec{n})$.

Intuition on $ENSH_k^d(n_0 \cdots n_{r-1})$

Proposition 15

$ENSH_2^2(22), ENSH_2^2(222)$ holds. $ENSH_2^2(n)$ holds for all $n > 0$.

Proof.

To see $ENSH_2^2(22)$, consider

$$f(\vec{a}) = \vec{a}(0) + \vec{a}(1) + \vec{a}(2) \ mod \ 2.$$

To see $ENSH_2^2(222)$, consider

$$f(\vec{a}) = I(\vec{a}(0) + \vec{a}(1) > 0) + \vec{a}(2) + \vec{a}(3) + \vec{a}(4) \ mod \ 2.$$

Where $I()$ is the indication function. To see $ENSH_2^2(n)$, simply consider $f(\vec{a}) = \vec{a}(0) \ mod \ 2$. □

Intuition on $ENSH_k^d(n_0 \cdots n_{r-1})$

Proposition 16

$ENSH_2^2(2222)$ *does not hold.*

Proof.

We don't know the proof. Adam P. Goucher at Mathoverflow examined this using SAT solver (https://mathoverflow.net/questions/293112/ramsey-type-theorem). It's easy to check that the following functions don't work:

$$f(\vec{a}) = I(\vec{a}(0) + \vec{a}(1) > 0) + \vec{a}(2) + \vec{a}(3) + \vec{a}(4) + \vec{a}(6) \ mod\ 2; \quad (3.1)$$

$$f(\vec{a}) = I(\vec{a}(0) + \vec{a}(1) > 0) + I(\vec{a}(2) + \vec{a}(3) > 0) + \\ + \vec{a}(4) + \vec{a}(5) + \vec{a}(6) \ mod\ 2;$$

□

Proof of theorem 13

($\Leftarrow$)

- A Turing functional Ψ^X computes a variable word if Ψ^X is an enumerable set (possibly finite) $\{v_0, v_1, \cdots\}$ of finitely long variable words such that $v_0 \preceq v_1 \preceq \cdots$.

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- Putting priority argument aside, assume each Turing functional is total. i.e.,
  for each $r \in \omega$, let $v_r \in \Psi_r^X$ be such that v_r contains $X(r)$ many variables whose first occurrence is after $|v_{r-1}|$.

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  for each $r \in \omega$, let $v_r \in \Psi_r^X$ be such that v_r contains $X(r)$ many variables whose first occurrence is after $|v_{r-1}|$.
- Suppose $(f_r : r \in \omega)$ witnesses $ENSH_k^d(X \upharpoonright r)$. We transform these f_r to a coloring c so that there is no $v \succeq v_r$ monochromatic for c.

Proof of theorem 13

- To define c on d^n, let $r(n)$ be the maximal integer such that $|v_{r(n)}| \leq n$. We ensure that c on d^n "oppress" v_r for all $r \leq r(n)$.

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- Define $c(\vec{a}) = f_{r(n)+1}(\vec{a} \upharpoonright \cup_{r \leq r(n)} P_r)$.

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($\Rightarrow$)

- Take advantage of some particular algorithms $\Phi_0, \Phi_1, \cdots$ and show that their failure (to compute a solution to c) gives rise to a sequence $X \in ENSH_k^d$.

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- $\Phi_0^c, \Phi_1^c, \cdots$ are greedy algorithms in the sense that they extend their current computation (which is a finitely long variable word) whenever possible. More precisely,

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- Φ_{r+1}^c extends its current computation from v_{r+1} to some $\hat{v} \succeq v_{r+1}$ where $\hat{v}$ has more variables than v_{r+1}, whenever it is found that for *some* $\vec{a} \in d^{|v_r|+1}$, $\hat{v}/\vec{a}$ is monochromatic for c.

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- Moreover, Φ_{r+1}^c will build its solution v_{r+1} based on $\Phi_0^c, \cdots, \Phi_r^c$ in the sense that all variables in v_{r+1} occur after $|v_r|$ and if some $\Phi_{\tilde{r}}^c$ extends its current computation, then all Φ_r^c (where $r > \tilde{r}$) will restart all over again.

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- there is no $\hat{v} \succeq \hat{v}_r$ such that for some $\vec{a} \in d^{|\hat{v}_{r-1}|}$, $\hat{v}/\vec{a}$ is monochromatic for c; moreover, all variables in v_r occur after $|v_{r-1}|$ and $|v_r| > |v_{r-1}|$.
- We show that $n_0 n_1 n_2 \cdots \in ENSH_k^d$.

Proof of theorem 13

- Fix an $r \in \omega$, let $N = n_0 + \cdots + n_r$. To define $f : d^N \to k$ witnessing $ENSH_k^d(n_0 \cdots n_r)$, for every $\vec{a} \in d^N$ we map $\vec{a}$ to a word $\vec{\hat{a}} = h(\vec{a}) \in d^{|\hat{v}_r|}$ and let $f(\vec{a}) = c(\vec{\hat{a}})$.

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- Intuitively, h is defined by connecting each element of N, say $n_0 + \cdots + n_{s-1} + m$, to a set $P_{x_m}(\hat{v}_s)$ and copy the value $\vec{a}(n_0 + \cdots + n_{s-1} + m)$ to $\hat{a}(t)$ for all $t \in P_{x_m}(\hat{v}_s)$. More precisely,

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- Suppose $\vec{a} = \vec{a}_0 \cdots \vec{a}_r$ where $|\vec{a}_s| = n_s$ for all $s \le r$. Let

$$\vec{\hat{a}}_s = \hat{v}_s(\vec{a}_s) \upharpoonright_{|\hat{v}_{s-1}|}^{|\hat{v}_s|-1} \text{ and } h(\vec{a}) = \vec{\hat{a}}_0 \cdots \vec{\hat{a}}_r.$$

Theorem 17

The following two classes of oracles are equal:

$$\{D \subseteq \omega : D' \text{ computes a member in } ENSH_k^d.\}$$

$$\{D \subseteq \omega : D \text{ computes a } \mathsf{VWI}(d,k)\text{-instance } c \text{ that does not admit a } c\text{-computable solution.}\}$$

Relation to Hales-Jewett theorem

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Theorem 18 (Hales-Jewett theorem)

For every $d, k, n \in \omega$, $HJ(d, k, n)$ holds.

- HJ theorem is of fundamental importance in combinatorics.
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- The density HJ theorem $\Rightarrow$ the density van der Waerden theorem, namely Szemerédi's theorem, which asserts that for every set A of integers of positive density (meaning $\limsup_{n\to\infty} |A \cap n|/n > 0$), every $r \in \omega$, there exists an arithmetical progression in A of length r (conjectured by Erdős and Turán).

- We show that: $\forall d, k, n[n^\omega \notin ENSH_k^d] \Leftrightarrow$ HJ theorem.
- Actually,

$$n^\omega \notin ENSH_k^d \Rightarrow HJ(d, k, n) \text{ and}$$
$$HJ(d^n, k, 1) \Rightarrow n^\omega \notin ENSH_k^d.$$

Proposition 19

For every $d, k, n \in \omega$, $n^\omega \notin ENSH_k^d$.

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- Code 2^{2r} into 4^r where $\vec{a}(2t)\vec{a}(2t+1)$ $(00, 01, 10, 11$ respectively) is coded into $\vec{\hat{a}}(t)$ $(0, 1, 2, 3$ respectively).

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- Given a coloring $c : 2^{2r} \to k$, consider $\hat{c} : 4^r \ni \vec{\hat{a}} \mapsto c(\vec{a})$.
- Let $\hat{v}$ be a 1-variable word monochromatic for $\hat{c}$ and consider v such that $v(2t)v(2t+1) = 00, 01, 10, 11, x_0x_1$ respectively if $\hat{v}(t) = 0, 1, 2, 3, x_0$ respectively.

□

The following theorem generalizes HJ theorem on $d = 2, k = 2, n = 2$.

Theorem 20 ([Liu, 2020])

For every sequence $n_0 n_1 \cdots$ of positive integers with $n_s = 2$ for some s, $n_0 n_1 \cdots \notin ENSH_2^2$.

Lemma 21

There exists a sequence $n_0 \cdots n_r$ *such that* $ENSH_2^2(n_0 \cdots n_r)$ *holds but* $ENSH_2^2(n_0 \cdots n_r n)$ *does not hold for all* n.

Proof.

For example, $n_0 \cdots n_r = 1$ and note that $ENSH_2^2(1)$ is true but $ENSH_2^2(1n)$ is not true for any n.

□

Some open questions

Question 22

- Does $ENSH_2^2(2223)$ holds? Does $ENSH_2^2(222n)$ holds for sufficiently large n?
- Is it true that for every n,$\hat{n}$ $ENSH_2^2(n\ \underbrace{\hat{n}\cdots\hat{n}}_{n+1 \text{ many}})$ does not hold.

Many thanks!

Carlson, T. J. and Simpson, S. G. (1984).
A dual form of ramsey's theorem.
Advances in Mathematics, 53(3):265–290.

Liu, L. (2020).
The combinatorial equivalence of a computability theoretic question.
arXiv preprint arXiv:2012.13588.

Liu, L., Monin, B., and Patey, L. (2019).
A computable analysis of variable words theorems.
Proceedings of the American Mathematical Society, 147(2):823–834.

Miller, J. S. and Solomon, R. (2004).
Effectiveness for infinite variable words and the dual ramsey theorem.
Archive for Mathematical Logic, 43(4):543–555.

Montalbán, A. (2011).
Open questions in reverse mathematics.

Bulletin of Symbolic Logic, 17(03):431–454.