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On A-computable Numberings

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Basic Definitions

Numbering of a countable set S is a surjective mapping $\nu : \mathbb{N} \longrightarrow S$.

Let $H(S)$ be the set of all numberings of S.

Definition

A numbering ν of a countable family $\mathcal{S} \subseteq 2^{\mathbb{N}}$ is computable (A-computable), if the set $G_\nu = \{\langle x, y\rangle : y \in \nu(x)\}$ is c.e. (A-c.e.). In this case, the family $\mathcal{S}$ is said to be also computable (A-computable).

If $A = \emptyset^{(n)}$, then A-computable numberings are called Σ^0_{n+1}-computable.

Let $\mathrm{Com}^A(\mathcal{S}) = \{\nu \in H(\mathcal{S}) : \nu \text{ is } A\text{-computable}\}$,
$\mathrm{Com}^0_{n+2}(\mathcal{S}) = \mathrm{Com}^{\emptyset^{(n+1)}}(\mathcal{S})$, $\mathrm{Com}(\mathcal{S}) = \mathrm{Com}^{\emptyset}(\mathcal{S})$.

History

Program of A.N. Kolmogorov to study numbered sets as a research subject started at 1954: V.A. Uspensky (1955-1957,1969), H. Rogers (1958), A.I. Maltsev (1961-1965), Yu.L. Ershov (since 1967), S.A. Badaev (since 1974), S.S. Goncharov (since 1980) . . .

Computable numberings in well-known hierarchies (arithmetic and analytical hierarchy, the Ershov hierarchy) began to be studied by S.S. Goncharov, A. Sorbi, S.A. Badaev, S.Yu. Podzorov, S. Lempp and others since 1997.

The first paper completely devoted to the study of A-computable numberings is the paper by S.A. Badaev and S.S. Goncharov: «Generalized computable universal numberings», Algebra and Logic, 53:5 (2014), 355–364.

Part I. Rogers Semilattices of *A*-computable Families

Let $\nu_0, \nu_1 \in H(S)$.

Definition

We say that ν_0 is reducible to ν_1 ($\nu_0 \leqslant \nu_1$) if $\nu_0 = \nu_1 \circ f$ for some computable function f. Numberings ν_0 and ν_1 are called equivalent ($\nu_0 \equiv \nu_1$) if $\nu_0 \leqslant \nu_1$ and $\nu_1 \leqslant \nu_0$.

Let $(\nu_0 \oplus \nu_1)(2x + i) = \nu_i(x)$, $i = 0, 1$.

Let $\mathcal{S}$ is an A-computable family. The quotient structure $\mathscr{L}^A(\mathcal{S}) = \langle \mathrm{Com}^A(\mathcal{S})_{/\equiv}; \leqslant \rangle$ is said to be the Rogers semilattice of the family $\mathcal{S}$.

Let $\mathscr{L}^0_{n+2}(\mathcal{S}) = \mathscr{L}^{\emptyset^{(n+1)}}(\mathcal{S})$, $\mathscr{L}(\mathcal{S}) = \mathscr{L}^{\emptyset}(\mathcal{S})$.

Cardinality and Latticeness of $\mathscr{L}(\mathcal{S})$

Questions (Ershov, 1967)

1. What can we say about the cardinalities of the Rogers semilattices?
2. When are they lattices?

Theorem (Khutoretskii, 1971)

If $|\mathscr{L}(\mathcal{S})| > 1$, then $\mathscr{L}(\mathcal{S})$ is infinite.

Theorem (Selivanov, 1976)

If $|\mathscr{L}(\mathcal{S})| > 1$, then $\mathscr{L}(\mathcal{S})$ is not a lattice.

Remark

There are computable families $\mathcal{S}$ such that $|\mathcal{S}| > 1$ and $|\mathscr{L}(\mathcal{S})| = 1$.

Cardinality and Latticeness of $\mathscr{L}^0_{n+2}(\mathcal{S})$

Theorem (Goncharov, Sorbi, 1997)

Let $\mathcal{S}$ be an infinite Σ^0_{n+2}-computable family. Then $\mathscr{L}^0_{n+2}(\mathcal{S})$ contains an infinite subset such that any two different elements of the subset form a minimal pair.

Theorem (Goncharov, Sorbi, 1997)

Let $\mathcal{S}$ be a finite family of Σ^0_{n+2}-sets such that $|\mathcal{S}| > 1$. Then $\mathscr{L}^0_{n+2}(\mathcal{S})$ contains an ideal that is isomorphic to the upper semilattice of c.e. m-degrees $\mathscr{L}^0$.

Corollary (Goncharov, Sorbi, 1997; Ershov, 1969)

Let $\mathcal{S}$ be a Σ^0_{n+2}-computable family such that $|\mathcal{S}| > 1$. Then $\mathscr{L}^0_{n+2}(\mathcal{S})$ is infinite and is not a lattice.

Cardinality and Latticeness of $\mathcal{L}^A(\mathcal{S})$

Let $\emptyset <_T A$.

Theorem

Let $\mathcal{S}$ be an infinite A-computable family. Then $\mathcal{L}^A(\mathcal{S})$ contains an infinite subset such that any two different elements of the subset form a minimal pair.

Theorem

1. Let $\mathcal{S}$ be a finite family of A-c.e. sets such that $|\mathcal{S}| > 1$. Then $\mathcal{L}^A(\mathcal{S})$ contains an ideal that is isomorphic to the following ideal of the upper semilattice of m-degrees: $I_T^m(A) = \{\deg_m(X) : X \leqslant_T A\}$.
2. The ideal $I_T^m(A)$ is not a lattice.

Cardinality and Latticeness of $\mathscr{L}^A(\mathcal{S})$

Let $\emptyset <_T A$.

Theorem (Jockush, 1969)

The ideal $I_T^m(A) = \{\deg_m(X) : X \leqslant_T A\}$ is infinite.

Corollary

Let $\mathcal{S}$ be an A-computable family such that $|\mathcal{S}| > 1$. Then $\mathscr{L}^A(\mathcal{S})$ is infinite and is not a lattice.

Distinguishing $\mathscr{L}$ from $\mathscr{L}^A$

Let $\mathbf{R}_0 = \{\mathscr{L}(\mathcal{S}) : \mathcal{S} \text{ is computable}\}$, $\mathrm{Th}(\mathbf{R}_0) = \bigcap_{\mathfrak{A} \in \mathbf{R}_0} \mathrm{Th}(\mathfrak{A})$,

$\mathbf{R}_1 = \bigcup_{\emptyset <_T A} \{\mathscr{L}^A(\mathcal{S}) : \mathcal{S} \text{ is } A\text{-computable}\}$, $\mathrm{Th}(\mathbf{R}_1) = \bigcap_{\mathfrak{A} \in \mathbf{R}_1} \mathrm{Th}(\mathfrak{A})$.

Is there a difference between $\mathrm{Th}(\mathbf{R}_0)$ and $\mathrm{Th}(\mathbf{R}_1)$?

Definition

An upper semilattice $\langle L; \vee, \leqslant \rangle$ is (weakly) distributive if for every $a_0, a_1, b \in L$, if $b \leqslant a_0 \vee a_1$ (and $b \nleqslant a_0$, $b \nleqslant a_1$), then there exist $b_0, b_1 \in L$ such that $b_0 \leqslant a_0$, $b_1 \leqslant a_1$ and $b = b_0 \vee b_1$.

Proposition (Folklore)

If $\mathcal{S}$ is a finite family of A-c.e. sets, then $\mathscr{L}^A(\mathcal{S})$ is distributive.

Distinguishing $\mathscr{L}$ from $\mathscr{L}^A$

Theorem (Badaev, Goncharov, Sorbi, 2003)

If $\mathcal{S}$ is an infinite Σ^0_{n+2}-computable family, then $\mathscr{L}^0_{n+2}(\mathcal{S})$ is not weakly distributive.

Theorem

If $\mathcal{S}$ is an infinite A-computable family, where $\emptyset <_T A$, then $\mathscr{L}^A(\mathcal{S})$ is not weakly distributive.

Theorem

There is a computable family $\mathcal{S}$ such that $\mathscr{L}(\mathcal{S})$ is weakly distributive but not distributive.

Corollary

$\mathrm{Th}(\mathbf{R}_0) \neq \mathrm{Th}(\mathbf{R}_1)$.

Universal A-computable Numberings

Definition

A numbering $\nu \in \mathrm{Com}^A(\mathcal{S})$ is universal if $\alpha \leqslant \nu$ for each $\alpha \in \mathrm{Com}^A(\mathcal{S})$.

Theorem (Lachlan, 1964)

Any finite family of c.e. sets has a universal computable numbering.

Theorem (Badaev, Goncharov, 2014)

Let $\emptyset' \leqslant_T A$. Let $\mathcal{S}$ be a finite family of A-c.e. sets. Then $\mathcal{S}$ has a universal A-computable numbering iff $\bigcap \mathcal{S} \in \mathcal{S}$.

Question (Badaev, Goncharov, 2014)

What happens if we replace the set $\emptyset' \leqslant_T A$ by $\emptyset <_T A <_T \emptyset'$ or $A \mid_T \emptyset'$?

Universal A-computable Numberings

If $\emptyset <_T A \leqslant_T \emptyset'$ or $\emptyset' \leqslant_T A$, then $\deg_T(A)$ is hyperimmune.

Theorem

For a set A the following conditions are equivalent.

1. $\deg_T(A)$ is hyperimmune;
2. Let $\mathcal{S}$ be a finite family of A-c.e. sets. Then $\mathcal{S}$ has a universal A-computable numbering iff $\bigcap \mathcal{S} \in \mathcal{S}$.
3. There is a finite family of A-c.e. sets without universal A-computable numberings.

Corollary

Let $\deg_T(A)$ is hyperimmune-free. Then any finite family of A-c.e. sets has a universal A-computable numbering.

Limitness of the Greatest Element of $\mathscr{L}^A(\mathcal{S})$

Theorem (Khutoretskii, 1971)

Let c be the greatest element of $\mathscr{L}(\mathcal{S})$. Then c is limitness, that is $\forall a \in \mathscr{L}(\mathcal{S}) \exists b \in \mathscr{L}(\mathcal{S}) \, [a < c \Rightarrow a < b < c]$.

Theorem (Podzorov, 2004)

Let c be the greatest element of $\mathscr{L}^0_{n+2}(\mathcal{S})$. Then c is limitness if one of the following conditions is met.

1. $\bigcap \mathcal{S} \in \mathcal{S}$;
2. $\mathcal{S}$ is finite;
3. $\mathcal{S}$ is Σ^0_{n+1}-computable.

Limitness of the Greatest Element of $\mathscr{L}^A(\mathcal{S})$

Question (Podzorov, 2004)

Let c be the greatest element of $\mathscr{L}^0_{n+2}(\mathcal{S})$. Is c limitness?

Proposition (Selivanov, 1982; Badaev, Goncharov, 2014)

Let $\emptyset' \leqslant_T A$. Then any universal A-computable numbering is complete and, therefore, non-splittable.

Theorem

Let $\emptyset' \leqslant_T A$. Let $\gamma \in \mathrm{Com}^A(\mathcal{S})$ is non-splittable. Then for any $a \in \mathscr{L}^A(\mathcal{S})$ with $a < c = [\gamma]$ there is a $b \in \mathscr{L}^A(\mathcal{S})$ such that $a < b$ and $c \nleqslant b$.

Corollary

The greatest elements of the semilattices $\mathscr{L}^0_{n+2}(\mathscr{R})$, if they exist, are limitness.

Definition

A numbering $\mu \in H(S)$ is minimal if $\alpha \leqslant \mu \Rightarrow \mu \leqslant \alpha$ for each $\alpha \in H(S)$.

Theorem (Badaev, Goncharov, 2001)

Any infinite Σ^0_{n+2}-computable family has infinitely many Σ^0_{n+2}-computable minimal numberings.

Definition (S. Goncharov, A. Yakhnis, V. Yakhnis, 1993)

A class $\mathscr{C} \subseteq H(\mathcal{S})$ is A-effectively infinite if there is a p.c. function ψ such that $\{\alpha^A_{\varphi_e(x)} : \varphi_e(x)\downarrow\} \subseteq \mathscr{C} \Rightarrow \forall x \in \mathrm{dom}\varphi_e\,[\mathscr{C} \ni \alpha^A_{\psi(e)} \not\equiv \alpha^A_{\varphi_e(x)}]$, for each e, where α^A_n is the A-computable numbering with the Gödel number n.

Distribution of Minimal A-computable Numberings

Questions (Badaev, Goncharov, 2001)

Let $\mathcal{S}$ be an infinite Σ^0_{n+2}-computable family.

1. Is the class $K_{min}(\mathcal{S}) = \{\mu \in H(\mathcal{S}) : \mu \text{ is minimal}\}$ $\emptyset^{(n+1)}$-effectively infinite?
2. Does $\mathcal{L}^0_{n+2}(\mathcal{S})$ contain an ideal without minimal elements?

Theorem

Let A be a high set ($\emptyset'' \leqslant_T A'$). Then for any infinite A-computable family $\mathcal{S}$ we have
$\overline{A'''} \leqslant_1 \mathrm{Min}^A(\mathcal{S}) = \{e : \alpha^A_e \in K_{min}(\mathcal{S})\}$.

Corollary

Let A be a high set (in particular, $A = \emptyset^{(n+1)}$) and $\mathcal{S}$ an infinite A-computable family. Then $K_{min}(\mathcal{S})$ is A-effectively infinite.

Ideals Without Minimal Elements

Theorem (Badaev, Goncharov, 2001)

If a family $\mathcal{S}$ has an single-valued Σ^0_{n+2}-computable numbering, then $\mathscr{L}^0_{n+2}(\mathcal{S})$ contains an ideal without minimal elements.

Theorem (Podzorov, 2003)

If a Σ^0_{n+2}-computable family $\mathcal{S}$ is infinite, then $\mathscr{L}^0_{n+2}(\mathcal{S})$ contains an ideal without minimal elements.

Theorem

Let $\deg_T(A)$ be a hyperimmune degree and $\mathcal{S}$ an infinite A-computable family. Then $\mathscr{L}^A(\mathcal{S})$ contains an ideal without minimal elements.

Ideals Without Minimal Elements

Let $\leqslant^{\mathbf{a}}$ be the reducibility of numberings by **a**-computable functions, and $\equiv^{\mathbf{a}}$ the corresponding equivalence.

Theorem (Podzorov, 2003)

Let $U \leqslant_T \emptyset^{(m)}$ be an immune set, where $1 \leqslant m \leqslant n+1$. Then for every $\alpha \in \mathrm{Com}^0_{n+2}(\mathcal{S})$ there are $\beta_0, \beta_1 \in \mathrm{Com}^0_{n+2}(\mathcal{S})$ such that $\beta_0 \equiv^{\mathbf{0}^{(m+1)}} \alpha$, $\beta_1 \equiv^{\mathbf{0}''} \alpha$ and

1. $\langle \widehat{[\beta_0]}; \leqslant \rangle \cong \langle \widehat{\mathrm{deg}_m(U)}; \leqslant \rangle$ if $\mathcal{S}$ is finite;
2. $\langle \widehat{[\beta_0]}; \leqslant \rangle \cong \langle \widehat{\mathrm{deg}_m(U)} \setminus \{\mathbf{0}\}; \leqslant \rangle$ if $\mathcal{S}$ is infinite;
3. $\langle \widehat{[\beta_1]}; \leqslant \rangle \cong \langle \mathcal{E}^*; \subseteq^* \rangle$ if $\mathcal{S}$ is finite;
4. $\langle \widehat{[\beta_1]}; \leqslant \rangle \cong \langle \mathcal{E}^* \setminus \{\mathbf{0}\}; \subseteq^* \rangle$ if $\mathcal{S}$ is infinite.

Part II. Upper Semilattice of *A*-computable Families

The Upper Semilattice Ω^A

Let $\Omega^A = \{\mathcal{S} \subseteq 2^{\mathbb{N}} : \mathcal{S} \text{ is } A\text{-computable}\}$. Then $\langle \Omega^A; \subseteq \rangle$ is an upper semilattice with the greatest element $\mathcal{E}^A$ and the least element $\emptyset$.

The families $\mathcal{S}_0 = \{\{2x\} : x \notin A'\} \cap \{\mathbb{N}\}$ and $\mathcal{S}_1 = \{\{2x\} : x \notin A'\} \cap \{2\mathbb{N}\}$ have no infimum in Ω^A. Therefore, Ω^A is not a lattice.

Degtev, A.N. The semilattice of computable families of recursively enumerable sets. Mathematical Notes of the Academy of Sciences of the USSR 50, 1027–1030 (1991).

Minuends

Definition

A family $\mathscr{A} \in \Omega^A$ is a minuend if $\mathscr{A} \setminus \mathscr{B} \in \Omega^A$ for any $\mathscr{B} \in \Omega^A$. Let Ω^A_M be the class of all minuends.

Remark (definability of Ω^A_M)

$\mathscr{A} \in \Omega^A_M$ iff $\forall \mathscr{B} \in \Omega^A \exists \mathscr{C} \in \Omega^A\, [\mathscr{C} \subseteq \mathscr{A} \,\&\, \mathscr{A} \cup \mathscr{B} = \mathscr{C} \cup \mathscr{B} \,\&$
$\forall \mathscr{D} \in \Omega^A\, [\mathscr{D} \subseteq \mathscr{C} \,\&\, \mathscr{D} \subseteq \mathscr{B} \Rightarrow \mathscr{D} = \emptyset]]$.

Theorem (Degtev, 1991)

Ω^A_M is an ideal of Ω^A that forms a lattice.

Theorem

$\Omega^A_M = \mathrm{Fin}^A = \{\mathscr{F} \in \Omega^A : \mathscr{F} \text{ is finite}\}$. Therefore, Fin^A is definable in Ω^A.

Minuends

Definition

A numbering ν is precomplete if for every p.c. function ψ there exists a computable function f such that for every n $\psi(n)\downarrow \Rightarrow \nu\psi(n) = \nu f(n)$.

Theorem (Ershov, 1977)

A numbering ν is precomplete iff there is a computable function fix such that for every n $\varphi_n(\mathrm{fix}(n))\downarrow \Rightarrow \nu\varphi_n(\mathrm{fix}(n)) = \nu\mathrm{fix}(n)$.

Definition

A numbering ν is positive if the set $\{\langle x, y\rangle : \nu(x) = \nu(y)\}$ is c.e.

Minuends

Theorem

Let $\mathcal{S}$ be a computable family such that $|\mathcal{S}| > 1$. If $\mathcal{S}$ has a precomplete, positive, universal computable numbering, then there is an infinite computable family $\mathcal{A} \subseteq \mathcal{S}$ such that

1. $\mathcal{A} \setminus \mathcal{B}$ is finite for each infinite computable family $\mathcal{B} \subseteq \mathcal{A}$;
2. $\mathcal{A} \setminus \mathcal{B} \in \Omega$ for each finite family of c.e. sets $\mathcal{B}$.

Subtrahends

Definition

A family $\mathscr{A} \in \Omega^A$ is a subtrahend if $\mathscr{B} \setminus \mathscr{A} \in \Omega^A$ for any $\mathscr{B} \in \Omega^A$. Let Ω^A_S be the class of all subtrahends.

Remark

Ω^A_S is also definable in Ω^A.

Theorem (Degtev, 1991)

Let $\mathscr{A} \in \mathrm{Fin}^A$. Then $\mathscr{A} \in \Omega^A_S$ iff any set $F \in \mathscr{A}$ is finite.

Subtrahends

Corollary

$D^A = \{\mathscr{A} \in \Omega^A : \forall F \in \mathscr{A}\, [F \text{ is finite}]\}$ is definable in Ω^A.

Indeed, $\mathscr{A} \in D^A$ iff $\forall \mathscr{C} \in \mathrm{Fin}^A\, [\mathscr{C} \subseteq \mathscr{A} \Rightarrow \mathscr{C} \in \Omega^A_S]$.

Corollary

The singleton $\mathscr{F} = \{F \subseteq \mathbb{N} : F \text{ is finite}\}$ is definable in Ω^A.

Indeed, $\mathscr{A} = \mathscr{F}$ iff $\mathscr{A} \in D^A \,\&\, \forall \mathscr{C} \in D^A\, [\mathscr{C} \subseteq \mathscr{A}]$.

Weak Minuends

Definition

A family $\mathscr{A} \in \Omega$ is a weak minuend if $\mathscr{A} \cap \mathscr{B} \in \Omega$ for any $\mathscr{B} \in \Omega$. Let Ω_{WM} be the class of all weak minuends.

Remark

$\Omega_M = \mathrm{Fin} \subseteq \Omega_{WM}$.

Definition

A family $\mathscr{A} \in \Omega$ is called a completely c.e. if the index set of $\mathscr{A}$ is c.e. By Rice-Shapiro Theorem, $\mathscr{A} \neq \emptyset$ is completely c.e. iff $\mathscr{A} = \{X : D_{f(x)} \subseteq X,\ x \in \mathbb{N}\}$ for some computable function f.

Proposition (Degtev, 1991)

If $\mathscr{A}$ is completely c.e., then $\mathscr{A} \in \Omega_{WM}$.

Weak Minuends

Question

How can we describe the weak minuends?

Question

Is there a family $\mathscr{A} \in \Omega_{WM}$ such that $\mathscr{A} \neq^* \mathscr{B}$ for any completely c.e. family $\mathscr{B}$?

Thank you for attention!