Perron-Frobenius in classical and max linear algebra: comparison and analysis

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NOTATION

Classical:

$$\boldsymbol{a}, \boldsymbol{b} \in \mathbb{R}_+$$

$$\boldsymbol{a} + \boldsymbol{b}, \ \boldsymbol{ab}$$

Max:

$$\boldsymbol{a} \oplus \boldsymbol{b} = \max(\boldsymbol{a}, \boldsymbol{b})$$

$$\boldsymbol{a} \otimes \boldsymbol{b} = \boldsymbol{ab}$$

Both:

$$\boldsymbol{a} \dagger \boldsymbol{b}, \quad \boldsymbol{a} * \boldsymbol{b}$$

$\mathbb{R}_+$ is a semiring under both ops

Commutative semigroups with identity under $+$, $\oplus$
Distributivity

DIFFERENCE

$$a + b = a \Longrightarrow b = 0$$

$$a \oplus a = a$$

$$\boldsymbol{A}, \boldsymbol{B} \in \mathbb{R}_{+}^{\boldsymbol{m} \times \boldsymbol{n}}$$

$$\boldsymbol{A} + \boldsymbol{B}, \boldsymbol{A}\boldsymbol{B},$$

$$\boldsymbol{A} \oplus \boldsymbol{B} = \max(\boldsymbol{A}, \boldsymbol{B})$$

$$\boldsymbol{C} = \boldsymbol{A} \otimes \boldsymbol{B}$$

$$\boldsymbol{c}_{\boldsymbol{ij}} = \max_{\boldsymbol{k}} \boldsymbol{a}_{\boldsymbol{ik}} \boldsymbol{b}_{\boldsymbol{kj}}$$

SIMILARITIES

Linearity

$$A * (\alpha x) = \alpha(A * x)$$

$$A * (x \dagger y) = A * x \dagger A * y$$

Monotonicity

$$x \leq y \Longrightarrow A * x \leq A * y$$

Difference: SEPARATION

$$A > 0, x \lneq y \Longrightarrow Ax < Ay$$

$$A > 0, x \lneq y \not\Longrightarrow A \otimes x < A \otimes y$$

Classic

$$\begin{bmatrix} 2 & 1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 4 \\ 4 \end{bmatrix}$$

Max

$$\begin{bmatrix} \mathbf{2} & \mathbf{1} \\ \mathbf{2} & \mathbf{1} \end{bmatrix} \otimes \begin{bmatrix} \mathbf{1} \\ \mathbf{1} \end{bmatrix} = \begin{bmatrix} \mathbf{2} \\ \mathbf{2} \end{bmatrix}$$

$$\begin{bmatrix} \mathbf{2} & \mathbf{1} \\ \mathbf{2} & \mathbf{1} \end{bmatrix} \otimes \begin{bmatrix} \mathbf{1} \\ \mathbf{2} \end{bmatrix} = \begin{bmatrix} \mathbf{2} \\ \mathbf{2} \end{bmatrix}$$

$\boldsymbol{A}$ **irreducible**

Wielandt style proof of P-F works in both classic and max

yields unique evalue in both

$$\rho^\dagger(A), \quad \rho(A), \quad \rho^\circ(A)$$

separation implies unique evector in classic not nec unique in max

DEFN: A reducible:
for some permutation
matrix P, $A' = P^T A P$

$$\begin{pmatrix} A'_{11} & A'_{12} \\ 0 & A'_{22} \end{pmatrix}$$

A'_{11} and A'_{22} nontrivial

.

irreducible = not reducible
Assume $A \neq [0] \in \mathbb{R}^{11}$

Spectral radius

$$\boldsymbol{\rho}(\boldsymbol{A}) = \mathbf{max}\{|\boldsymbol{\lambda}|;\ \boldsymbol{\lambda} \in \mathbf{spec(A)}\}$$

CLASSICAL PERRON-FROBENIUS
Perron (1907, 1907)
Frobenius(1908,1909,1912)

Theorem: $A \geq 0$ irreducible

1. **$\rho(A)$ is an algebraically simple eigenvalue**
2. **Its (essentially) unique eigenvector u is positive.**
3. **u is the only nonnegative eigenvector for A**

P-F in MAX(For the max savvy)

Cunningham-Green (1962, 1979), Gondrian-Minoux (1977)

Theorem: $A \geq 0$ irreducible.

1. **The unique evalue ρ° is the max cycle mean of the graph of A.**
2. **For each crit cpt of this graph there is an ess unique positive evector for ρ°.**
3. **These evectors form a basis for the max cone of evecs**

EXAMPLES:

Classic

$$\begin{bmatrix} \mathbf{2} & \mathbf{1} \\ \mathbf{1} & \mathbf{2} \end{bmatrix} \begin{bmatrix} \mathbf{1} \\ \mathbf{1} \end{bmatrix} = \mathbf{3} \begin{bmatrix} \mathbf{1} \\ \mathbf{1} \end{bmatrix}$$

Max

$$\begin{bmatrix} \mathbf{2} & \mathbf{1} \\ \mathbf{1} & \mathbf{2} \end{bmatrix} \otimes \begin{bmatrix} \mathbf{2} & \mathbf{1} \\ \mathbf{1} & \mathbf{2} \end{bmatrix} = \mathbf{2} \begin{bmatrix} \mathbf{2} & \mathbf{1} \\ \mathbf{1} & \mathbf{2} \end{bmatrix}$$

.	CLASSIC	MAX
eval	**unique**	**unique**
evec	**unique**	**non-uq**

Wielandt (1950)
Analysis of his proof of P-F

$\boldsymbol{A \geq 0}$ **positive (irreducible)**

$$\boldsymbol{A \dagger x \leq \lambda x \Longrightarrow x > 0}$$

$$\boldsymbol{\rho^{\dagger}(A) := \min\{\lambda : \exists x > 0,\ A \dagger x \leq \lambda x\}}$$

$$\boldsymbol{\exists u > 0 : A \dagger u \leq \rho u}$$

$\boldsymbol{u}$ **extremal vec (subevector)**

LEMMA: $\rho^\dagger$ is the only possible evalue.

$$A * u \leq \rho^\dagger u, \;\; u > 0$$

$$A * v = \sigma v, \;\; v > 0$$

$$\rho^\dagger \leq \sigma$$

$$\gamma := \min\{\alpha : \;\; v \leq \alpha u\}$$

$$(\gamma = 1)$$

$$\sigma v = A * v \leq A * u \leq \rho^\dagger u$$

$$v \leq (\rho^\dagger / \sigma) u$$

$$\rho^\dagger \geq \sigma$$

$$\rho^\dagger = \sigma$$

Have faith:
$\boldsymbol{\rho}^\dagger$ is an eigenvalue

$$\begin{bmatrix} \mathbf{2} & \mathbf{1} \\ \mathbf{2} & \mathbf{1} \end{bmatrix} \otimes \begin{bmatrix} \mathbf{1} \\ \mathbf{1} \end{bmatrix} = \begin{bmatrix} \mathbf{2} \\ \mathbf{2} \end{bmatrix}$$

$$\begin{bmatrix} \mathbf{2} & \mathbf{1} \\ \mathbf{2} & \mathbf{1} \end{bmatrix} \otimes \begin{bmatrix} \mathbf{1} \\ \mathbf{2} \end{bmatrix} = \begin{bmatrix} \mathbf{2} \\ \mathbf{2} \end{bmatrix}$$

Not every extr vec is an evec

$$\rho(A) := \min\{\lambda : \exists x > 0,\ Ax \leq \lambda x\}$$

$$u > 0 : \ Au \leq \rho u$$

u extremal

THEOREM: $A > 0$

1. u extremal $\Longrightarrow$ u evector
2. eigenvector and eigenvalue unique

Proof:

1.

$$\boldsymbol{Au \lneq \rho u \Longrightarrow A(Au) < \rho Au}$$

separation!

$$\boldsymbol{\exists \sigma < \rho : A(Au) \leq \sigma Au}$$

$$\Longrightarrow\Longleftarrow$$

2.

$$\boldsymbol{Au = \rho v,\ Av = \sigma v, u \neq \lambda v}$$

$$\boldsymbol{\alpha =: \min\{\lambda : v \leq \lambda u\}}$$

$$\boldsymbol{(\alpha = 1)}$$

$$\boldsymbol{u \lneq v}$$

$$\boldsymbol{\sigma v = Av < Au = \rho u}$$

$$\boldsymbol{\sigma < \rho}$$

$$\Rightarrow\!\Leftarrow$$

$$\boldsymbol{u = v, \quad \rho = \sigma}$$

Extends to irred in class lin

$$A^\sharp = I + A + A^2 + \cdots + A^{n-1} > 0$$

$$A^\sharp A = A A^\sharp$$

$$Au = \lambda u \Longrightarrow A^\sharp u = \lambda^\sharp u$$

$$Au \lneq \rho u \Longrightarrow A A^\sharp u = A^\sharp A u < \rho A^\sharp u$$

Wielandt-Perron-Frobenius

Back to basics

SIMILARITY

$$\lambda^p \longrightarrow 0 \text{ if } \lambda < 1$$

$$\lambda^p = 1 \text{ if } \lambda = 1$$

$$\lambda^p \longrightarrow \infty \text{ if } \lambda > 1$$

DIFFERENCE:

$$1 \dagger \lambda \dagger \lambda^2 \dagger \cdots$$

cvges if $\lambda < 1$
dvges if $\lambda > 1$

dvges in class if $\lambda = 1$
cvges in max if $\lambda = 1$

$$A^p \to 0 \iff \rho^\dagger(A) < 1$$

$$I \dagger A \dagger A^2 \cdots \text{ cvges if } \rho^\dagger(A) < 1$$

$$I \dagger A \dagger A^2 \cdots \text{ dvges if } \rho^\dagger(A) > 1$$

$$I \oplus A \oplus A^2 \cdots \text{ cvges if } \rho^\circ(A) \leq 1$$

$$I + A + A^2 \cdots \text{dvges if} \rho(A) = 1$$

Z-matrix equations

A **irreducible**

Classical AND max

$$A * x \dagger b = \lambda x$$

$$(\lambda = 1)$$

$$A * x \dagger b = x$$

$$x = A(A*x\dagger b)\dagger b = A^2*x\dagger(I\dagger A)*b$$

$$x = A^k*x+(I\dagger A\dagger A^2\dagger\cdots)*b$$

and conversely if cvgce

Ostrowski (1937) Lemma:

$\boldsymbol{A} \in \mathbb{R}_{+}^{n\times n}$ **irreducible**

$\boldsymbol{\lambda} > 0$

.

$$\longrightarrow \quad \boldsymbol{A} * \boldsymbol{x} \dagger \boldsymbol{b} = \boldsymbol{\lambda x}$$

$\boldsymbol{\lambda} > \rho^{\dagger}(\boldsymbol{A});$ **unique soln** $\boldsymbol{x}^{0}$

$$\boldsymbol{x}^{0} = (\boldsymbol{I} \oplus \boldsymbol{A}/\boldsymbol{\lambda} \oplus (\boldsymbol{A}/\boldsymbol{\lambda})^{2} \oplus \cdots) * \boldsymbol{b}$$

$\boldsymbol{x}^{0} = 0$ **if** $\boldsymbol{b} = 0$

$\boldsymbol{x}^{0} > 0$ **if** $\boldsymbol{b} \gneq 0$

.

$$\boldsymbol{\lambda < \rho(A)}$$

no solution

$$\longrightarrow \quad A * x \dagger b = \rho^{\dagger} x$$

Classical:

soln iff $b = 0$

$$Au = \rho u$$

Max:

$$x = x^{0} + u$$

$$x^{0} = (I \oplus A/\rho^{\circ} \oplus (A/\rho^{\circ})^{2} \oplus \cdots) * b$$

$$A * u = \rho^{\circ} u$$

The operative difference when going from irreducible to reducible

Frobenius trace down method

$$A * x = \lambda x$$

$$A * x \dagger b = \lambda x$$

Frobenius (1912)
Determines all nonneg evectors
of a reducible nonneg matrix
Carlson (1963), Hershkowitz-S (1985)
Determine all solution of

$$Ax + b = \lambda x$$

Repeated application of the irreducible case

Precisely the same arguments work in max alg

Forces consideration of

$$A * x \dagger b = \lambda x$$

Sharp inequs become weak inequs

The difference lies in existence of sols of

$$\boldsymbol{Ax + b = \rho(A)x}$$

$$\boldsymbol{A \otimes x \oplus b = \rho^{\circ}(A)}$$

for irred $\boldsymbol{A}$