NOTES ON VECTOR BUNDLES WITH INTEGRABLE CONNECTION AND CLASSICAL RIEMANN-HILBERT CORRESPONDENCE

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Abstract. These are the notes for the second talk in Brian Lawrence's reading group $\mathbb{P}^1$ *minus three points* which happened in fall 2022. We discuss the basic notion of vector bundles with connection on complex varieties, state and prove Riemann-Hilbert correspondence in the cases of projective variety and (affine) algebraic curve. A good reference is [Con]. We also provide a concise review of relative de Rham cohomology and Gauss-Manin connections, with emphasise on examples.

Contents

1. Vector bundles with integrable connection

We discuss the basic notion of (algebraic and analytic) vector bundles with connection on complex varieties. Tons of examples will be given.

1.1. **Definitions and examples.** Let $X/\mathbb{C}$ be a smooth variety and $\mathcal{E}$ over X be an algebraic vector bundle. A *connection* on $\mathcal{E}$ is an $\mathcal{O}_X$-linear sheaf morphism

$$\nabla : \mathcal{E} \to \Omega_X \otimes_{\mathcal{O}_X} \mathcal{E} \tag{1.1.1}$$

satisfying the Leibniz rule $\nabla(fs) = df \otimes s + f \cdot \nabla(s)$ for local sections f of $\mathcal{O}_X$ and s of $\mathcal{E}$. An equivalent way of rewriting (1.1.1) is

$$\nabla : \mathcal{T}_X \to \mathcal{E}nd_{\mathcal{O}_X}(\mathcal{E}). \tag{1.1.2}$$

Writing $\nabla(\frac{\partial}{\partial f})s = \frac{\partial s}{\partial f}$, we can then express the Leibniz rule as $\frac{\partial gs}{\partial f} = \frac{\partial g}{\partial f}s + g\frac{\partial s}{\partial f}$. Therefore one can think of the connection as a mechanism of differentiating a vector bundle along vector fields. The category of algebraic vector bundles over X with connection is denoted $\mathrm{VC}(X)$. For each $i \geq 0$, ∇ induces a unique $\mathcal{O}_X$-linear morphism

$$\nabla^i : \Omega^i_X \otimes_{\mathcal{O}_X} \mathcal{E} \to \Omega^{i+1}_X \otimes_{\mathcal{O}_X} \mathcal{E}$$

by the rule $\nabla^i(\omega \otimes s) = d\omega \otimes s + (-1)^i \omega \wedge \nabla(s)$. The composition

$$\mathcal{R} = \nabla^1 \circ \nabla : \mathcal{E} \to \Omega^2_X \otimes_{\mathcal{O}_X} \mathcal{E} \tag{1.1.3}$$

is called the *curvature form* of $\mathcal{E}$. $\mathcal{E}$ is called *integrable* (or *flat*) if its curvature is 0. The full subcategory of $\mathrm{VC}(X)$ consisting of vector bundles with integrable connection is denoted $\mathrm{VIC}(X)$. A *flat section* (or *horizontal section*) of $(\mathcal{E}, \nabla)$ is an analytical local section s such that $\nabla(s) = 0$. When restricting the connection to a curve $\gamma : \mathbf{I} \to X$, we recover the notion of *parallel transport*, i.e. moving local geometric data along curves. We will see in Lemma 2.4 that the flat sections of $(\mathcal{E}, \nabla)$ form a constructible sheaf (in analytic topology) of $\mathbb{C}$-vector spaces, denoted $\ker \nabla$. If ∇ is furthermore integrable, then the holonomies are trivial, i.e. if two paths connecting the same starting and ending points are homotopic, then the parallel transportation of a section of $\mathcal{E}$ from the starting point to the end point along both curves are equal. In this case, $\ker \nabla$ is a locally constant sheaf of rank $\operatorname{rk} \mathcal{E}$. If X is connected and $x \in X$ is a point, $\ker \nabla$ gives rise to the *monodromy representation*

$$\pi_1(X, x) \to \mathrm{GL}(\mathcal{E}(x)), \gamma \to \{e \to \text{parallel transport of } e \text{ along } \gamma\}.$$

The integrability condition guarantees that the map only depends on the homotopy class of the loop.

Remark 1.1. The above constructions have their analytic counterparts. For example, a holomorphic vector bundle with connection is a holomorphic vector bundle with an $\mathcal{O}_{X^{\mathrm{an}}}$-linear sheave morphism ∇. The categories of holomorphic vector bundle with connection and integrable connection are denoted $\mathrm{VC}(X^{\mathrm{an}})$ and $\mathrm{VIC}(X^{\mathrm{an}})$. There are analytification functors

$$F^{\mathrm{an}} : \mathrm{VC}(X) \to \mathrm{VC}(X^{\mathrm{an}}),$$
$$F^{\mathrm{an}} : \mathrm{VIC}(X) \to \mathrm{VIC}(X^{\mathrm{an}}).$$

When X is projective, F^{an} is an equivalence by GAGA (Géométrie algébrique et géométrie analytique, see [Wika]). However when X is not projective, the algebraic categories are usually "larger" than their analytic counterparts, in the sense that two non-isomorphic algebraic vector bundles with connection may be holomorphically isomorphic, see Example 1.5.

Remark 1.2. Confusion may arise if the readers are more familiar with the notion of connections in differential geometry. In differential geometry, a vector bundle with connection and its curvature are still defined as (1.1.1) and (1.1.3). We shall make a clarification here.

Let X be a complex variety and $(\mathcal{E}, \nabla) \in \mathrm{VC}(X^{\mathrm{an}})$. We can forget the complex structure and consider X has a smooth manifold over $\mathbb{R}$. We shall call the smooth manifold as X^{diff}, the vector bundle as $\mathcal{E}^{\mathrm{diff}}$ and the tangent bundle as $\Omega_{X^{\mathrm{diff}}}$. The readers should be aware that the dimensions of X^{diff}, $\mathcal{E}^{\mathrm{diff}}$ and $\Omega_{X^{\mathrm{diff}}}$ are doubled. More concretely, let $\{z^\alpha\}$ be a set of local coordinates of X. Then X^{diff} has local coordinates $\{z^\alpha, \overline{z}^\alpha\}$ and $\Omega_{X^{\mathrm{diff}}}$ has local coordinates $\{dz^\alpha, d\overline{z}^\alpha\}$. The holomorphic connection ∇ also gives rise to a smooth connection

$$\nabla^{\mathrm{diff}} : \mathcal{E}^{\mathrm{diff}} \to \Omega_{X^{\mathrm{diff}}} \otimes_{\mathbb{R}} \mathcal{E}^{\mathrm{diff}}.$$

Let $\mathcal{R}^{\mathrm{diff}} = \nabla^{\mathrm{diff},1} \circ \nabla^{\mathrm{diff}}$. The readers shall try to establish a relation between $\mathcal{R}$ and $\mathcal{R}^{\mathrm{diff}}$. In particular, one shall verify that if $\mathcal{R} = 0$ then $\mathcal{R}^{\mathrm{diff}} = 0$, so there is no ambiguity in defining the integrability. However, many frequently used connections in differential geometry are not holomorphic. For example, the *Levi-Civita connection*, which gives rise to Riemannian curvature, is usually not holomorphic.

Example 1.3. $\mathcal{O}_X$ is canonically equipped with a integrable connection $\nabla : \mathcal{O}_X \to \Omega_X$ which sends f to df. The flatness follows from the fact that $d^2 = 0$.

Example 1.4. A connection on a smooth curve is integrable since $\Omega_X^2 = 0$. Geometrically, this means parallel transport for a holomorphic (or algebraic) vector bundle has trivial holonomy.

Another decent perspective is from complex analysis. Consider a vector bundle $(\mathcal{E}, \nabla) \in \mathrm{VIC}(X^{\mathrm{an}})$. Let U be an open subset where $\mathcal{E}$ is trivial. Pick a global section $e \in \mathcal{E}(U)$. Now let $x \in U$, and consider a path $\gamma : \mathbf{I} \to U$ with $\gamma(0) = x$. Let $\mathfrak{e}(t)$ be the parallel transport of $e(x)$ along γ with $\mathfrak{e}(0) = e(x)$. Then we have

$$\mathfrak{e}(t) = e - \int_0^t de.$$

To see why this is true, simply differentiate both sides with respect to t. Now suppose γ is a nulhomotopic loop in U. Since de is holomorphic, we see that

$$\mathfrak{e}(1) - \mathfrak{e}(0) = \int_\gamma de = 0.$$

As a result, parallel transports of a given section along homotopic curves give the same section at the end point. Note that being holomorphic is essential here.

Example 1.5. Take $X = \operatorname{Spec} \mathbb{C}[z]$ and $\mathcal{E} = \mathcal{O}_X e$. For any $g \in \mathbb{C}[z]$, let ∇_g be the unique algebraic connection such that $\nabla_g(e) = g(z)dz \otimes e$.
Claim: in the category of holomorphic vector bundles with connection, all $(\mathcal{E}, \nabla_g)$ are isomorphic to $(\mathcal{E}, \nabla_0)$; in the category of algebraic vector bundles with connection, $(\mathcal{E}, \nabla_g) \simeq (\mathcal{E}, \nabla_h)$ only when $g = h$.

In fact, an isomorphism between $(\mathcal{E}, \nabla_g)$ and $(\mathcal{E}, \nabla_h)$ must be of the form $f : e \to \alpha e$. By compatibility we have

$$\nabla_h(\alpha e) = \alpha \nabla_g(e),$$

hence $\alpha = Ce^{\int_0^z (g(v) - h(v))dv}$ where $C \in \mathbb{C}^*$. This isomorphism is never algebraic unless $g = h$.

Example 1.6. Local in analytic topology, one takes an open subset $U \subseteq X$ such that U admits an open embedding into $\mathbb{C}^n$ and $\mathcal{E}|_U$ is trivial. Let $\{z^\alpha\}$ be a basis of $\mathbb{C}^n$ and $\{e_\beta\}$ be a basis of $\mathcal{E}|_U$. *Christoffel symobls* $\Gamma^\gamma_{\alpha\beta}$ of a connection ∇ are analytic sections of $\mathcal{O}_X(U)$ such that

$$\nabla(e_\beta) = \Gamma^\gamma_{\alpha\beta} dz^\alpha \otimes e_\gamma \text{ (Einstein's summation convention).}$$

The *connection 1-form* is a matrix

$$\Gamma = (\Gamma^\gamma_{\alpha\beta} dz^\alpha)_{\gamma,\beta} \in \mathfrak{gl}\left(\Omega_U^{\oplus \operatorname{rk} \mathcal{E}}\right).$$

So under the basis $\{e_\beta\}$ we can write $\nabla = d + \Gamma$. If $\{e'_\beta\}$ is another basis of the vector bundle $\mathcal{E}|_U$ and $\{e'_\beta\} = g\{e_\beta\}$ is the matrix of change of basis. Then under the new basis

$$\nabla = d + g\Gamma g^{-1} + gd(g^{-1}).$$

The map $g \circ \Gamma : \Gamma \to g\Gamma g^{-1} + gd(g^{-1})$ is called the *gauge transformation*. Gauge transformations do not satisfy the cocycle relation, i.e. $h \circ g \circ \Gamma \neq (hg) \circ \Gamma$. This shows that Γ is a completely local construction, i.e. we can not glue them along local patches in the same sense as gluing vector bundles from local charts.

Now we discuss flat sections under these explicit coordinates. For $s = s^\beta e_\beta$, an easy computation shows that

$$\begin{aligned} \nabla(s) &= \left(\frac{\partial s^\beta}{\partial z^\alpha} + \Gamma^\gamma_{\alpha\beta} s^\beta\right) dz^\alpha \otimes e_\beta, \\ \mathcal{R} &= d\Gamma + \Gamma \wedge \Gamma. \end{aligned}$$

A flat section s is then a solution of a system of first order PDEs

$$\frac{\partial s^\beta}{\partial z^\alpha} + \Gamma^\gamma_{\alpha\beta} s^\beta = 0. \tag{1.1.4}$$

See Example 1.7 for a concrete computation.

Example 1.7. Let $X = \mathbb{G}_m = \mathbb{C} - \{0\}$ and $\mathcal{E} = \mathcal{O}_X e_1 \oplus \mathcal{O}_X e_2$. Writing $X = \operatorname{Spec} \mathbb{C}[z, z^{-1}]$, we have $\Omega_X = \mathcal{O}_X dz$. Define a connection as follows:

$$\Gamma = \begin{bmatrix} 0 & -\frac{dz}{z} \\ 0 & 0 \end{bmatrix} \in \mathfrak{gl}\left(\Omega_X^{\oplus 2}\right),$$
$$\nabla = d + \Gamma.$$

For a sufficiently small analytical open disk $U \subseteq X$, write a global section s of $\mathcal{E}(U)$ as $s = x_1 e_1 + x_2 e_2$ for $x_1, x_2 \in \mathcal{O}_{X^{\mathrm{an}}}(U)$. Then $\nabla(s) = 0$ gives an equation

$$\frac{dx_1}{dz} e_1 + \left(\frac{dx_2}{dz} - \frac{x_1}{z}\right) e_2 = 0.$$

Solving it, we see that $\ker \nabla(U)$ consist of elements

$$s = C_1 e_1 + (C_1 \log z + C_2) e_2, \; C_1, C_2 \in \mathbb{C}. \tag{1.1.5}$$

It is clear that $\ker \nabla$ is locally constant. Let U be a small disk around the point 1, and $s_1, s_2 \in \ker \nabla(U)$ such that $s_1(1) = e_1$ and $s_2(1) = e_2$. We see that $s_1 = e_1 + e_2 \log z$ and $s_2 = e_2$. Let γ be the unit circle, with counterclockwise direction, representing the element $1 \in \pi_1(\mathbb{G}_m, 1)$. The sections s_1, s_2 analytically continue along the path, giving the following unipotent monodromy representation:

$$\pi_1(\mathbb{G}_m, 1) \to \mathrm{GL}(\mathcal{E}(1)), \; 1 \to \begin{bmatrix} 1 & 2\pi i \\ 0 & 1 \end{bmatrix}.$$

In general, a strict upper-triangular Γ always yields a unipotent representation of $\pi_1(\mathbb{G}_m, 1)$.

Example 1.8 (Example 1.7 in a different flavor). Now we give a complex analytic perspective towards Example 1.7, in the same spirit as Example 1.4. Suppose γ is a curve based at 1. Let $e = C_1 e_1 + C_2 e_2$ be a global section of $\mathcal{E}$. As explained in Example 1.4, the parallel transport $e(1)$ along γ is

$$\begin{aligned} s &= e - \int_\gamma de \\ &= e + \int_\gamma C_1 e_2 \frac{dz}{z} \\ &= e + \int_\gamma C_1 e_2 \; d\log z \\ &= e + C_1 e_2 \log z - C_1 \int_\gamma \log z \; de_2 \\ &= C_1 e_1 + (C_1 \log z + C_2) e_2. \end{aligned}$$

This recovers (1.1.5).

Example 1.9 (important!). As a follow up of Example 1.7, we build, for a given finite dimensional representation $\rho : \pi_1(\mathbb{G}_m, 1) \to \mathrm{GL}(V)$, an element $(\mathcal{E}, \nabla) \in \mathrm{VIC}(\mathbb{G}_m)$ such that $\rho = \ker \nabla$. Let $\rho(1) = A \in \mathrm{GL}(V)$, write $A = e^{2\pi i \mathbf{a}}$ for some $\mathbf{a} \in \mathfrak{gl}(V)$. Consider

$$\Gamma = -\mathbf{a} \frac{dz}{z} \in \mathfrak{gl}(\Omega_{\mathbb{G}_m} \otimes V).$$

Consider $(\mathcal{O}_X \otimes_{\mathbb{C}} V, d + \Gamma) \in \mathrm{VIC}(\mathbb{G}_m)$. For a basis $\mathbf{e}$ of V and a row vector $\mathbf{x}$ in $\mathbb{C}^{\dim V}$, we have

$$\nabla(\mathbf{x}\mathbf{e}^{\mathrm{T}}) = \left(d\mathbf{x} - \mathbf{x}\mathbf{a} \frac{dz}{z}\right) \mathbf{e}^{\mathrm{T}}.$$

Therefore flat sections are $s = \mathbf{x}\mathbf{e}^{\mathrm{T}}$ satisfying $\frac{d\mathbf{x}}{dz} = \frac{\mathbf{x}\mathbf{a}}{z}$. Solving the equation, we find that flat sections are of the form

$$s = \mathbf{C}e^{\log(z)\mathbf{a}}\mathbf{e}^{\mathrm{T}},\ \mathbf{C} \in \mathbb{C}^{\dim V}.$$

Therefore the monodromy representation of $\ker(d+\Gamma)$ is exactly given by $\rho(1) = e^{2\pi i\mathbf{a}} = A$.

Exercise: explain this using contour integration, as in Example 1.8.

2. Riemann-Hilbert correspondence

Let X be connected and $x \in X$. As already illustrated in Example 1.7, it is a general phenomenon that a vector bundle with integrable connection gives rise to a complex representation of $\pi_1(X, x)$ via parallel transport. Conversely, for a finite dimensional complex vector space V and a representation $\rho : \pi_1(X, x) \to \mathrm{GL}(V)$, we obtain a sheaf $\mathcal{K}_\rho$ of complex vector spaces on X by quotienting $\tilde{X} \times V$ by $\pi_1(X, x)$, where $\tilde{X}$ is the universal cover of X. The vector bundle $\mathcal{K}_\rho \otimes_{\mathbb{C}} \mathcal{O}_X$ with $\nabla = 1 \otimes d$ is a vector bundle with integrable connection such that $\ker \nabla = \mathcal{K}_\rho$.

Theorem 2.1. *There is an anti-equivalence of categories:*

$$\mathrm{VIC}(X^{\mathrm{an}}) \Leftrightarrow \begin{Bmatrix} \textit{Finite dimensional complex} \\ \pi_1(X,x)\textit{-representations} \end{Bmatrix}$$
$$(\mathcal{K}_\rho \otimes_{\mathbb{C}} \mathcal{O}_{X^{\mathrm{an}}}, 1 \otimes d) \leftarrow \rho$$
$$(\mathcal{E}, \nabla) \to \ker \nabla$$

The situation is somewhat different for algebraic vector bundles. When X is projective, one can replace $\mathrm{VIC}(X^{\mathrm{an}})$ by $\mathrm{VIC}(X)$, as a consequence of GAGA. The problem arises when X is not projective, as already noted in Remark 1.1. To make Theorem 2.1 still true in the algebraic setting, one need to restrict to a suitable subcategory of $\mathrm{VIC}(X)$.

Definition 1 (Connection having regular singularities)**.** *A logarithmic compatification is an embedding $j : X \hookrightarrow \overline{X}$ into a smooth proper variety $\overline{X}$ such that $D = \overline{X} - X$ is a normal crossing divisor. Such compatification always exists (Hironaka). Let $\Omega_{\overline{X}}(\log D)$ be the subsheaf of $j_*\Omega_X$ whose local sections have at most simple poles along D. An (algebraic or analytic) logarithmic vector bundle with connection is a vector bundle $\mathcal{E}'$ over $\overline{X}$ with a morphism*

$$\nabla' : \mathcal{E}' \to \Omega_{\overline{X}}(\log D) \otimes_{\mathcal{O}_{\overline{X}}} \mathcal{E}'$$

satisfying the Leibniz rule. An $(\mathcal{E}, \nabla) \in \mathrm{VIC}(X)$ is said to have regular singularities, if it has an algebraic logarithmic extension $(\mathcal{E}^{\mathrm{ext}}, \nabla^{\mathrm{ext}})$ to some $\overline{X}$ (hence to all $\overline{X}$). The full subcategory of $\mathrm{VIC}(X)$ *consists of vector bundles with connection having regular singularities is denoted* $\mathrm{VIC}_{\mathrm{reg}}(X)$.

Example 2.2. Consider the connection $(\mathcal{E}, \nabla_g)$ in Example 1.5. The only possible compatification is $\mathbb{P}^1$. Let $s = \frac{1}{z}$. Then the connection can be expressed as $\nabla_g(e) = -g(\frac{1}{s})\frac{ds}{s^2} \otimes e$. Therefore ∇_g has a regular singularity at ∞ only when $g = 0$. The argument also shows that in general, a connection $\nabla = d + \Gamma$ on $\mathbb{A}^1$ has a regular singularity at ∞ if and only if $\Gamma = f(z)dz$ where $f(z) = O(\frac{1}{z})$.

Also note that the connections in Example 1.7 and Example 1.9 have regular singularities.

Theorem 2.3 (Riemann-Hilbert correspondence)**.** *There is an anti-equivalence of categories:*

$$\mathrm{VIC}_{\mathrm{reg}}(X) \Leftrightarrow \begin{Bmatrix} \textit{Finite dimensional complex} \\ \pi_1(X,x)\textit{-representations} \end{Bmatrix}$$
$$(\mathcal{K}_\rho \otimes_{\mathbb{C}} \mathcal{O}_X, \mathrm{Id} \otimes d) \leftarrow \rho$$
$$(\mathcal{E}, \nabla) \to \ker \nabla$$

Note that the left hand side is totally algebraic, while the right hand side is transcendental. Therefore Riemann-Hilbert correspondence builds a bridge between the algebraic and analytic world. In the following, we will sketch a proof of 2.1, and a proof of 2.3 only in the case of curves. After all, $\mathbb{P}^1 - \{0, 1, \infty\}$ is all we care about.

Lemma 2.4. *Let* $(\mathcal{E}, \nabla) \in \mathrm{VC}(X^{\mathrm{an}})$. *There is a natural map* $\ker \nabla \to \mathcal{E}$. *Let* $x \in X$ *be a point, the following are true:*
(1) $(\ker \nabla)_x \to \mathcal{E}(x)$ *is an injection.*
(2) The function $\dim : X \to \mathbb{Z},\ x \to \dim(\ker \nabla)_x$ *is lower semi-continuous in analytic topology.*
(3) The sheaf $\ker \nabla|_{\dim^{-1}(d)}$ *is locally constant. In particular,* $\ker \nabla$ *is constructible.*
(4) If ∇ *is integrable, then* $X = \dim^{-1}(\mathrm{rk}\, \mathcal{E})$. *In particular,* $\ker \nabla$ *is locally constant of rank* $\mathrm{rk}\, \mathcal{E}$.

Proof. All of the statements are local, so we might restrict to a sufficiently small open neighbourhood U containing x. Choose a trivialization of $\mathcal{E}|_U$, we see from Example 1.6 that $\ker \nabla$ is given by the solution of the PDEs (1.1.4). For any curve $x \in \gamma \subseteq U$, the restriction of (1.1.4) to γ is an ODE, which by Picard's theorem has unique solution once the initial condition is given. Since for any point in U there is a γ connecting this point to γ, this shows (1).

Suppose $\dim_x \ker \nabla$ has dimension d, i.e. in a small neighbourhood V of x the equation (1.1.4) has d solutions $s_1, s_2, ..., s_d$, whose initial conditions $s_1(x), s_2(x)..., s_d(x)$ are linearly independent. The restricting to curve trick again implies that $\{s_i(y)\}$ are linearly independent at any $y \in V$, showing that $\dim(\ker \nabla)_y \geq d$. Therefore $\dim^{-1}([d, \infty))$ is open, this gives (2).

Let $x \in \dim^{-1}(d)$. Shrinking U if necessary, we can find a subspace $\Lambda \in \ker \nabla(U)$ such that $\Lambda \to (\ker \nabla)_x$ is an isomorphism. Consider the trivial local system $i : \underline{\Lambda} \hookrightarrow \mathcal{E}|_U$. By (2), i factors through $\ker \nabla$ in a small neighbourhood of x. Shrinking U if necessary, we see that $\underline{\Lambda}|_{\dim^{-1}(d) \cap U} \simeq \ker \nabla|_{\dim^{-1}(d) \cap U}$, and this gives (3).

If ∇ is flat, then a section $v \in \mathcal{E}(x)$ can be extended to a flat section over U by parallel transport. This shows that $\dim(\ker \nabla)_x = \mathrm{rk}\, \mathcal{E}$, and yields (4). ☐

Proof of 2.1. It suffices to show that the compositions of the given functors are isomorphic to identities. Composition on one direction is

$$\rho \to (\mathcal{K}_\rho \otimes_{\mathbb{C}} \mathcal{O}_{X^{\mathrm{an}}}, 1 \otimes d) \to \ker(1 \otimes d).$$

It is clear that $\ker(1 \otimes d) = \mathcal{K}_\rho$. Composition on the other direction is

$$(\mathcal{E}, \nabla) \to \ker \nabla \to (\ker \nabla \otimes_{\mathbb{C}} \mathcal{O}_{X^{\mathrm{an}}}, 1 \otimes d).$$

Lemma 2.4(4) guarantees that $\ker \nabla$ is locally constant of rank $\mathrm{rk}\, \mathcal{E}$. By rank considerations, the natural inclusion $\ker \nabla \otimes_{\mathbb{C}} \mathcal{O}_{X^{\mathrm{an}}} \hookrightarrow \mathcal{E}$ is an isomorphism of vector bundles. The inclusion furthermore commutes with the connections, i.e. for local sections $s \otimes f \in \ker \nabla \otimes_{\mathbb{C}} \mathcal{O}_{X^{\mathrm{an}}}$ we have

$$(1 \otimes d)(s \otimes f) = s \otimes df = \nabla(fs).$$

Therefore the inclusion induces an isomorphism $(\ker \nabla \otimes_{\mathbb{C}} \mathcal{O}_{X^{\mathrm{an}}}, 1 \otimes d) \simeq (\mathcal{E}, \nabla)$. ☐

Proof of 2.3 when X is an (affine) curve. By Theorem 2.1 it suffices to show that the analytification functor

$$F^{\mathrm{an}} : \mathrm{VIC}_{\mathrm{reg}}(X) \to \mathrm{VIC}(X^{\mathrm{an}})$$

is an equivalence. We show this by introducing an inverse functor $G : \mathrm{VIC}(X^{\mathrm{an}}) \to \mathrm{VIC}_{\mathrm{reg}}(X)$.

In fact, pick an embedding X into a smooth projective curve C. For any $P \in C - X$, pick a sufficiently small disk $P \in D_P \subseteq C(\mathbb{C})$. Consider $(\mathcal{E}, \nabla) \in \mathrm{VIC}^{\mathrm{an}}(X)$, the restriction $(\mathcal{E}, \nabla)|_{D_P - \{0\}}$ is in $\mathrm{VIC}(\mathbb{G}_m^{\mathrm{an}})$. By Theorem 2.1, the restriction $(\mathcal{E}, \nabla)|_{D_P - \{0\}}$ gives rise to a $\mathbb{C}$-representation of $\mathbb{Z} = \pi_1(D_P - \{0\})$. Now by Example 1.9 and Example 2.2 we obtain a vector bundle with logarithmic connection $(\mathcal{E}_P^{\mathrm{ext}}, \nabla_P^{\mathrm{ext}})$ over D_P extending $(\mathcal{E}, \nabla)|_{D_P - \{0\}}$. Glueing $(\mathcal{E}_P^{\mathrm{ext}}, \nabla_P^{\mathrm{ext}})$ to $(\mathcal{E}, \nabla)$

along $(\mathcal{E},\nabla)|_{D_P-\{0\}}$, we obtain a holomorphic vector bundle with logarithmic connection $(\mathcal{E}^{\text{ext}},\nabla^{\text{ext}})$ over C extending $(\mathcal{E},\nabla)$.

By GAGA $(\mathcal{E}^{\text{ext}},\nabla^{\text{ext}})$ is isomorphic to an algebraic vector bundle with logarithmic connection $(\mathcal{E}',\nabla')$ over C. We have $(\mathcal{E}',\nabla')|_X \in \mathrm{VIC}_{\text{reg}}(X)$. Define G sending $(\mathcal{E},\nabla)$ to $(\mathcal{E}',\nabla')|_X$. The above discussion also applies to extension of morphisms in $\mathrm{VIC}(X^{\text{an}})$ and we can upgrade G into a functor, which is easily seen inverse to F^{an}. $\square$

3. Vector bundles with connection arising from geometry

Let S be a smooth variety. An important family of vector bundles with integrable connection over S arise from geometry. Precisely, these vector bundles arise as the relative de Rham cohomology $\mathcal{H}^i_{\text{dR}}(X/S)$ of a smooth morphism $\pi: X\to S$. There is a canonical connection ∇_{GM} on $\mathcal{H}^i_{\text{dR}}(X/S)$, called the Gauss-Manin connection. In order to keep the review concise and simple, we will always work on base field $\mathbb{C}$, with π projective. The letter n will be used to denote the relative dimension of X over S. We won't give precise proof of the statements we make. Rather, we put more emphasise on examples.

3.1. **Algebraic de Rham cohomology.** With notation and assumption as above, we define a complex $\Omega^\bullet_{X/S}\in D_{\text{Coh}}(\pi^{-1}\mathcal{O}_S)$ as follows:

$$0\to \mathcal{O}_{X/S}\xrightarrow{d}\Omega_{X/S}\xrightarrow{d}\Omega^2_{X/S}\xrightarrow{d}\Omega^3_{X/S}\xrightarrow{d}\dots \tag{3.1.1}$$

The *relative algebraic de Rham cohomology* $\mathcal{H}^i_{dR}(X/S)$ is an $\mathcal{O}_S$-module defined by the i-th hypercohomology

$$H^i(R\Gamma(S,\Omega^\bullet_{X/S})).$$

When $S=\operatorname{Spec}\mathbb{C}$, $\mathcal{H}^i_{dR}(X/S)$ is just a vector space over $\mathbb{C}$, conventionally denoted $H^i_{dR}(X/\mathbb{C})$. *Poincare lemma* says that the analytification $\Omega^\bullet_{X^{\text{an}}/S^{\text{an}}}$ is a resolution of the constant sheaf $\underline{\mathbb{C}}_X$. Therefore via GAGA,

$$H^i_{\text{dR}}(X/\mathbb{C})\simeq H^i_{\text{B}}(X,\mathbb{C}).$$

A more motivated way of viewing this is by *de Rham's theorem*, which asserts the existence of a canonical isomorphism

$$\begin{aligned} H^i_{\text{dR}}(X/\mathbb{C}) &\xrightarrow{\simeq} H^i_{\text{B}}(X,\mathbb{C}),\\ \omega &\to \left(\gamma\to\int_\gamma\omega\right), \text{ where } \gamma\in H^{\text{B}}_{n-i}(X/\mathbb{C}).\end{aligned}$$

For general X, $\Omega^\bullet_{X^{\text{an}}/S^{\text{an}}}$ is a resolution of $\pi^{-1}\mathcal{O}_{S^{\text{an}}}$. It follows from *projection formula* that

$$\mathcal{H}^i_{\text{dR}}(X/S)^{\text{an}}:=\mathcal{H}^i_{\text{dR}}(X/S)\otimes_{\mathcal{O}_S}\mathcal{O}_S^{\text{an}}\simeq R^i\pi_*(\underline{\mathbb{C}}_X\otimes_{\mathbb{C}}\pi^{-1}\mathcal{O}_S^{\text{an}})\simeq R^i\pi_*\underline{\mathbb{C}}_X\otimes_{\mathbb{C}}\mathcal{O}_{S^{\text{an}}},$$

where $R^i\pi_*\underline{\mathbb{C}}_X$ is the relative Betti cohomology, which is a locally constant sheaf of $\mathbb{C}$ vector spaces. Again, a more motivated perspective of viewing this is to apply de Rham's theorem to the whole family, i.e. we have a canonical isomorphism

$$\begin{aligned} \mathcal{H}^i_{\text{dR}}(X/S)^{\text{an}} &\xrightarrow{\simeq} R^i\pi_*\underline{\mathbb{C}}_X\otimes_{\mathbb{C}}\mathcal{O}_{S^{\text{an}}},\\ \omega &\to \left(\gamma\to\int_\gamma\omega\right).\end{aligned} \tag{3.1.2}$$

Note that $\int_\gamma\omega$ is a holomorphic function over S. This is also the reason why we base change to $\mathcal{O}_{S^{\text{an}}}$. An important consequence of this isomorphism is that $R^i\pi_*\underline{\mathbb{C}}_X$ canonically sits inside $\mathcal{H}^i_{\text{dR}}(X/S)^{\text{an}}$ as the sheaf of differential forms ω such that $\int_\gamma\omega$ is constant for all γ.

Remark 3.1. Algebraic de Rham cohomology can be defined, not only over $\mathbb{C}$, but over any field k of characteristic 0. When $S = \operatorname{Spec} k$, the cohomology assigning a smooth projective scheme X with the graded vector space $H^{\bullet}_{\mathrm{dR}}(X/k)$, is a *Weil cohomology theory*. This means that $H^{\bullet}_{\mathrm{dR}}(X/k)$ behaves well, i.e. has expected dimension and properties (Poincare duality, Kunneth formula and Lefchetz axioms).

Remark 3.2. The complex (3.1.1) admits a canonical bi-complex resolution $\Omega^{\bullet}_{X^{\mathrm{an}}/S^{\mathrm{an}}} \to \Omega^{\bullet;\bullet}_{X^{\mathrm{an}}/S^{\mathrm{an}}}$, called *Dolbeault complex*. The sheaf $\Omega^{p,q}_{X^{\mathrm{an}}/S^{\mathrm{an}}}$ has local sections a wedge of p holomorphic differential forms and q anti-holomorphic differential forms. The resulting cohomology is classically called Dolbeault cohomology. As a consequence of GAGA, Dolbeault cohomology equals the algebraic de Rham cohomology when X is projective.

Remark 3.3. There is a *Hodge to de Rham spectral sequence* that comes from double complex resolution

$$E_1^{p,q} = R^q\pi_*\Omega^p_{X/S} \Rightarrow \mathcal{H}^{p+q}_{dR}(X/S). \tag{3.1.3}$$

When $S = \operatorname{Spec}\mathbb{C}$, (3.1.3) degenerates at the first page, yielding the classical *Hodge decomposition*. In general case, (3.1.3) gives rise to a *Hodge filtration* $\mathrm{Fil}^{\bullet}$ of $\mathcal{H}^{p+q}_{dR}(X/S)$ compatible with the Hodge decomposition on each fibre. This provides an example of a *variation of Hodge structure*. As it will not be used, we won't go any further.

3.2. **Gauss-Manin connection.** In contrast to $H^i_{\mathrm{dR}}(X/\mathbb{C})$, the relative cohomology $\mathcal{H}^i_{\mathrm{dR}}(X/S)$ only contains fiberwise data. However, it is possible to recover the data of the total space from $\mathcal{H}^i_{\mathrm{dR}}(X/S)$ by putting a connection ∇_{GM} over $\mathcal{H}^i_{\mathrm{dR}}(X/S)$, which tells how the fiberwise data vary "horizontally". This is the intuition of Gauss-Manin connection.

In fact, let ω be a differential form in $\mathcal{H}^i_{\mathrm{dR}}(X/S)$. Then the Gauss-Manin connection is the "most natural" way of differentiating ω along vector fields. To motivate it, let S be an open subset of $\operatorname{Spec}\mathbb{C}[t]$. Let γ denote a cycle in the $(n-i)$-th relative Betti homology of X/S. Then the differential $\frac{d\omega}{dt}$ given by ∇_{GM} is the unique cocycle in $\mathcal{H}^i_{\mathrm{dR}}(X/S)^{\mathrm{an}}$ satisfying

$$\int_\gamma \frac{d\omega}{dt} = \frac{d}{dt}\int_\gamma \omega,\ \forall\gamma \in \mathcal{H}^{\mathrm{B}}_{n-i}(X/S).$$

It is easy to see that $\frac{d\omega}{dt} = 0$ if and only if for every γ, the function $\int_\gamma \omega$ is a constant. It follows immediately that

$$\ker\nabla_{\mathrm{GM}} = R^i\pi_*\underline{\mathbb{C}}_X.$$

To give a formal definition of the Gauss-Manin connection, note that (3.1.2) equips the analytic vector bundle $\mathcal{H}^i_{\mathrm{dR}}(X/S)^{\mathrm{an}}$ with an integrable connection

$$\nabla(s\otimes f) := s\otimes df \in R^i\pi_*\underline{\mathbb{C}}_X \otimes_{\mathbb{C}} \Omega_{S^{\mathrm{an}}} = \mathcal{H}^i_{\mathrm{dR}}(X/S)^{\mathrm{an}} \otimes \Omega_{S^{\mathrm{an}}},$$

which is exactly the Gauss-Manin connection. Though we defined it analytically, the Gauss-Manin connection is actually an algebraic connection with regular singularities. In other words, $(\mathcal{H}^i_{\mathrm{dR}}(X/S), \nabla_{\mathrm{GM}})$ is an object of $\mathrm{VIC}_{\mathrm{reg}}(S)$. There are several ways to show this. For example, it follows from Riemann-Hilbert correspondence (Theorem 2.3), as $\ker\nabla_{\mathrm{GM}} = R^i\pi_*\underline{\mathbb{C}}_X$ is clearly a local system over X. Or one can stick to the theory of algebraic de Rham cohomology, defining Gauss-Manin connection algebraically, then show it coincides with the analytic definition (see [KO68]).

3.3. **Concrete examples.** In practice, one may compute ∇_{GM} directly is as follows: given a relative local section $\omega \in \Omega^i_{X/S}$, let $\overline{\omega}$ be its class in $\mathcal{H}^i_{\mathrm{dR}}(X/S)$. One can always lift ω to an absolute section $\tilde{\omega} \in \Omega^i_{X/\mathbb{C}}$, then projects $d\tilde{\omega}$ to $\mathcal{H}^i_{\mathrm{dR}}(X/S) \otimes \Omega_X$. This computes $\nabla_{\mathrm{GM}}(\overline{\omega})$. The readers shall find out how this method fit into our motivating picture of Guass-Manin connection given in the last section.

Example 3.4. The example is taken from [Ked]. Let $U \subseteq \operatorname{Spec} \mathbb{C}[t]$ be a Zariksi open subset, and $a(t), b(t) \in \mathbb{C}[t]$ are chosen such that the elliptic fiberation

$$E_t : y^2 = x^3 + a(t)x + b(t)$$

is smooth projective over U (So U need to avoid points of bad reduction, i.e. points such that $\Delta = 4a^2 + 27b^3 = 0$). Even though we express the curve as an affine curve, what we really mean is a projective elliptic curve, i.e. there is a point at infinity. Our goal in this example is to explicitly compute $(\mathcal{H}^1_{\mathrm{dR}}(E_t/U), \nabla_{\mathrm{GM}})$.

Let $\mathcal{U} \subseteq E_t$ be the affine open chart excluding the infinite section, and $\mathcal{V} \subseteq E_t$ be the affine open chart avoiding the locus $x = 0$. As a consequence, x and y are global sections of $\mathcal{O}_{\mathcal{U}/U}$, while $\omega = \frac{dx}{y}$ is a global section of $\Omega_{E_t/U}$. The Cech resolution of the de Rham complex $\Omega^\bullet_{E_t/U}$ associated to the cover $\{\mathcal{U}, \mathcal{V}\}$ is

$$\begin{array}{ccc} \mathcal{O}_{\mathcal{U}\cap\mathcal{V}/U} & \longrightarrow & \Omega_{\mathcal{U}\cap\mathcal{V}/U} \\ \uparrow & & \uparrow \\ \mathcal{O}_{\mathcal{U}/U} \oplus \mathcal{O}_{\mathcal{V}/U} & \longrightarrow & \Omega_{\mathcal{U}/U} \oplus \Omega_{\mathcal{V}/U} \end{array}$$

The total complex is then

$$\mathcal{O}_{\mathcal{U}/U} \oplus \mathcal{O}_{\mathcal{V}/U} \xrightarrow{\partial_1} \Omega_{\mathcal{U}/U} \oplus \Omega_{\mathcal{V}/U} \oplus \mathcal{O}_{\mathcal{U}\cap\mathcal{V}/U} \xrightarrow{\partial_2} \Omega_{\mathcal{U}\cap\mathcal{V}/U},$$
$$\partial_1(a, b) = (da, db, b - a), \ \partial_2(\alpha, \beta, c) = dc + \beta - \alpha.$$

As a result we get

$$\mathcal{H}^1_{\mathrm{dR}}(E_t/U) = \frac{\ker \partial_2}{\operatorname{im} \partial_1}.$$

Note that $\ker \partial_2$ contains an $\mathcal{O}_U$-linear combination of following 1-cocycles

$$(\omega, \omega, 0), \ (x\omega, \ x\omega - d(y^2/x^2), \ -y^2/x^2),$$

here one needs to check $x\omega - d(y^2/x^2) \in \Omega_{\mathcal{U}\cap\mathcal{V}/U}$ has a continuation to a section in $\Omega_{\mathcal{V}/U}$. It is easy to check that $\ker \partial_2 / \operatorname{im} \partial_1$ is generated by the two 1-cocycles listed. Projecting to the first direct summand, we have

$$\mathcal{H}^1_{\mathrm{dR}}(E_t/U) = \mathcal{O}_U\langle \omega, x\omega \rangle.$$

We now compute the Gauss-Manin connection. Let $A, B \in \mathcal{O}_U[x]$ such that $\omega = Aydx + 2Bdy$. This is always doable, for example let $P(t) = x^3 + a(t)x + b(t)$, then one can choose A, B such that

$$AP + BP_x = 1.$$

The process of computing ∇_{GM} is summarized in the following diagram:

$$\begin{array}{ccc} \Omega_{E_t/\mathbb{C}} & \xrightarrow{\quad d \quad} & \Omega^2_{E_t/\mathbb{C}} \\ \downarrow & & \downarrow \\ \mathcal{H}^1_{\mathrm{dR}}(E_t/U) & \xrightarrow{\nabla_{\mathrm{GM}}} & \mathcal{H}^1_{\mathrm{dR}}(E_t/U) \otimes \Omega_{U/\mathbb{C}} \end{array}$$

Note that ω itself can be regarded as living in $\Omega_{E_t/\mathbb{C}}$. Differentiating, we find

$$d\omega = d(Aydx + 2Bdy) = Ady \wedge dx + A_t ydt \wedge dx + 2B_x dx \wedge dy + 2B_t dt \wedge dy$$

Since $dx = y\omega, dy = \frac{1}{2}P_x\omega$ and $2ydy = P_x dx + P_t dt$, we obtain that

$$\begin{aligned} dx \wedge dt &= y\omega \wedge dt, \\ dy \wedge dt &= \frac{1}{2}P_x\omega \wedge dt, \\ dx \wedge dy &= \frac{1}{2}P_t\omega \wedge dt. \end{aligned}$$

As a result, we get the following explicit formulas

$$\begin{aligned} d\omega &= (B_x P_t - \frac{1}{2}AP_t - A_t P - B_t P_x)\omega \wedge dt, \\ d(x\omega) &= xd\omega + BP_t\omega \wedge dt. \end{aligned}$$

Projecting down to $\mathcal{H}^1_{\mathrm{dR}}(E_t/U) \otimes \Omega_{U/\mathbb{C}}$, we find

$$\begin{aligned} \nabla_{\mathrm{GM}}(\omega) &= (B_x P_t - \frac{1}{2}AP_t - A_t P - B_t P_x)\omega \otimes dt, \\ \nabla_{\mathrm{GM}}(x\omega) &= x\nabla\omega + BP_t\omega \otimes dt. \end{aligned}$$

Example 3.5. With the notation as in Example 3.4, consider the elliptic fiberation

$$E_t : y^2 = x^3 + t$$

over $\mathbb{G}_m = \operatorname{Spec}\mathbb{C}[t, \frac{1}{t}]$. Let $A = \frac{1}{t}, B = -\frac{x}{3t}$ so that $\omega = \frac{dx}{y} = Aydx + 2Bdy$. The computation in Example 3.4 shows that

$$\nabla(\omega) = \frac{1}{6t}\omega \otimes dt, \;\; \nabla(x\omega) = -\frac{1}{6t}x\omega \otimes dt.$$

Therefore under the basis $\{\omega, x\omega\}$, the matrix of connection 1-form (Example 1.6) is given by

$$\Gamma_{\mathrm{GM}} = \begin{bmatrix} \frac{dt}{6t} & 0 \\ 0 & -\frac{dt}{6t} \end{bmatrix}.$$

A flat section on a sufficiently small analytic disk U is of the form $s = a\omega + bx\omega$ where $a, b \in \mathcal{O}_{\mathbb{G}_m^{\mathrm{an}}}(U)$. Therefore we get differential equations

$$0 = \frac{ds}{dt} = a_t\omega + a\frac{d\omega}{dt} + b_t x\omega + b\frac{dx\omega}{dt}.$$

Solving, we get $a = C_1 t^{-\frac{1}{6}}, b = C_2 t^{\frac{1}{6}}$. Therefore we get $\ker \nabla_{\mathrm{GM}}$ as a rank two local system whose monodromy representation at the point 1 is given by

$$\pi_1(\mathbb{G}_m, 1) \to \mathrm{GL}(\mathbb{C}\omega \oplus \mathbb{C}x\omega), \;\; 1 \to \begin{bmatrix} \zeta_6 & 0 \\ 0 & \zeta_6^{-1} \end{bmatrix}.$$

The monodromy is trivial after passing to the etale cover $t \to t^6$, corresponding to the fact that the fiberation $E_t : y^2 = x^3 + t$ becomes trivial by adjoining $t^{\frac{1}{6}}$.

Example 3.6. With the notation as in Example 3.4, consider the elliptic fiberation

$$E_t : y^2 = (x^2 - t)(x - 1)$$

over $\mathbb{P}^1 - \{0,1,\infty\} = \operatorname{Spec}\mathbb{C}[t, \frac{1}{t}, \frac{1}{t-1}]$. By a similar argument one shows that the connection 1-form of ∇_{GM} is given by

$$\Gamma_{\mathrm{GM}} = \begin{bmatrix} \frac{1}{4(t-1)} & \frac{1}{4(t-1)} \\ \frac{-1}{4t(t-1)} & \frac{-1}{4(t-1)} \end{bmatrix} dt.$$

Near a point $z \in U(\mathbb{C})$, we expand $-\Gamma_{\mathrm{GM}}$ as Laurent series

$$-\Gamma_{\mathrm{GM}} = \sum_{n \geq -1} (t-z)^n \Gamma_n.$$

A flat section $s = a\omega + bx\omega$ in a small analytic disk V is given by the ODEs

$$(a_t, b_t) + (a,b)\Gamma_{\mathrm{GM}} = 0,$$

and it is easy to see that a solution (a,b) is given by

$$(a,b) = (C_1, C_2)e^{\Gamma_{-1}\log(t-z) + \sum_{n\geq 0} \frac{(t-z)^{n+1}}{n+1}\Gamma_n}, \ C_1, C_2 \in \mathbb{C}.$$

Therefore the monodromy around z is completely determined by the term $e^{\Gamma_{-1}\log(t-z)}$. Around points 0 and 1, the corresponding Γ_{-1} are given by

$$\frac{1}{4}\begin{bmatrix} 0 & 0 \\ -1 & 0 \end{bmatrix} \text{ and } \frac{1}{4}\begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}$$

respectively. Let $z_0 \neq 0, 1$ and γ_0, γ_1 be simple loops based at z_0 and travel around 0 and 1, so that $\pi_1(U, z_0)$ is the free group generated by γ_0 and γ_1. The monodromy representation is then given by

$$\pi_1(U, z_0) \to \mathrm{GL}(\mathbb{C}\omega \oplus \mathbb{C}x\omega), \ \gamma_0 \to \begin{bmatrix} 1 & 0 \\ -\frac{\pi i}{2} & 1 \end{bmatrix}, \gamma_1 \to \mathbf{I} + \frac{\pi i}{2}\begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}.$$

The local monodromies are unipotent, while the Zariski closure of the global monodromy is SL_2.

There are strong restrictions on the monodromy of local systems that come from geometry. The following are some big theorems underlying Example 3.5 and Example 3.6. We simply state them without any comment. The readers shall find out how they fit into the examples.

Theorem 3.7 (Picard-Lefschetz, [Wikb])**.** *Let $X \to \mathbb{P}^1$ be a proper flat family of relative dimension n. Suppose all critical points only have A_1-singularities (singularities that locally look like nodes) and lie in different fibres, with image $x_1, ..., x_k$. Let $x \notin \{x_1, ..., x_k\}$, then the monodromy representation*

$$\pi_1(\mathbb{P}^1 - \{x_1, ..., x_k\}, x) \to \mathrm{GL}(H^n(X_x, \mathbb{C}))$$

is given by

$$\gamma_i(\omega) = \omega + (-1)^{\frac{(n+1)(n+2)}{2}}\langle \omega, \delta_i \rangle \delta_i,$$

where γ_i is the simple loop based at x and going around x_i, and δ_i is the vanishing cycle corresponding to x_i. The monodromy actions on cohomologies of other degrees are trivial.

Theorem 3.8 (Griffith-Landman-Grothendieck-Katz, [Kat70])**.** *Let $\mathcal{X} \to X$ be a proper flat morphism with X a smooth curve, suppose the morphism is smooth over the open subset $X - \{x_1, ..., x_k\}$. Let $x \notin \{x_1, ...x_k\}$ and γ_i be the simple loop based at x and going around x_i. Then the γ_i action on $\mathrm{GL}(H^i(X_x, \mathbb{C}))$ is given by a linear operator T which admits a decomposition of the form $T = DU = UD$ such that*
1) U is unipotent with $(I-U)^{i+1} = 0$.
2) D is semisimple and all of its eigenvalues are roots of unity.

Theorem 3.9 (Deligne, [Del71])**.** *Let U be a smooth (affine) curve, and $\pi : X \to U$ be a smooth projective morphism. Then $R^n\pi_*\underline{\mathbb{C}}_X$ is semisimple.*

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