Relativized Depth

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Depth

In many cases, the same information may be organized in different ways, making it more or less useful for various computational purposes. The notion of *depth* was introduced by Bennett as an attempt to separate useful and organized information from random noise and trivial information.

Definition 1.

A set X is *deep* if, for every computable time-bound t and $c \in \mathbb{N}$,

$$\left(\overset{\infty}{\forall} n\right) \left[K^t(X \upharpoonright n) - K(X \upharpoonright n) \geq c\right].$$

Otherwise, we say that X is *shallow*.

Fact.

- *The halting problem $\emptyset'$ is deep.*
- *If a set is ML-random or computable, then it is shallow.*
- *(Slow Growth Law) If X is deep and $X \leq_{tt} Y$, then Y is deep.*

Lower-semicomputable discrete semimeasures

Definition 2.

- A *discrete semimeasure* is a function $m : \mathbf{2}^{<\mathbb{N}} \to [0, \infty)$ such that $\sum_\sigma m(\sigma) \leq 1$. It is *lower-semicomputable* if it is approximable from below. We will write *lss* for lower-semicomputable discrete semimeasure.
- A lss $\mathbf{m}$ is *universal* if, for each lss m,

$$(\forall \sigma)\left[m(\sigma) \leq^{\times} \mathbf{m}(\sigma)\right].$$

Fact (Coding Theorem).

There exists a universal lss $\mathbf{m}$. *In particular,* $\sigma \mapsto 2^{-K(\sigma)}$ *is a universal lss. Hence,*

$$K(\sigma) =^{+} -\log \mathbf{m}(\sigma).$$

Depth in terms of discrete semimeasures

It is often convenient to use the following equivalent characterization of depth in terms of discrete semimeasures.

A set X is deep if and only if, for every computable time-bound t,

$$\lim_{n\to\infty} \frac{\mathbf{m}(X \upharpoonright n)}{\mathbf{m}^t(X \upharpoonright n)} = \infty.$$

or equivalently, iff for every computable semimeasure m ,

$$\lim_{n\to\infty} \frac{\mathbf{m}(X \upharpoonright n)}{m(X \upharpoonright n)} = \infty.$$

Relativized depth

Both the unbounded and the t-time-bounded prefix-free complexities of a string σ relative to an oracle A, denoted, respectively, by $K^A(\sigma)$ and $K^{A,t}(\sigma)$ are defined analogously to the unrelativized case.

The relativized notion of depth is meant to better understand the power of oracles in organizing information.

Definition 3.

Given an oracle A, we say that a set X is *A-deep* if, for every computable time-bound t and $c \in \mathbb{N}$,

$$\left(\overset{\infty}{\forall} n\right) \left[K^{A,t}(X \upharpoonright n) - K^A(X \upharpoonright n) \geq c\right].$$

Otherwise, we say that X is *A-shallow*.

The main properties of depth mentioned above relativize.

When depth and relativized depth are incomparable: the case of $\emptyset'$

For some oracles, depth and relativized depth are incomparable. An example is given by the halting problem $\emptyset'$. Clearly, $\emptyset'$ is $\emptyset'$-shallow.

Moreover, we can build a ML-random (hence shallow) but $\emptyset'$-deep set. In order to do so, we need the following technical lemma (basically, a rephrasing of the Space Lemma (Gács, 1986 and Merkle and Mihailović, 2004)).

Lemma 4.

Let $l(n) \geq^+ \log n$ be a computable function. There exists a Δ^0_2 perfect tree T such that:

- *every string at level n of T has length $l(n)$;*
- *every infinite path of T is ML-random.*

Moreover, if $l(n) \geq^+ n^2$, then every string at level n of T has at least 2^{n+1} children.

When depth and relativized depth are incomparable: the case of $\emptyset'$ (continue)

Theorem 5.

There exists a Δ^0_2 set X which is ML-random and $\emptyset'$-deep.

Proof (Sketch). Let T be a tree as in Lemma 4 and $F \leq_T \emptyset'$ dominate every computable time-bound.
There exists a path $(\tau_n)_n \subset T$ such that, for almost all n, $K^{\emptyset',F}(\tau_n) \geq n$, as every string σ at level $n-1$ has at least 2^n children. Let $X = \bigcup_n \tau_n$.
Being a path of T, X is ML-random, hence shallow. Moreover, X is Δ^0_2, hence, for almost all n, $K^{\emptyset'}(X \upharpoonright n) \leq 2 \log n$.
Then, for $n^2 < m \leq (n+1)^2$,

$$K^{\emptyset',F}(X \upharpoonright m) - K^{\emptyset'}(X \upharpoonright m) \geq n - 8 \log n,$$

which is eventually larger than any constant. This shows that X is $\emptyset'$-deep, as F is dominating. □

Deep sets remain deep relative to ML-random oracles

There are also oracles which do not make any deep set shallow relatively to them. We show that this is the case for ML-random oracles.
We will make use of the following characterization of ML-randomness.

Definition 6.

$\Psi : \mathbf{2}^{\mathbb{N}} \to [0, \infty]$ is an integral test if

- Ψ is *lower-semicomputable*, i.e. the supremum of a computable sequence of computable functions $\Psi_n : \mathbf{2}^{\mathbb{N}} \to [0, \infty)$, and
- $\int_{\mathbf{2}^{\mathbb{N}}} \Psi d\mu \leq 1$.

Fact.

X is not ML-random if and only if there is an integral test Ψ such that $\Psi(X) = \infty$.

Deep sets remain deep relative to ML-random oracles (continue)

Theorem 7.

Let A be ML-random. If a set X is deep, then X is also A-deep.

Proof's sketch. We prove more: if A is ML-random, then, for every computable time-bound t, there is a computable time-bound t' with

$$(\forall\sigma)\left[K^{t'}(\sigma) - K(\sigma) \leq^{+} K^{A,t}(\sigma) - K^{A}(\sigma)\right]. \quad (\dagger)$$

The map $\sigma \mapsto \int_{\mathbf{2}^{\mathbb{N}}} \mathbf{m}^{A,t}(\sigma)d\mu$ is a computable discrete semimeasure. Hence,

$$\int_{\mathbf{2}^{\mathbb{N}}} \mathbf{m}^{A,t}(\sigma)d\mu \leq^{\times} \mathbf{m}^{t'}(\sigma),$$

for some computable time-bound t'.

Deep sets remain deep relative to ML-random oracles (continue)

Proof's sketch (continue). Consider the map $\Psi : \mathbf{2}^{\mathbb{N}} \to [0, \infty]$ given by

$$\Psi(A) = \sum_{\sigma} \frac{\mathbf{m}^{A,t}(\sigma)\mathbf{m}(\sigma)}{\mathbf{m}^{t'}(\sigma)}.$$

Ψ is lower-semicomputable and, being only lss involved, $\int_{\mathbf{2}^{\mathbb{N}}} \Psi d\mu \leq 1$.
Then Ψ is an integral test.
So, if A is ML-random, $\Psi(A) < c$, for some c. But then the map

$$\sigma \mapsto \frac{\mathbf{m}^{A,t}(\sigma)\mathbf{m}(\sigma)}{\mathbf{m}^{t'}(\sigma)}$$

is an A-lss. Then, for any string σ,

$$\frac{\mathbf{m}^{A,t}(\sigma)\mathbf{m}(\sigma)}{\mathbf{m}^{t'}(\sigma)} \leq^{\times} \mathbf{m}^{A}(\sigma),$$

which implies (†) by the Coding Theorem.

Shallow sets remain shallow relatively to almost every oracle

Theorem 8.

If X is shallow, then $\mu(\{A : X \text{ is } A\text{-deep}\}) = 0$.

The idea to prove this theorem is that, if t is a computable time-bound such that $K^t(X \upharpoonright n) =^+ K(X \upharpoonright n)$ i.o., then

$$\lim_{d\to\infty} \mu\left(\left\{Y : \left(\overset{\infty}{\forall} n\right)\left[\frac{\mathbf{m}^Y(X \upharpoonright n)}{\mathbf{m}^t(X \upharpoonright n)} \geq d\right]\right\}\right) = \lim_{d\to\infty} \mu(\mathcal{L}_d) = 0.$$

$\mathcal{L}_d$ is, in fact, a test for 2-randomness relative to X. Hence, in particular, X remains shallow relatively to every X-2-random oracle.

Question. Does every shallow set remain shallow relatively to any n-random oracle, for some n?

Depth relative to ML-random oracles is strictly weaker than depth

We answer the previous question in the negative.

Theorem 9.

For every ML-random set A, there exist a shallow set X which is A-deep.

Intuitively, the proof of this fact is similar to the one-time pad protocol in cryptography: we can "mix" together some important piece of information x with some random string a we know, so that the output $x \boxplus a$ still looks important for us (as we can distinguish the added random noise a), while looking random to the others.

Depth relative to ML-random oracles is strictly weaker than depth (continue)

Fact (Moser and Stephan, 2017).

There exists a non-empty Π^0_1 class consisting of deep sets.

Hence, we can use well-known basis theorems to obtain deep sets with some desired properties.

Fact (Randomness Basis Theorem).

Let A be ML-random. Every non-empty Π^0_1 class contains a set X such that A is X-ML-random.

So, if A is ML-random, there is a deep set X such that A is X-ML-random. Consider the set $Y = A \boxplus X$. Y is X-ML-random, as A is, and hence shallow. Moreover, X is deep, hence A-deep. Then, by the relativized version of the Slow Growth Law, Y is A-deep, as $X \leq_{tt} Y \oplus A$. □

Digression: PA-complete degrees are the join of two ML-random degrees

As a consequence of Theorem 9, it is possible to give a short proof of the following result.

Fact (Barmpalias, Lewis and Ng, 2010).

Every PA-complete degree is the join of two ML-random degrees.

The key point of the proof is the following lemma, whose proof uses techniques due to Kučera and Slaman (2006).

Lemma 10.

Let $\mathcal{C}$ be a non-empty Medvedev-complete Π^0_1 class (i.e., there is a tt-reduction Φ such that $\Phi(X)$ is DNC_2 for every $X \in \mathcal{C}$). For every A of PA-complete Turing degree, there exists $B \in \mathcal{C}$ such that $B \equiv_T A$.

Digression: PA-complete degrees are the join of two ML-random degrees (continue)

Since any DNC_2 function is deep, by Theorem 9 we get that the following Π^0_1 class is non-empty (for large enough d)

$$\mathcal{C} = \{\langle A, X, Y\rangle : \; A \in DNC_2, X \in MLR_d, Y \in MLR_d, X \boxplus Y = A\},$$

where $MLR_d = \{X : (\forall n)[K(X \upharpoonright n) \geq n - d]\}$. Moreover, the first projection witnesses that Lemma 10 applies to our class. Hence, for every B with PA-complete degree there is a triple $\langle A, X, Y\rangle \in \mathcal{C}$ such that $B \equiv_T \langle A, X, Y\rangle$. Moreover, since $A = X \boxplus Y$, clearly $B \equiv_T \langle A, X, Y\rangle \equiv_T \langle X, Y\rangle$. □

Shallowness is preserved by K-trivial oracles

Recall that a set A is *K-trivial* if $K(A \upharpoonright n) \leq^+ K(n)$ for all n. Nies (2005) proved that a set A is K-trivial if and only if it is *low for K*, namely if $K(\sigma) \leq^+ K^A(\sigma)$ for every string σ.

Theorem 11.

Let A be K-trivial. Then every shallow set is A-shallow.

Proof. Let t be a computable time-bound such that $K^t(X \upharpoonright n) =^+ K(X \upharpoonright n)$ i.o. Then, for any such n,

$$K^{A,t}(X \upharpoonright n) \leq^+ K^t(X \upharpoonright n) =^+ K(X \upharpoonright n) \leq^+ K^A(X \upharpoonright n),$$

so that X is A-shallow. □

Then depth relative to K-trivial oracles is either strictly stronger than or equal to depth.

Open question. Which of the above possibilities do actually happen? Do all K-trivial oracles yield the same answer?