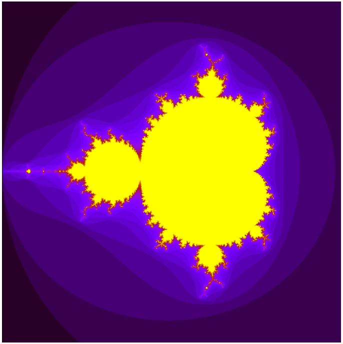
Is the Mandelbrot set computable?



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Oberwolfach Workshop “Computability Theory”
April 27, 2021, Problem Session

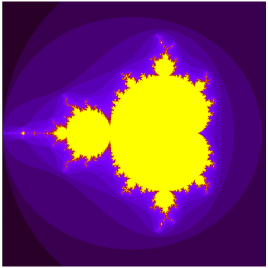
The Mandelbrot set M

For $c \in \mathbb{C}$: $p_c(z) := z^2 + c$

$$\begin{aligned} p_c^{\circ 0}(z) &:= z, \\ p_c^{\circ k+1}(z) &:= p_c(p_c^{\circ k}(z)) \end{aligned}$$

$$M := \{c \in \mathbb{C} \mid p_c^{\circ k}(0) \not\mapsto \infty \text{ for } k \mapsto \infty\}$$

- M is the closure of its interior.
- $M \subseteq \{c \in \mathbb{C} \mid |c| \leq 2\}$.
- $M = \{c \in \mathbb{C} \mid |p_c^{\circ k}(0)| \leq 2 \text{ for all } k\}$



Mandelbrot set and Julia sets

The Mandelbrot set "describes" the behaviour of all quadratic polynomials with complex coefficients under iteration:

- Every quadratic polynomial with complex coefficients is affinely conjugated to a uniquely determined polynomial of the form p_c.
- The Julia set of p_c:

$$J_c := \text{Boundary}(\{z \in \mathbb{C} \mid p_c^{\circ k}(z) \mapsto \infty\})$$

Known:

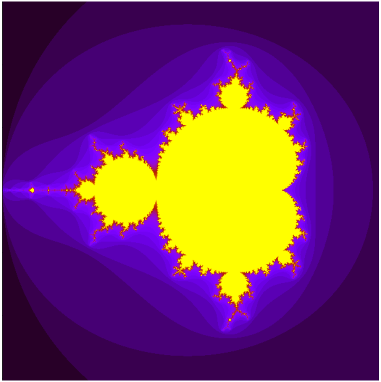
$$c \in M \quad \Rightarrow \quad J_c \text{ is connected}$$
$$c \notin M \quad \Rightarrow \quad J_c \text{ is a Cantor set (hence, totally disconnected)}$$

History of the Mandelbrot set

- 1978/1980: Brooks and Matelski: first pictures
- 1980: Mandelbrot: further pictures
- 1982: Douady and Hubbard: call the set M Mandelbrot set,
  show that M is simply connected,
  give a parametrization of the complement of M,
  give parametrizations of the hyperbolic components of M,
  $\ldots$,
- Since then: many results by Branner, Douady, Hubbard, Lavaurs, Lyubich, Mc Mullen, Milnor, Shishikura, Sullivan, Tan Lei, Yoccoz, and many more.
- 1991/1998: Shishikura: shows that the Hausdorff dimension of the boundary of M is equal to 2.

Is the Mandelbrot set computable?

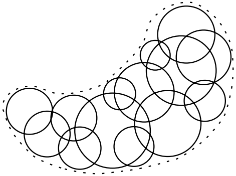
Question posed by Penrose (1989, *The Emperor's New Mind*)



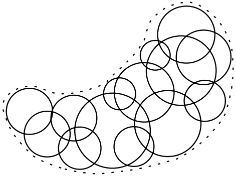
In what sense "computable"?

Computably enumerable open subsets of $\mathbb{R}^n$

- An open set $U \subseteq \mathbb{R}^n$ is called c.e. open, if one can effectively produce a list of open rational (i.e., with rational midpoint and rational radius) balls that cover exactly U.



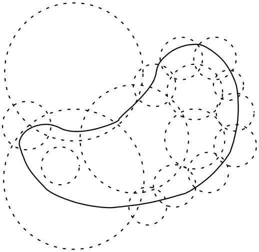
- An open set $U \subseteq \mathbb{R}^n$ is c.e. open iff one can enumerate all closed rational balls that are contained in U.



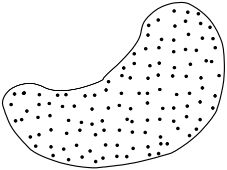
- An open set $U \subseteq \mathbb{R}^n$ is c.e. open iff there is a Turing machine T such that, for any $x \in \mathbb{R}^n$ and any sequence $(q_0, q_1, q_2 \ldots)$ of rational numbers q_i with $|q_i - x| \leq 2^{-i}$

$$x \in U \iff \text{On input } (q_0, q_1, q_2, \ldots),\ T \text{ stops after finitely many steps.}$$

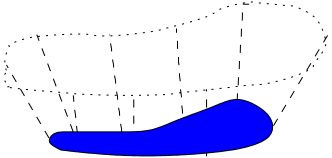
Computably enumerable closed subsets of $\mathbb{R}^n$



- A closed set $A \subseteq \mathbb{R}^n$ is called c.e. closed if one can effectively enumerate all open rational balls B with $B \cap A \neq \emptyset$.
- A closed set $A \subseteq \mathbb{R}^n$ is c.e. closed iff one can compute a list of points dense in A.



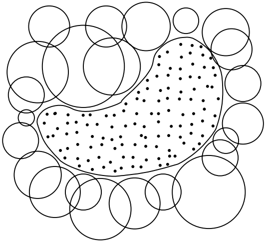
Computable subsets of $\mathbb{R}^n$



Lemma:

For a closed subset $A \subseteq \mathbb{R}^n$ the following are equivalent:

- The distance function d_A of A is computable.
- 1. A is c.e. closed and
  2. $\mathbb{R}^n \setminus A$ ist c.e. open.
- One can draw a pixel image of A, with any precision.

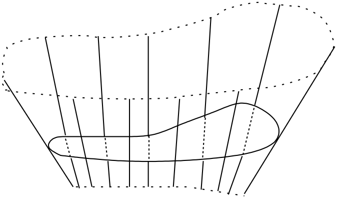


Pixel image of A with precision 2^{-n}:

Every pixel p of side length 2^{-n} in a sufficiently large rectangle has a correct color, where correct color means:

- black if $d(\text{midpoint}(p), A) < 2^{-n}$,
- white if $d(\text{midpoint}(p), A) > 2 \cdot 2^{-n}$,
- black or white, otherwise.

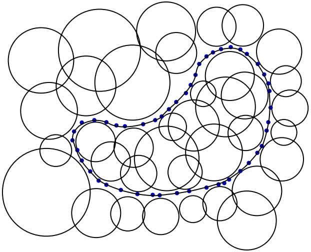
Strongly computable subsets of $\mathbb{R}^n$



Lemma:

For a closed subset $A \subseteq \mathbb{R}^n$ the following are equivalent:

- The two-sided distance function $d_{\text{two-sided},A}$ of A is computable.
- 1. the interior of A is c.e. open,
  2. the boundary of A is c.e. closed, and
  3. $\mathbb{R}^n \setminus A$ is c.e. open.



Computable enumerability of various sets

Proposition (Weihrauch: "Computable Analysis", 2000)

The complement $\mathbb{C} \setminus M$ of the Mandelbrot set is c.e. open.

Proposition (H. 2005)

The boundary ∂M of the Mandelbrot set is c.e. closed.

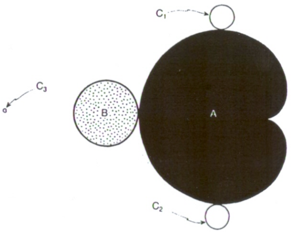
Proposition (H. 2005)

The union $H(M)$ of the hyperbolic components of the interior of the Mandelbrot set is c.e. open.

The hyperbolic components of M

A point $z_0 \in \mathbb{C}$ is a periodic point of a polynomial p if there is a $k > 0$ such that $p^{\circ k}(z_0) = z_0$. Then, k is its period, and the k points z_0 $z_1 = p(z_0)$, $\ldots$, $z_{k-1} = p^{\circ k-1}(z_0)$ form a cycle. Then, $(p^{\circ k})'(z_0)$ is the multiplier of the cycle. A cycle is called attracting if $|\text{multiplier}| < 1$.

The set $H(M) := \{c \in \mathbb{C} \mid p_c \text{ has an attracting cycle}\}$ is open and a subset of the interior of M.
The connected components of $H(M)$ are called hyperbolic components.



Four conjectures about the Mandelbrot set

Conjecture 1: The Mandelbrot set M is locally connected.

$\Downarrow$ [Douady, Hubbard 1982]

Conjecture 2: (Hyperbolicity conjecture) The union $H(M)$ of the hyperbolic components of the interior of the Mandelbrot set M is equal to the interior of M.

$\Downarrow$ (H. 2005)

Conjecture 3: The two-sided distance function $d_{\mathrm{two-sided},M}$ of M is computable.

$\iff$ Conjecture 3': The interior of the Mandelbrot set is c.e. open.

$\Downarrow$ (Clear)

Conjecture 4: The distance function d_M of M is computable.

$\iff$ Conjecture 4': The Mandelbrot set is c.e. closed

Escape time

Let the *escape time function* $e : \mathbb{N} \to \mathbb{N}$ be defined by

$$e(k) := \min \left\{ m \in \mathbb{N} \;:\; (\forall c \in \mathbb{C}) \; \left(d_M(c) \geq 2^{-k} \Rightarrow |p_c^{\circ m}(0)| \geq 3 \right) \right\}.$$

Theorem (Carleson, Gamelin, 1993)

If e satisfies $\sum_{k=0}^{\infty} e(k) \cdot 2^{-k} < \infty$ then M is even locally connected.

Lemma (H. 2005)

The following are equivalent:

- *The distance function d_M of M is computable (*Conjecture 4*).*
- *There exists a computable function $b : \mathbb{N} \to \mathbb{N}$ with $(\forall k \in \mathbb{N}) \; e(k) \leq b(k)$.*