Open questions on effective-dimension proofs of classical theorems: packing dimension and regularity

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Seminar on Computability Theory, Oberwolfach April 30, 2021
Warning

These questions were suggested by Ted Slaman’s talk this Monday, so his talk may give a better motivation and intuition
A set is regular if its Hausdorff and packing dimension coincide.

Can we prove Besicovitch-Davies results for analytic regular sets using computability?
Hausdorff dimension

Definition

Let $A \subseteq 2^\omega$, $d \in [0, 1],

$$H^d(A) = \lim_{\delta \to 0} \inf \left\{ \sum_i 2^{-|\sigma_i|d} \mid \text{there is a cover of } A \right\}$$

by balls $B(\sigma_i)$ with $2^{-|\sigma_i|} < \delta$}

$$\dim_H(A) = \inf \left\{ d \geq 0 \mid H^d(A) = 0 \right\}$$
Packing dimension

Definition
Let $A \subseteq 2^\omega$, $d \in [0, 1]$,

$$\mathcal{P}_0^d(A) = \limsup_{\delta \to 0} \sup \left\{ \sum_i 2^{-|\sigma_i|d} \mid \text{there is a packing of } A \right.$$ 

by disjoint $B(\sigma_i)$ with $2^{-|\sigma_i|} < \delta$ \}

$$\mathcal{P}^d(A) = \inf \left\{ \sum_{U \in \mathcal{U}} \mathcal{P}_0^d(U) \mid \mathcal{U} \text{ is a countable cover of } A \right\}$$

$$\dim_p(A) = \inf \{ d \geq 0 \mid \mathcal{P}^d(A) = 0 \}$$
Let $x \in 2^\omega$.

**Definition**

\[
\dim(x) = \liminf_n \frac{K(x \upharpoonright n)}{n},
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Theorem (Lutz Lutz 2018)

Let $A \subseteq 2^\omega$. Then

$$\dim_H(A) = \min_{B \subseteq \mathbb{N}} \dim^B(A).$$

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Let $A \subseteq 2^\omega$. Then

$$\dim_P(A) = \min_{B \subseteq \mathbb{N}} \operatorname{Dim}^B(A).$$
Question 1

Definition
A is regular if $\dim_H(A) = \dim_P(A)$.

Can computability (partially) characterize regularity for $A \subseteq 2^\omega$?
Capacitability

Theorem (Besicovitch-Davies / Frostman)
Let $A$ be analytic, $\dim_H(A) = d$. For every $s < d$ there is a closed $C \subseteq A$ with $s \leq \dim_H(C)$.

Theorem (Joyce Preiss)
Let $A$ be analytic, $\dim_P(A) = d$. For every $s < d$ there is a compact $K \subseteq A$ with $s \leq \dim_P(K)$. 
Question 2

Theorem

Let $A \subseteq 2^\omega$ be analytic with $\dim_P(A) = \dim_H(A)$. For every $s < d$ there is a closed $C \subseteq A$ with $s \leq \dim_H(C)$.

Can we prove this result using the point-to-set principles?
Extensions

- **Question 3** The same question for other separable spaces
- **Question 4** The same question for other gauge functions/families