INTRODUCTION TO MULTIPLE ZETA VALUES

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Abstract. This is the introductory talk in a learning seminar on Deligne's paper [5], which gave evidence that the pro-unipotent completion of the fundamental group of $\mathbb{P}^1 - \{0, 1, \infty\}$ should be regarded as a "motive". The ideas in [5] are related to multiple zeta values, which arise as "periods" of the motive attached to $\pi_1(\mathbb{P}^1 - \{0, 1, \infty\})$.

1. Multiple zeta values and periods

1.1. Some infinite series.

Let

$$\zeta(k) = \sum_{n=1}^{\infty} \frac{1}{n^k},$$

for integers $k \geq 2$.

Theorem 1.1.1. *(Euler [8])*

$$\zeta(2) = \frac{\pi^2}{6}.$$

Remark 1.1.2. *In fact, a similar method allows one to determine $\zeta(k)$ for all even k. The values are related to Bernoulli numbers, which are beyond the scope of our seminar.*

Proof. (Following Euler, we will omit some complex-analytic details needed to make this argument rigorous.)

The idea is to "factor $\sin x$ like a polynomial" and apply Vièta's formula to the x^3 coefficient.

Consider the sine function $\sin x$. On the one hand, we have the Taylor expansion

$$\sin x = x - \frac{x^3}{6} + \frac{x^5}{120} - \cdots .$$

On the other hand, since $\sin x$ has zeroes at precisely the points $x = n\pi$, with $n \in \mathbb{Z}$, we expect it to admit a factorization

$$\sin x = x \prod_{n=1}^{\infty} \left(1 - \frac{x^2}{n^2\pi^2}\right). \tag{1}$$

Suppose for a moment that this factorization has been proven. The x^3-term of the partial product

$$x \prod_{n=1}^{N} \left(1 - \frac{x^2}{n^2\pi^2}\right)$$

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is

$$\sum_{n=1}^{N} \frac{-1}{n^2\pi^2}.$$

Taking the limit as $N \to \infty$ (the justification for which is a routine exercise in complex analystis), we find that

$$\frac{-1}{6} = x^3\text{-coefficient of } \sin x = \sum_{n=1}^{\infty} \frac{-1}{n^2\pi^2},$$

which is precisely what we wanted to prove.

Now we outline the proof of Equation (1).

Let us temporarily introduce the bizarre notation

$$\widetilde{\sin} x = x \prod_{n=1}^{\infty} \left(1 - \frac{x^2}{n^2\pi^2}\right).$$

A calculation shows that this product converges uniformly on compact subsets of $\mathbb{C}$, and gives an asymptotic bound of the form

$$\left|\widetilde{\sin} x\right| \ll e^{C|x| \ln |x|}$$

for $|x|$ large.

The quotient

$$\frac{\widetilde{\sin} x}{\sin x}$$

is a nowhere-vanishing entire holomorphic function, so it admits a global holomorphic logarithm

$$\frac{\widetilde{\sin} x}{\sin x} = e^{f(x)}.$$

Now f is an entire function, and our bound on $\left|\widetilde{\sin} x\right|$ implies

$$|f(x)| \ll |x| \ln |x|$$

for $|x|$ large.

By complex analysis, our growth bound on f implies that f is a polynomial of degree at most 1. We see that f is an even function, so in fact f is constant; comparing x-coefficients shows that $f(x) = 1$ uniformly. □

Euler did not manage to find closed-form expressions for $\zeta(3)$ or other odd zeta values. (We now expect that there is no such expression; we'll discuss some algebraic-independence conjectures shortly.)

Some decades later, Euler managed to relate $\zeta(3)$ to the sum of a "strange series" now known as a multiple zeta value. For integers k_1, k_2 with $k_1 \geq 2$ and $k_2 \geq 1$, let

$$\zeta(k_1, k_2) = \sum_{n_1 > n_2 \geq 1} \frac{1}{n_1^{k_1} n_2^{k_2}}.$$

Theorem 1.1.3. *(Euler* [7]*)*

$$\zeta(3) = \zeta(2, 1).$$

Proof. We will mimic the "partial fractions and telescoping series" trick familiar from an elementary calculus course.

We rewrite

$$\zeta(2,1) = \sum_{n,k \geq 1} \frac{1}{(n+k)^2 n}.$$

(You might think of this as "summing along the diagonals".)

By partial fractions, we have

$$\frac{1}{(n+k)^2 n} = \frac{-1}{(n+k)^2 k} + \frac{1}{nk^2} - \frac{1}{(n+k)k^2}.$$

Hence, for each fixed k, we obtain a telescoping sum:

$$\sum_{n \geq 1} \frac{1}{(n+k)^2 n} = \frac{1}{k^2} + \frac{1}{2k^2} + \cdots + \frac{1}{k k^2} - \sum_{n \geq 1} \frac{1}{(n+k)^2 k}.$$

Summing over all k, we obtain

$$\zeta(2,1) = \zeta(2,1) + \zeta(3) - \zeta(2,1),$$

which is what we wanted to prove. □

We conclude with a third (and much easier) relation among multiple zeta values.

Theorem 1.1.4. *Whenever $k_1, k_2 \geq 2$, we have*

$$\zeta(k_1)\zeta(k_2) = \zeta(k_1 + k_2) + \zeta(k_1, k_2) + \zeta(k_2, k_1).$$

Proof. Evaluate the double sum

$$\sum_{n_1, n_2 \geq 1} \frac{1}{n_1^{k_1} n_2^{k_2}}$$

in two different ways. The details are left as an exercise to the reader. □

1.2. **Relations among multiple zeta values.** The theorems above are special cases of a more general picture, which is not yet completely understood.

First, by analogy with $\zeta(k_1, k_2)$, one can define multiple zeta values

$$\zeta(k_1, k_2, \ldots, k_r) = \sum_{n_1 > n_2 > \cdots > n_r \geq 1} \frac{1}{n_1^{k_1} \cdots n_r^{k_r}}.$$

Both Theorems 1.1.3 and 1.1.4 generalize to give a variety of relations among these multiple zeta values.

We can define a grading on multiple zeta values by declaring that

$$\zeta(k_1, k_2, \ldots, k_r)$$

is in degree $k_1 + k_2 + \cdots + k_r$, and the degree of a product is the sum of the degrees. (So, for example, Theorem 1.1.4

$$\zeta(k_1)\zeta(k_2) = \zeta(k_1 + k_2) + \zeta(k_1, k_2) + \zeta(k_2, k_1)$$

is a linear relation among multiple zeta values of degree $k_1 + k_2$.) With this grading, all known relations between multiple zeta values respect the degree.

Conjecture 1.2.1. *All polynomial relations among the values $\zeta(k_1, k_2, \ldots, k_r)$ are generated by relations homogenous in the degree.*

1.3. **Reinterpretation as integrals.** We want to interpret multiple zeta values as integrals. We begin with two operations on power series. Suppose

$$f(x) = \sum_{n=1}^{\infty} a_n x^n.$$

Then an easy calculation shows that

$$\int \frac{f(x)\,dx}{x} = \sum_{n=1}^{\infty} \frac{a_n}{n} x^n$$

and

$$\int \frac{x f(x)\,dx}{1-x} = \sum_{n=1}^{\infty} \left(\sum_{k=1}^{n-1} a_k \right) x^n.$$

In other words, by evaluating integrals, we can perform the following manipulations on power series:

- Divide the n-th coefficient by n.
- Replace each coefficient by the cumulative sum of all prior coefficients.

It is not hard to see that, starting from the constant function

$$f(x) = 1 = 1 + 0x + 0x^2 + \cdots$$

and applying the above two operations, we can obtain any power series of the form

$$f_{k_1,\ldots,k_r}(x) = \sum_{n_1 > n_2 > \cdots > n_r \geq 1} \frac{x^{n_1}}{n_1^{k_1} \cdots n_r^{k_r}};$$

substituting $x = 1$ then recovers the multiple zeta value

$$f_{k_1,\ldots,k_r}(1) = \zeta(k_1, k_2, \ldots, k_r).$$

For example,

$$\zeta(2) = \int_0^1 \left(\int_0^{t_2} \frac{dt_2}{1-t_2} \right) \frac{dt_1}{t_1},$$

and

$$\zeta(2,1) = \int_0^1 \left(\int_0^{t_3} \left(\int_0^{t_2} \frac{dt_1}{1-t_1} \right) \frac{dt_2}{1-t_2} \right) \frac{dt_3}{t_3}.$$

Such expressions are known as *iterated integrals.*

1.4. **Cohomology and periods.** We have just seen that multiple zeta values can be interpreted as certain definite integrals of algebraic functions. Definite integrals of algebraic functions arise often in algebraic geometry, where they are known as "periods". We begin with some abstract nonsense.

Let X be a smooth algebraic variety defined over $\mathbb{Q}$. We consider two cohomology theories, each of which assigns to X a $\mathbb{Q}$-vector space.

1.4.1. *Betti cohomology.* The first is singular (or "Betti") cohomology

$$H^i_B(X,\mathbb{Q}),$$

defined using classical topology by viewing $X(\mathbb{C})$ as a topological space. In fancy language: one starts with the scheme X, performs a base-change from $\mathbb{Q}$ to $\mathbb{C}$, applies the analytification functor to make a complex-analytic space $X^{\mathrm{an}}_{\mathbb{C}}$, and then applies the forgetful functor to arrive at a topological space. In plain language: Elements of $H^i_B(X,\mathbb{Q})$ are dual to actual physical topological cycles on the complex manifold X.

1.4.2. *Algebraic de Rham cohomology.* The second is Grothendieck's algebraic de Rham cohomology

$$H^i_{dR}(X/\mathbb{Q}),$$

defined using sheaves of differentials on X. Grothendieck observed that de Rham cohomology can be computed purely algebraically, without leaving the category of coherent sheaves.

In fancy language: the algebraic de Rham cohomology of X is the hypercohomology of the de Rham complex

$$0 \to \mathcal{O}_X \to \Omega^1_X \to \Omega^2_X \to \cdots$$

of sheaves of differentials on X. This "hypercohomology" combines the cohomology of the complex (closed differentials modulo exact differentials) with the Zariski cohomology of the individual sheaves on X (i.e. $H^i(X,-)$).

In plain language: de Rham cohomology classes are cooked up from algebraic differentials on open covers of X. For example, there is an injection

$$\Gamma(X,\Omega^i_X) \hookrightarrow H^i_{dR}(X,\mathbb{Q}),$$

so any global differential of order i on X gives a class in $H^i_{dR}(X,\mathbb{Q})$.

Remark 1.4.1. *(Hodge filtration; I did not mention this during the talk, and it won't be needed for the rest of these notes.)*

In general, the machinery of "hypercohomology" gives rise to a filtration on $H^i_{dR}(X,\mathbb{Q})$*, known as the Hodge filtration, for which* $\Gamma(X,\Omega^i_X)$ *is the first filtered piece.*

Specifically, we have a descending filtration

$$H^i_{dR}(X,\mathbb{Q}) = \mathrm{Fil}^0 H^i_{dR} \supseteq \mathrm{Fil}^1 H^i_{dR} \supseteq \mathrm{Fil}^2 H^i_{dR} \supseteq \cdots \supseteq \mathrm{Fil}^{i+1} H^i_{dR} = 0,$$

where

$$\mathrm{Fil}^p H^i / \mathrm{Fil}^{p+1} H^i_{dR} \cong H^p(X,\Omega^{i-p}_X).$$

1.4.3. *The comparison theorem.* Grothendieck showed [9] that algebraic de Rham cohomology agrees with Betti cohomology over $\mathbb{C}$. In other words, the Betti numbers (a topological invariant) of a variety X can be computed purely algebraically from the complex of differentials on X.

Let

$$H_i^B(X/\mathbb{Q}) = H^i_B(X/\mathbb{Q})^\vee$$

be the singular homology of X. Write

$$H_i^B(X/\mathbb{C}) = H_i^B(X/\mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{C},$$

and similarly for other (co)homology theories.

Integration defines a pairing

$$H^i_B(X/\mathbb{C}) \times H^i_{dR}(X/\mathbb{C}) \to \mathbb{C},$$

given by

$$(\gamma, \omega) \mapsto \int_\gamma \omega.$$

Remark 1.4.2. *Technically, this definition only works for*

$$\omega \in \Gamma(X, \Omega^i_X) \subseteq H^i_{dR}(X, \mathbb{Q});$$

in order to give a general definition, one would need to work with Čech cocycles in hypercohomology.

Theorem 1.4.3. *(Grothendieck's de Rham theorem)*
The linear map

$$H^i_{dR}(X/\mathbb{C}) \to H^i_B(X/\mathbb{C})$$

defined by the integration pairing is an isomorphism of vector spaces.

Remark 1.4.4. *So far we have not used the hypothesis that X is defined over $\mathbb{Q}$: Theorem 1.4.3 holds for all varieties over $\mathbb{C}$.*

1.4.4. *Periods.*

Definition 1.4.5. *Let X be a smooth algebraic variety over $\mathbb{Q}$. A* period *on X is a complex number in the image of the $\mathbb{Q}$-bilinear pairing*

$$H^i_B(X/\mathbb{Q}) \times H^i_{dR}(X/\mathbb{Q}) \to \mathbb{C}.$$

Notice that this makes essential use of the fact that X is a variety over $\mathbb{Q}$! The rational structure on $H^i_B(X/\mathbb{Q})$ comes from the topology of X; the rational structure on $H^i_{dR}(X/\mathbb{Q})$ comes from considering algebraic differentials with rational coefficients.

Example 1.4.6. *Let $X = \mathbb{P}^1 - \{0, \infty\}$. Then $H^B_1(X, \mathbb{Q})$ is one-dimensional, generated by a counterclockwise loop γ around the origin; $H^1_{dR}(X/\mathbb{Q})$ is also one-dimensional, generated by*

$$\omega = \frac{dx}{x}.$$

Integration gives

$$\int_\gamma \omega = 2\pi i,$$

and hence $2\pi i$ is a period on X.

Example 1.4.7. *Let X be an elliptic curve $y^2 = x^3 + ax + b$ in Weierstrass form, and choose some $\gamma \in H^B_1(X, \mathbb{Q})$. Let*

$$\omega = \frac{dx}{y}.$$

Then

$$\int_\gamma \omega$$

is a period on X.

1.4.5. *Historical note: why do we call them "periods"?* Let's go back to the naive perspective of single-variable calculus, and imagine trying to integrate algebraic functions. For example, consider

$$\int \frac{dx}{\sqrt{1-x^2}}.$$

(Secretly, we know this integral is $\sin^{-1}(x)$.) We may as well work on the complex plane; then the integrand is a "holomorphic two-valued function" for $x \neq \pm 1$. In other words, the integrand is defined on a double cover X of $\mathbb{P}^1 - \{-1, 1\}$. If we try to define a global antiderivative of $\frac{1}{\sqrt{1-x^2}}$, we will run into trouble: going around a loop in $\pi_1(X)$ has the effect of translating our antiderivative by 2π.

In other words, the integral $\int \frac{dx}{\sqrt{1-x^2}}$ is the inverse of the periodic function $\sin x$. The fact that $\sin x$ is periodic (rather than one-to-one) implies that the integral is only defined modulo addition of $2\pi\mathbb{Z}$.

A similar story comes up for

$$\int \frac{dx}{\sqrt{x^3-x}}:$$

the inverse function of the indefinite integral is now a *doubly* periodic function on $\mathbb{C}$, known as an *elliptic function*; the two periods are precisely the integrals of the differential $\frac{dx}{y}$ on the elliptic curve

$$y^2 = x^3 - x.$$

1.5. **Multiple zeta values and periods.** We have seen that multiple zeta values can be interpreted as certain definite integrals of algebraic functions. In fact, all multiple zeta values arise as periods.

Theorem 1.5.1. *Every $\zeta(k_1, k_2, \ldots, k_r)$ arises as a period on a variety X defined over $\mathbb{Q}$.*

For a proof, see [6, SS3.1-3.3] and [4, §2.1]. The key technical difficulty is to reinterpret an iterated integral as a period coming from classes in cohomology. Note that an n-fold iterated integral on $\mathbb{P}^1$ can be seen as an integral over an n-dimensional simplex in $(\mathbb{P}^1)^n$; the idea is to take $X = (\mathbb{P}^1)^n - Z$, where Z is the union of hyperplanes in $(\mathbb{P}^1)^n$ defining the faces of this simplex.

2. Motives

2.1. **Motivation for motives.** We want to pass from the category of algebraic varieties to a larger category of *motives*. The motivating ideas behind [5] are as follows.

- A *motive* should be a "cohomological piece" of a variety. Each cohomology theory (Betti, de Rham, étale, etc.) should define functors H^i from the category of motives to ($\mathbb{Q}$-vector spaces, Hodge structures, Galois representations, etc.). Every motive should "come from geometry": the cohomology $H^i(X)$ of any motive X should appear as a subquotient of the cohomology of some bona fide algebraic variety.
- If X is an algebraic variety (or at least in the special case $X = \mathbb{P}^1 - \{0, 1, \infty\}$), then the prounipotent completion $\pi_1(X)^{un}$ of $\pi_1(X)$ should have the structure of a motive. (We will discuss this "prounipotent completion" at length in coming weeks.)
- One can define periods of a motive by comparing its Betti and de Rham cohomology. Periods of the motive $\pi_1(X)^{un}$ are multiple zeta values.

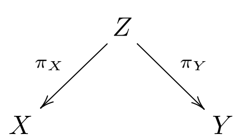
2.2. **Attempts at a definition.** The category of motives should be an abelian category, admitting the category of varieties as a full subcategory. We begin with the question of morphisms.

2.2.1. *Correspondences.*

Definition 2.2.1. *Let X and Y be finite-type schemes over a field K. A* correspondence *from X to Y is a subscheme Z of $X \times Y$ such that $Z \to X$ is finite and surjective onto at least one component of X.*

(Some authors omit the "finite and surjective" condition, or impose other conditions.)

A correspondence from X to Y gives rise to a morphism on the level of cohomology: if π_X and π_Y are the projections



then $\pi_{X*}\pi_Y^*$ defines a map from the cohomology of Y to the cohomology of X.

We can regard a correspondence as a sort of multivalued generalization of the notion of morphism from X to Y. Indeed, $Z \to X$ is finite of degree 1 (i.e. an isomorphism) precisely when it is the graph of a morphism $X \to Y$; in this case, the action on cohomology is simply the pullback.

2.2.2. *Morphisms in the category of motives?* We want our category of motives to have the following properties.

- The category should be a Tannakian category over $\mathbb{Q}$. In particular, it should be an abelian category: kernels and cokernels should exist.
- The set of morphisms $\mathrm{Hom}(X, Y)$ should be a finite-dimensional $\mathbb{Q}$-vector space, for any two motives X and Y.
- For any two varieties X and Y, every correspondence from X to Y should give rise to a morphism from X to Y.

- For each cohomology theory (Betti, de Rham, etc.), "taking cohomology" should be a well-defined functor on motives: $H^i(X)$ should be defined for any motive X.

The problem is that it's not clear how to define morphisms to make this happen. Roughly speaking, the Hom-set $\mathrm{Hom}(X, Y)$ should be a quotient of the set of correspondences from X to Y: if two correspondences Z_1 and Z_2 give rise to the same map $H^*(Y) \to H^*(X)$, they should correspond to the same morphism in the category of motives. But we don't know that the condition "Z_1 and Z_2 give rise to the same map on cohomology" is independent of the cohomology theory! This would follow from the "standard conjectures" (e.g. the Hodge and Tate conjectures), but as long as those conjectures remain unknown, it seems to be difficult to give an unconditional definition of the category of motives. For a detailed introduction to this whole mess, see [1, Part I].

Voevodsky [10] defined a category of motives by a derived-category approach. Voevodsky first defines a triangulated category, which is supposed to be the derived category; the abelian category of motives is then constructed as the core of a certain t-structure on this triangulated category. We will not discuss Voevodsky's approach further.

2.3. **Deligne's non-definition: systems of realizations.** Deligne makes an adhoc definition of motive: a motive is nothing more than a package of cohomological data coming from geometry. We will sketch the definition here; for the details, see [5, 1.11].

Definition 2.3.1. *A system of realizations (for a motive over $\mathbb{Q}$) consists of the following objects:*

- *A $\mathbb{Q}$-vector space M_B (the "Betti realization"), with a "weight" filtration;*
- *A $\mathbb{Q}$-vector space M_{dR} (the "de Rham realization"), with "weight" and "Hodge" filtrations;*
- *For each prime number ℓ, a $\mathbb{Q}_\ell$-vector space M_ℓ (the "étale realization"), with a "weight" filtration and an action of the Galois group $\mathrm{Gal}_{\mathbb{Q}}$; and*
- *A $\mathbb{Q}$-Hodge structure M_H (the "Hodge realization");*

equipped with various "standard comparison maps" that satisfy various compatibility axioms.

Remark 2.3.2. *In some sense, the precise details of the definition are unimportant. Deligne chose a list of comparison maps and compatibility axioms in order that:*

(1) *the axioms be provably satisfied for cohomology of a variety, and*
(2) *the category of systems of realizations be a Tannakian category.*

Definition 2.3.3. *A* motive *is a system of realizations "coming from geometry".*

Remark 2.3.4. *Deligne makes no attempt to define the phrase "coming from geometry". Of course, a typical example of a system of realizations coming from geometry is the cohomology $H^i(X)$ of an algebraic variety, for each of the various cohomology theories: $(H^i_B(X), H^i_{dR}(X), H^i_{et}(X, \mathbb{Q}_\ell), H^i_H(X))$.*

The whole paper [5] is a heuristic argument that one should allow other "geometric" constructions as well. In particular, π_1 of an algebraic variety (or more precisely its prounipotent completion) should be regarded as a motive, even though (at least at first glance) it is not the cohomology of any algebraic variety.

However, see [6, SS3.1-3.3]*, where the prounipotent completion of* $\pi_1(X)$ *is related to the cohomology of certain open subvarieties of powers of* X*.*

2.4. **Motives and periods.** Since periods on a variety (Definition 1.4.5) depend only on the (Betti and de Rham) cohomology of the variety, we can extend the notion of period to arbitrary motives.

Definition 2.4.1. *On any motive* X*, with Betti realization* M_B *and de Rham realization* M_{dR}*, the Betti-de Rham comparison isomorphism*

$$M_B \otimes_{\mathbb{Q}} \mathbb{C} \to M_{dR} \otimes_{\mathbb{Q}} \mathbb{C}$$

gives rise to a $\mathbb{Q}$*-linear pairing*

$$M_B \times M_{dR}^{\vee} \to \mathbb{C}.$$

An element of the image of this pairing is called a period *on the motive* X*.*

Note that periods form a $\mathbb{Q}$-algebra. The sum of two periods, on motives X and Y, is a period on the disjoint union $X \sqcup Y$; their product is a period on the product $X \times Y$.

Conjecture 2.4.2. *All algebraic relations between periods are "explained by geometry".*

By the above remarks, this conjecture is equivalent to the following statement: on any motive X, the $\mathbb{Q}$-linear pairing

$$M_{dR} \otimes_{\mathbb{Q}} M_B^{\vee} \to \mathbb{C}$$

is injective.

Remark 2.4.3. *Conjecture 2.4.2 appears to be far out of reach. Since* π *is a period (Section 1.4.5), it implies the transcendence of* π *as a first special case. Some modest partial results have been proven; for a survey of the problem, see* [3]*.*

According to Theorem 1.5.1, all multiple zeta values are periods on motives. Applying Conjecture 2.4.2 to the multiple zeta values, we expect all the many relations among them to be "explained by geometry". More precisely, Brown [4] defines a ring $\mathcal{H}^{\mathcal{MT}+}$ of "motivic zeta values"; this ring comes with an "evaluation" map

$$\mathcal{H}^{\mathcal{MT}+} \to \mathbb{C}.$$

Conjecture 2.4.2 then implies that this evaluation map is injective.

Thus, Conjecture 2.4.2 implies that Theorems 1.1.1 ($\zeta(2) = \pi^2/6$) and 1.1.3 ($\zeta(3) = \zeta(2,1)$) admit motivic proofs. In Section **??** we will give a motivic proof of Theorem 1.1.1.

Remark 2.4.4. *(The reader can safely skip this: we will not use it again.)*

In fact the motives that give rise to multiple zeta values are of a special form: they are all iterated extensions of Tate motives. Let $\mathbb{Q}(1)$ *be the motive* $H^1(\mathbb{P}^1 - \{0, \infty\})$ *(one has to check that this is a well-defined motive!). This object has one-dimensional realization for each of the cohomology theories; its étale representation is the cyclotomic Galois representation* $\mathbb{Q}_\ell(1)$*. A* Tate motive *is a tensor power* $\mathbb{Q}(n) = \mathbb{Q}(1)^{\otimes n}$ *or its dual* $\mathbb{Q}(-n) = \mathbb{Q}(n)^{\vee}$*. An* iterated extension of Tate motives *is a motive* X *admitting a filtration*

$$0 = F_0 \subseteq F_1 \subseteq \cdots \subseteq F_n = X$$

such that each quotient F_i/F_{i-1} *is a Tate motive* $\mathbb{Q}(k)$ *(for some* k *depending on* i*).*

When we discuss unipotent π_1*, we will see that it has a similar structure of iterated Tate motive.*

3. Unipotent π_1

In later talks, we will discuss the prounipotent completion π_1^{un} of $\pi_1(\mathbb{P}^1 - \{0, 1, \infty\})$ in some detail. As we have mentioned, Deligne's paper [5] argues that this π_1^{un} should be regarded as a motive. I will give a few possible interpretations of this statement. (My understanding is that these "possible interpretations" are all equivalent. We will discuss this in more detail in Asvin's talk.)

Let $X = \mathbb{P}^1 - \{0, 1, \infty\}$. Considering the complex analytification of X as a topological space (the plane with two points removed), we obtain an abstract group $\pi_1(X)$. It is free on two generators.

Definition 3.1. *Let G be an abstract group. (We are only interested in the case $G = \pi_1(X)$.) A representation of G on a vector space V is said to be* unipotent *if either of the following equivalent conditions is satisfied:*

1. *There is a basis for V with respect to which every element of G is upper-triangular, with 1's on the diagonal.*
2. *There is a filtration*

$$0 = W_0 \subseteq W_1 \subseteq \cdots \subseteq W_n = V$$

of V by G-stable subspaces, such that the action of G on each W_i/W_{i-1} is trivial.

The "prounipotent completion" G^{un} of any group G is a proalgebraic group. Its defining property is that there is a map $G \to G^{un}(\mathbb{Q})$, such that every unipotent representation of G factors through $G^{un}(\mathbb{Q})$. (See [5, §9] for discussion of the prounipotent completion.)

The prounipotent completion is closely related to the lower central series of G. Let $G^{[0]} = G$, and define $G^{[n]}$ inductively as the commutator subgroup

$$G^{[n+1]} = (G, G^{[n]}).$$

Hence, $G/G^{[n+1]}$ is the largest quotient of G for which the image of $G^{[n]}$ is central. Then any representation of $G/G^{[n]}$ is unipotent, and in fact [5, §9.8] G^{un} can be obtained as a sort of "algebraization" of the completion $\lim_{\leftarrow} G/G^{[n]}$.

With this preamble, here are several formulations of the claim that "π_1^{un} is a motive". Let

$$\pi_1 = \pi_1(\mathbb{P}^1 - \{0, 1, \infty\}).$$

1. Let $\Gamma^n = \pi_1/\pi_1^{[n]}$ be the largest n-step nilpotent quotient of π_1 (that is, a quotient of π_1 by the n-th group in its lower central series). One can define (see [5, §9]) the Lie algebra of such a group; a priori, $\operatorname{Lie}\Gamma^n$ is a vector space over $\mathbb{Q}$.

   Then for each n, $\operatorname{Lie}\Gamma^n$ is the Betti realization of a motive.
2. The prounipotent completion π_1^{un} is a pro-algebraic group; its affine Hopf algebra (ring of functions) can be expressed as an inductive limit $\lim_{\to} R_n$.

   Each R_n is the Betti realization of a motive.
3. Fix a basepoint $o \in X$.

   For every $x \in X$, the set of homotopy classes of path from o to x form a torsor under $\pi_1 = \pi_1(X, o)$. Algebraizing, this gives a torsor $P^n_{o,x}$ under $\Gamma^{n,alg}$. As it is a torsor for an algebraic group, we can write $P^n_{o,x} = \operatorname{Spec} A$.

   Then A is the Betti realization of a motive.

(4) The torsors $P^n_{o,x}$, as x varies over X, glue together to give a torsor P^n_o over X.

This torsor is the Betti realization of a motive over X.

(5) Every unipotent representation of $\pi_1(X)$ is the Betti realization of a motive over X.

In particular, every unipotent representation of $\pi_1(X)$ underlies a variation of mixed Hodge structure on X.

Remark 3.2. *I am not sure about the last two items.*

We have not defined motives over X or variations of mixed Hodge structure. We will define them as needed.

4. A motivic proof of an identity involving periods

Conjecture 2.4.2, predicts that there should be some "geometric" or "motivic" reason for any relation among multiple zeta values. In other words, identities like

$$\zeta(2) = \frac{\pi^2}{6}$$

should admit proofs based on the interpretation of each side as a period of a motive. In this section we will give such a proof, in the hopes that it will clarify the meaning of Conjecture 2.4.2.

Concretely, a period of a motive is the integral of an algebraic function over a topological cycle. The action of correspondences on cohomology is just a fancy word for integration by substitution, familiar to any calculus student.

In Section 1.3, we interpreted multiple zeta values as iterated integrals. But of course any iterated integral can be understood as a multiple integral on a higher-dimensional variety. For example,

$$\zeta(2) = \int_0^1 \left(\int_0^{t_2} \frac{dt_2}{1-t_2} \right) \frac{dt_1}{t_1} = \iint \frac{dt_1\, dt_2}{t_1(1-t_2)},$$

the double integral being taken over a triangle in the (real) (t_1, t_2)-plane.

Unfortunately I do not know a proof of Theorem 1.1.1 using this particular integral representation. Instead, I will give a proof (due to Apostol [2]) using the double integral relation below.

Theorem 4.1. *We have*

$$\zeta(2) = \frac{\pi^2}{6},$$

where

$$\zeta(2) = \iint \frac{dx\, dy}{1-xy},$$

the integral being taken over the unit square.

Proof. (That $\zeta(2) = \iint \frac{dx\,dy}{1-xy}$ can be verified by expanding $\frac{1}{1-xy}$ as a geometric series.)

First we rotated by 45 degrees; that is, we perform the substitution

$$x = u + v, \qquad y = u - v,$$

which transforms the integral (up to a factor of 2) into

$$\iint \frac{du\, dv}{1 - u^2 + v^2},$$

the integral being taken over the rotated square

$$0 \le u \le 1, \qquad -u \le v \le u, \qquad -(1-u) \le v \le 1-u.$$

Next we perform the substitution (if this feels unmotivated, see Remark 4.2 below)

$$(u, v) = (\cos 2\theta, \sin 2\theta \tan \phi).$$

The bounds of integration become

$$0 \le \theta \le \pi/4, \qquad -(\pi/2 - \theta) \le \phi \le \pi/2 - \theta, \qquad -\phi \le v \le 1-u;$$

here the last inequality follows from the half-angle formula

$$\tan\theta = \frac{1-\cos 2\theta}{\sin 2\theta}.$$

We compute

$$du \wedge dv = 2\sin^2 2\theta \sec^2\phi \, d\theta \wedge d\phi$$

so

$$\frac{du \wedge dv}{1-u^2+v+2} = 2\, d\theta \wedge d\phi.$$

Hence, we are reduced to the straightforward problem of integrating $d\theta \wedge d\phi$ over a quadrilateral in the (θ, ϕ)-plane; the result follows. □

Remark 4.2. *If the substitution*

$$(u, v) = (\cos 2\theta, \sin 2\theta \tan\phi)$$

feels unnatural, imagine integrating

$$\iint \frac{du\, dv}{1-u^2+v^2}$$

by standard techniques of elementary calculus. Integrating in the v-direction first, one is led to the trigonometric substitution

$$v = \sqrt{1-u^2}\tan\phi.$$

Given the appearance of $\sqrt{1-u^2}$, it is natural to take

$$u = \cos\psi$$

for some ψ. Computing the boundary of the region in terms of ϕ and ψ, one notices the key half-angle formula; for this reason we wrote the integral in terms of $\theta = \psi/2$.

Remark 4.3. *Trigonometric functions may appear to be non-algebraic, but in fact it is easy to rewrite any integral involving trigonometric functions as an integral of a rational function, by means of $s = e^{i\theta}$.*

If we take

$$(s, t) = (e^{i\theta}, e^{i\phi}),$$

the substitution above becomes

$$(u, v) = \left(\frac{s^2+s^{-2}}{2}, \frac{s^2-s^{-2}}{2} \cdot \frac{t-t^{-1}}{t+t^{-1}}\right),$$

and the integral

$$\iint \frac{ds\, dt}{st}$$

is evaluated over some subset of the torus

$$|s| = |t| = 1.$$

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