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NORTH-HOLLAND

On the Existence of Sequences and Matrices With Prescribed Partial Sums of Elements

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ABSTRACT

We prove necessary and sufficient conditions for the existence of sequences and matrices with elements in given intervals and with prescribed lower and upper bounds on the element sums corresponding to the sets of an orthogonal pair of partitions. We use these conditions to generalize known results on the existence of nonnegative matrices with a given zero pattern and prescribed row and column sums. We also generalize recently proven results on the existence of (a real or nonnegative) square matrix A with a given zero pattern and with prescribed row sums such that $A + A^T$ is

*The research of these authors was supported by their joint grant 90-00434 from the United States–Israel Binational Science Foundation, Jerusalem, Israel.
†The research of this author was supported in part by NSF grant DMS-9123318.

 0024-3795/97/$17.00

prescribed. We also introduce Hadamard adjustments, by means of which we generalize known results on the scaling of matrices with a given pattern to achieve prescribed row and column sums.

1. INTRODUCTION

In the last twenty years there have been about a hundred papers dealing with matrices with given sign pattern or zero pattern, e.g. [5], [6], [7], and [12]. Our paper generalizes some known results of this kind to the case of matrices with elements in given intervals.

A well-known result, due to Brualdi [1], gives necessary and sufficient conditions for the existence of a nonnegative rectangular matrix with a given zero pattern and with prescribed row and column sums. More recently, in Da Silva, Hershkowitz, and Schneider [2], we found necessary and sufficient conditions for the existence of a real or nonnegative square matrix A with a given zero pattern and with prescribed row sums and such that $A + A^T$ is prescribed. In both of these results one prescribes the element sums corresponding to two partitions of positions in the matrix (rows and columns in one case, rows and pairs of symmetrically located elements in the other), where a set of one partition intersects a set of the other partition in at most one element.

In this paper we generalize the results mentioned above in different directions. We look at sequences instead of matrices, and we consider pairs of general partitions with the above properties, which we call orthogonal partitions. We then obtain results where lower and upper bounds on the element sums corresponding to the sets of the partitions are prescribed, and we extend results from nonnegative patterns to the case that the elements of the sequence are in given intervals. Our applications are to the case of matrices. We also generalize classical results on the existence of diagonal scalings of a nonnegative matrix to achieve prescribed row and column sums (see Menon [8], Menon and Schneider [9], and many other references) to the case of general sign patterns in sequences.

We now describe our paper in more detail.

In Section 2 we prove our main results concerning the existence of sequences with elements in given intervals and with prescribed lower and upper bounds on the element sums corresponding to the sets of orthogonal pair of partitions. Our principal tool here comes from network flow theory, as we heavily use the Hoffman circulation theorem [3, p. 51]. This technique is similar to that used in [1].

In Section 3 we discuss sign pattern $m \times n$ matrices and matrices of given sign patterns. Application of the results of Section 2 to a sequence of mn numbers, arranged in an $m \times n$ matrix, with one partition being the rows of the matrix and the other partition being the columns of the matrix, yields new theorems, with known special cases where the sign pattern is nonnegative and all row and column sums are prescribed; see [1], [9], [10], and [11].

In Section 4 we generalize results of [2] on the existence of a square matrix B with given row sums and such that $B + B^T$ is prescribed to the case where only some of the row sums are given, and where B has a partially prescribed sign pattern. Our results are obtained by applying the results of Section 2 to a sequence of n^2 numbers, arranged in an $n \times n$ matrix, with one partition being the rows of the matrix and the other partition being the pairs of symmetrically located elements of the matrix. We discuss further applications to both the zero pattern case and the nonnegative sign pattern case, which contain known results in [2].

The discussion in Section 5 is independent of the previous section, and is motivated by the fact that the results of [10] and [11], some of which are generalized in Section 3, also contain a statement on the existence of row and column scalings that scale a nonnegative matrix A to have the same row and column sums as a matrix B having the same pattern as A. We introduce Hadamard adjustments, by means of which we generalize that scaling result to apply to our main results. In contrast to the previous sections, where known results on nonnegative sign patterns are derived as corollaries of our independently proven theorems, in this section we derive the generalizations using known scaling results on nonnegative sign patterns.

We remark that it would be possible first to prove results on the existence of matrices with bounds on the row and column sums and with elements in given intervals, and then to derive the results on sequences. However, the direct approach seems more natural, and applications such as in Section 4 are easier to derive.

2. MAIN RESULTS

NOTATION 2.1. Let n be a positive integer. We denote by $\langle n \rangle$ the set $\{1, \dots, n\}$.

DEFINITION 2.2. Let p be a positive integer, and let $\mathscr{S} = \{S_1, \dots, S_s\}$ and $\mathscr{T} = \{T_1, \dots, T_t\}$ be partitions of $\langle p \rangle$. Then $\mathscr{S}$ and $\mathscr{T}$ are said to be *orthogonal partitions* if

$$|S_i \cap T_j| \leqslant 1, \qquad i \in \langle s \rangle, \quad j \in \langle t \rangle.$$

NOTATION 2.3. Let A be a p-vector and let $k \in \langle p \rangle$. We denote by A_k the kth element of A.

DEFINITION 2.4. A vector P is said to be a (*close*) *interval vector* if every entry of P is a close interval in the real axis (for this purpose, intervals of type $(-\infty, a]$, $[b, \infty)$, and $(-\infty, \infty)$ are also considered close).

In the sequel we identify p-vectors and sequences of p elements.

DEFINITION 2.5. Let $P = \{[l_1, u_1], \dots, [l_p, u_p]\}$ be an interval p-vector.

(i) A sequence A of p real elements is said to be *in the interval* P if $l_k \leqslant A_k \leqslant u_k$, $k \in \langle p \rangle$.
(ii) A sequence A of p real elements is said to be *in the open interval* P if $l_k < A_k < u_k$ whenever $l_k < u_k$, $k \in \langle p \rangle$.

NOTATION 2.6. For subsets α of $\langle s \rangle$ and β of $\langle t \rangle$, we denote by α^C and β^C the set complements of α and β in $\langle s \rangle$ and $\langle t \rangle$ respectively.

We now state our main theorem concerning the existence of sequences with prescribed partial sums of elements.

THEOREM 2.7. *Let* $P = \{[l_1, u_1], \dots, [l_p, u_p]\}$ *be an interval* p-*vector, let* $\mathscr{S} = \{S_1, \dots, S_s\}$ *and* $\mathscr{T} = \{T_1, \dots, T_t\}$ *be orthogonal partitions of* $\langle p \rangle$, *and let* $r_i \leqslant R_i$, $i \subseteq \langle s \rangle$, *and* $c_j \leqslant C_j$, $j \subseteq \langle t \rangle$, *be real numbers (where* R_i *and* C_j *can be* ∞, *and* r_i *and* c_j *can be* $-\infty$). *The following are equivalent.*

(i) *There exists a sequence* A *in the interval* P *and such that*

$$r_i \leqslant \sum_{k \in S_i} A_k \leqslant R_i, \qquad i \in \langle s \rangle,$$

$$c_j \leqslant \sum_{k \in T_j} A_k \leqslant C_j, \qquad j \in \langle t \rangle.$$

(ii) *For any subsets α of $\langle s\rangle$ and β of $\langle t\rangle$ we have*

$$\min\left\{\sum_{i\in\alpha} R_i - \sum_{j\in\beta} c_j,\ \sum_{j\in\beta^c} C_j - \sum_{i\in\alpha^c} r_i\right\} \tag{2.8}$$

$$\geqslant \sum_{k\in\bigcup_{i\in\alpha}S_i\setminus\bigcup_{j\in\beta}T_j} l_k - \sum_{k\in\bigcup_{j\in\beta}T_j\setminus\bigcup_{i\in\alpha}S_i} u_k.$$

Proof. (i) $\Rightarrow$ (ii): Assume that $\alpha\subseteq\langle s\rangle$, $\beta\subseteq\langle t\rangle$. We have

$$\sum_{i\in\alpha} R_i - \sum_{j\in\beta} c_j \geqslant \sum_{k\in\bigcup_{i\in\alpha}S_i} A_k - \sum_{k\in\bigcup_{j\in\beta}T_j} A_k$$

$$= \sum_{k\in\bigcup_{i\in\alpha}S_i\setminus\bigcup_{j\in\beta}T_j} A_k - \sum_{k\in\bigcup_{j\in\beta}T_j\setminus\bigcup_{i\in\alpha}S_i} A_k.$$

Since A is in the interval P, we have $l_k \leqslant A_k \leqslant u_k$ and the first inequality of (2.8) follows. Similarly, we prove the second inequality of (2.8).

(ii) $\Rightarrow$ (i): By a technique similar to that used in [1], we associate a digraph D with the interval vector P which has the vertices

$$\{x_1,\ldots,x_s,y_1,\ldots,y_t,\xi,\psi\}.$$

There is an arc (x_i, y_j) from x_i to y_j if and only if $S_i\cap T_j\neq\varnothing$. There are also arcs (ξ, x_i) from ξ (the source) to each x_i and arcs (y_j,ψ) from y_j to ψ (the sink) for each y_j. Finally, there is an arc (ψ,ξ) from ψ to ξ. These are the only arcs in D. We assign an upper bound $c(v,w)$ and a lower bound $l(v,w)$ for the weights of the arcs (v,w) of D as follows:

$$\begin{aligned}
&c(x_i,y_j)=u_k, && l(x_i,y_j)=l_k, && \text{where } S_i\cap T_j=\{k\},\\
&c(\xi,x_i)=R_i, && l(\xi,x_i)=r_i, && i\in\langle s\rangle,\\
&c(y_j,\psi)=C_j, && l(y_j,\psi)=c_j, && j\in\langle t\rangle,\\
&c(\psi,\xi)=\infty, && l(\psi,\xi)=-\infty.
\end{aligned}\tag{2.9}$$

In order to use Hoffman's circulation theorem (e.g. [3, p. 51]), the condition

$$\sum_{\substack{(v,w)\in E(D)\\ v\notin\mathcal{N},\, w\in\mathcal{N}}} c(v,w) \geqslant \sum_{\substack{(v,w)\in E(D)\\ v\in\mathcal{N},\, w\notin\mathcal{N}}} l(v,w) \tag{2.10}$$

should be satisfied for every set $\mathcal{N}$ of vertices of D. Let $\alpha = \{i \in \langle s\rangle : x_i \in \mathcal{N}\}$ and $\beta = \{j \in \langle t\rangle : y_j \in \mathcal{N}\}$. Distinguish four cases:

(1) $\xi \in \mathcal{N}$, $\psi \notin \mathcal{N}$. Here the left hand side of (2.10) contains the term $c(\psi, \xi)$, and so it is equal to ∞ and (2.10) holds.

(2) $\xi \notin \mathcal{N}$, $\psi \in \mathcal{N}$. Here the right hand side of (2.10) contains the term $l(\psi, \xi)$, and so it is equal to $-\infty$ and (2.10) holds.

(3) $\xi, \psi \in \mathcal{N}$. Here we have

$$\begin{aligned}
&\{(v,w) \in E(D) : v \notin \mathcal{N},\, w \in \mathcal{N}\}\\
&\qquad = \{(x_i, y_j) \in E(D) : i \in \alpha^C,\, j \in \beta\} \cup \{(y_j, \psi) : j \in \beta^C\},\\
&\{(v,w) \in E(D) : v \in \mathcal{N},\, w \notin \mathcal{N}\}\\
&\qquad = \{(\xi, x_i) : i \in \alpha^C\} \cup \{(x_i, y_j) \in E(D) : i \in \alpha,\, j \in \beta^C\},
\end{aligned}$$

and so

$$\begin{aligned}
&\sum_{\substack{(v,w)\in E(D)\\ v\notin\mathcal{N},\, w\in\mathcal{N}}} c(v,w) - \sum_{\substack{(v,w)\in E(D)\\ v\in\mathcal{N},\, w\notin\mathcal{N}}} l(v,w)\\
&\qquad = \sum_{k \in \bigcup_{j\in\beta} T_j \setminus \bigcup_{i\in\alpha} S_i} u_k + \sum_{j\in\beta^c} C_j\\
&\qquad\quad - \sum_{k \in \bigcup_{i\in\alpha} S_i \setminus \bigcup_{j\in\beta} T_j} l_k - \sum_{i\in\alpha^c} r_i.
\end{aligned}$$

By (2.8), the right hand side of our equation is nonnegative, and (2.10) follows.

(4) $\xi, \psi \notin \mathcal{N}$. Here we have

$$\begin{aligned}
&\{(v,w) \in E(D) : v \notin \mathcal{N},\, w \in \mathcal{N}\}\\
&\qquad = \{(\xi, x_i) : i \in \alpha\} \cup \{(x_i, y_j) \in E(D) : i \in \alpha^C,\, j \in \beta\},\\
&\{(v,w) \in E(D) : v \in \mathcal{N},\, w \notin \mathcal{N}\}\\
&\qquad = \{(x_i, y_j) \in E(D) : i \in \alpha,\, j \in \beta^C\} \cup \{(y_j, \psi) : j \in \beta\},
\end{aligned}$$

and hence

$$\sum_{\substack{(v,w)\in E(D)\\ v\notin\mathcal{N},\, w\in\mathcal{N}}} c(v,w) - \sum_{\substack{(v,w)\in E(D)\\ v\in\mathcal{N},\, w\notin\mathcal{N}}} l(v,w)$$
$$= \sum_{k\in \bigcup_{j\in\beta}T_j\setminus\bigcup_{i\in\alpha}S_i} u_k + \sum_{i\in\alpha} R_i - \sum_{k\in\bigcup_{i\in\alpha}S_i\setminus\bigcup_{j\in\beta}T_j} l_k - \sum_{j\in\beta} c_j.$$

By (2.8), the right hand side of our equation is nonnegative, and (2.10) follows.

Therefore, (2.10) is satisfied for every set $\mathcal{N}$ of vertices of D, and by Hoffman's circulation theorem there exist weights (flows) $f(v,w)$ on the arcs (v,w) of D such that

$$l(u,v) \leqslant f(u,v) \leqslant c(u,v),$$

$$\text{(2.11)}\qquad \begin{aligned} &\sum_{\substack{j\in\langle t\rangle\\ (x_i,y_j)\in E(D)}} f(x_i,y_j) = f(\xi,x_i), \qquad i\in\langle s\rangle,\\ &\sum_{\substack{i\in\langle s\rangle\\ (x_i,y_j)\in E(D)}} f(x_i,y_j) = f(y_j,\psi), \qquad j\in\langle t\rangle. \end{aligned}$$

Let $k\in\langle p\rangle$. Since $\{S_1,\dots,S_s\}$ and $\{T_1,\dots,T_t\}$ are partitions of $\langle p\rangle$, it follows that k belongs to exactly one intersection $S_i\cap T_j$. Therefore, we can define a sequence A by $A_k = f(x_i, y_j)$, whenever $k\in S_i\cap T_j$. It now follows from (2.9) and (2.11) that (i) holds. ■

The existence of a sequence A in the open interval P is covered in the following theorem.

THEOREM 2.12. *Let $P = ([l_1,u_1],\dots,[l_p,u_p])$ be an interval p-vector, let $\mathcal{S}=\{S_1,\dots,S_s\}$ and $\mathcal{T}=\{T_1,\dots,T_t\}$ be orthogonal partitions of $\langle p\rangle$, and let $r_i \leqslant R_i$, $i\subseteq\langle s\rangle$, and $c_j\leqslant C_j$, $j\subseteq\langle t\rangle$, be real numbers (where R_i and C_j can be ∞ and r_i and c_j can be $-\infty$). The following are equivalent.*

(i) *There exists a sequence A in the open interval P and such that*

$$r_i \leqslant \sum_{k\in S_i} A_k \leqslant R_i, \qquad i\in\langle s\rangle,$$

$$c_j \leqslant \sum_{k\in T_j} A_k \leqslant C_j, \qquad j\in\langle t\rangle.$$

(ii) *For any two subsets α of $\langle s\rangle$ and β of $\langle t\rangle$ we have*

$$\min\left\{\sum_{i\in\alpha} R_i - \sum_{j\in\beta} c_j,\ \sum_{j\in\beta^c} C_j - \sum_{i\in\alpha^c} r_i\right\} \tag{2.13}$$

$$\geqslant \sum_{k\in\bigcup_{i\in\alpha}S_i\setminus\bigcup_{j\in\beta}T_j} l_k - \sum_{k\in\bigcup_{j\in\beta}T_j\setminus\bigcup_{i\in\alpha}S_i} u_k,$$

where strict inequality holds whenever

$$\sum_{k\in\bigcup_{i\in\alpha}S_i\setminus\bigcup_{j\in\beta}T_j} (u_k - l_k) - \sum_{k\in\bigcup_{j\in\beta}T_j\setminus\bigcup_{i\in\alpha}S_i} (u_k - l_k) > 0.$$

Proof. The proof is very similar to the proof of Theorem 2.7, with (2.9) replaced by

$$c(x_i, y_j) = u_k, \qquad l(x_i, y_j) = l_k, \qquad \text{where} \quad S_i \cap T_j = \{k\},\ u_k = l_k,$$

$$c(x_i, y_j) = u_k - \epsilon, l(x_i, y_j) = l_k + \epsilon, \qquad \text{where} \quad S_i \cap T_j = \{k\},\ u_k > l_k,$$

$$c(\xi, x_i) = R_i, \qquad l(\xi, x_i) = r_i, \qquad i \in \langle s\rangle,$$

$$c(y_j, \psi) = C_j, \qquad l(y_j, \psi) = c_j, \qquad j \in \langle t\rangle,$$

$$c(\psi, \xi) = \infty, \qquad l(\psi, \xi) = -\infty,$$

where ϵ is an unspecified positive number. Later on ϵ is chosen sufficiently small so that (2.13) is satisfied. ■

We now apply our results to sign patterns.

DEFINITION 2.14.

(i) An interval vector P is said to be a *sign pattern vector* if every entry of P is an element of the set $\{[0,\infty), (-\infty, 0], [0,0], (-\infty,\infty)\}$. We say that an element P_k of P satisfies $P_k \geqslant 0$ if $P_k = [0,\infty)$ or $P_k = [0,0]$. We say that $P_k \leqslant 0$ if $P_k = (-\infty, 0]$ or $P_k = [0,0]$. We say that $P_k > 0$ if $P_k = [0,\infty)$. We say that $P_k < 0$ if $P_k = (-\infty, 0]$.

(ii) A sequence in the interval P is said to be *of weak sign pattern P*. A sequence in the open interval P is said to be *of strong sign pattern P*.

(iii) An interval vector P is said to be a *zero pattern vector* if every entry of P is either $[0,0]$ or $(-\infty,\infty)$.

THEOREM 2.15. *Let P be a sign pattern p-vector, let $\mathcal{S} = \{S_1, \dots, S_s\}$ and $\mathcal{T} = \{T_1, \dots, T_t\}$ be orthogonal partitions of $\langle p \rangle$, and let $r_i \leqslant R_i$, $i \subseteq \langle s \rangle$, and $c_j \leqslant C_j$, $j \subseteq \langle t \rangle$, be real numbers (where R_i and C_j can be ∞ and r_i and c_j can be $-\infty$). The following are equivalent.*

(i) *There exists a sequence A with weak sign pattern P and such that*

$$r_i \leqslant \sum_{k \in S_i} A_k \leqslant R_i, \qquad i \in \langle s \rangle,$$

$$c_j \leqslant \sum_{k \in T_j} A_k \leqslant C_j, \qquad j \in \langle t \rangle.$$

(ii) *For every subsets α of $\langle s \rangle$ and β of $\langle t \rangle$ such that*

$$P_k \geqslant 0 \text{ whenever } k \in \bigcup_{i \in \alpha} S_i \setminus \bigcup_{j \in \beta} T_j \quad \text{and} \tag{2.16}$$

$$P_k \leqslant 0 \text{ whenever } k \in \bigcup_{j \in \beta} T_j \setminus \bigcup_{i \in \alpha} S_i$$

we have

$$\sum_{i \in \alpha} R_i \geqslant \sum_{j \in \beta} c_j, \tag{2.17}$$

$$\sum_{i \in \alpha^c} r_i \leqslant \sum_{j \in \beta^c} C_j.$$

Proof. Note that if (2.16) does not hold, then the right hand side of (2.8) is equal to $-\infty$ and hence (2.8) trivially holds. If (2.16) holds, then the right hand side of (2.8) is equal to 0 and thus our theorem follows from Theorem 2.7. ∎

REMARK 2.18. A similar application of Theorem 2.12 yields a new result asserting the equivalence of two statements similar to those in Theorem 2.15, where in statement (i) "weak sign pattern" is replaced by "strong sign pattern" and in statement (ii) the following condition is added: *Strict inequalities hold in* (2.17) *whenever $P_k > 0$ for some $k \in \bigcup_{i \in \alpha} S_i \setminus \bigcup_{j \in \beta} T_j$ or $P_k < 0$ for some $k \in \bigcup_{j \in \beta} T_j \setminus \bigcup_{i \in \alpha} S_i$.*

By considering the case where $R_i = r_i$, $i \in \langle s \rangle$, and $C_j = c_j$, $j \in \langle t \rangle$, one can obtain a corollary of Theorem 2.15 in which one prescribes all the element sums corresponding to the sets S_i and all the element sums corresponding to the sets T_j. In those corollaries, there is another equivalent condition, in which the equality

$$\sum_{i=1}^{p} r_i = \sum_{j=1}^{q} c_i$$

replaces either inequality of (2.17).

REMARK 2.19. Note that the proof in [3] for the Hoffman circulation theorem holds for flows with values in ordered abelian groups. Therefore, one can apply a generalized version of the circulation theorem to prove Theorems 2.7 and 2.15, as well as some results of the following section for matrices over ordered abelian groups; see a similar remark in Gale [4]. In particular, when the r_i's and the c_j's are integers, then the matrices whose existence is asserted in our results can be chosen to have integer elements.

3. APPLICATIONS TO MATRICES WITH PRESCRIBED ROW AND COLUMN SUMS

In this section we discuss sign pattern $m \times n$ matrices and matrices of given sign patterns, where the definitions of these correspond to Definitions 2.4 and 2.5 with vectors and sequences replaced by matrices.

NOTATION 3.1. Let A be an $m \times n$ matrix, and let $\alpha \subseteq \langle m \rangle$ and $\beta \subseteq \langle n \rangle$. We denote by $A(\alpha, \beta)$ the submatrix of A whose rows are indexed by α and whose columns are indexed by β. By convention we say that $A(\alpha, \beta) = 0$ whenever $\alpha = \varnothing$ or $\beta = \varnothing$.

Application of Theorem 2.15 to a sequence of mn numbers, arranged in an $m \times n$ matrix, with $S_1, \dots, S_m$ being the rows of the matrix and $T_1, \dots, T_n$ being the columns of the matrix, yields the following new theorem. Here we prescribe some of the row sums and some of the column sums of the matrix. Obviously, if the ith row sum is prescribed, then we have $r_i = R_i$. Otherwise, we have $r_i = -\infty$ and $R_i = \infty$. A similar comment holds for columns.

THEOREM 3.2. *Let P be a sign pattern $m \times n$ matrix and let r_i, $i \in R \subseteq \langle m \rangle$, and c_j, $j \in C \subseteq \langle n \rangle$, be real numbers. The following are equivalent.*

(i) *There exists an $m \times n$ matrix A with weak sign pattern P and such that*

$$\sum_{j=1}^{n} a_{ij} = r_i, \qquad i \in R,$$

$$\sum_{i=1}^{m} a_{ij} = c_j, \qquad j \in C.$$

(ii) *For any two subsets α of $\langle m \rangle$ and β of $\langle n \rangle$ such that $P(\alpha, \beta^C) \geqslant 0$ and $P(\alpha^C, \beta) \leqslant 0$ we have*

$$\text{(3.3)} \qquad \begin{aligned} &\sum_{i \in \alpha} r_i \geqslant \sum_{j \in \beta} c_j \qquad \text{whenever} \quad \alpha \subseteq R, \quad \beta \subseteq C, \\ &\sum_{i \in \alpha^c} r_i \leqslant \sum_{j \in \beta^c} c_j \qquad \text{whenever} \quad \alpha^C \subseteq R, \quad \beta^C \subseteq C. \end{aligned}$$

It is easy to verify that in the special case that $R = \langle m \rangle$ and $C = \langle n \rangle$, that is, when all row and column sums are prescribed, either of the conditions in (3.3) may be replaced by

$$\text{(3.4)} \qquad \sum_{i=1}^{m} r_i = \sum_{j=1}^{n} c_j.$$

REMARK 3.5. In view of Remark 2.18, we also have a new result asserting the equivalence of two statements similar to those in Theorem 3.2, where in statement (i) "weak sign pattern" is replaced by "strong sign pattern" and in statement (ii) the following condition is added: *Strict inequalities hold in* (3.3) *whenever $P(\alpha, \beta^C) \neq 0$ or $P(\alpha^C, \beta) \neq 0$.*

The special cases of Theorem 3.2 and Remark 3.5 where P is a nonnegative sign pattern and all row and column sums are prescribed are known. In the case of Theorem 3.2 it is a part of Theorem 3 in [10] and of Theorem 3 in [11], where, however, the condition (3.4) should be added. In the case of Remark 3.5 the result is Theorem 2.1 in [1] under the hypothesis that the sign pattern matrix is chainable; see also Corollary 4.2 in [9]. The result can be found in the current form as a part of Theorem 2 in [10] and of Theorem 2 in [11].

4. APPLICATIONS TO REAL SUM DECOMPOSITIONS OF SYMMETRIC MATRICES

Let A be a symmetric $n \times n$ matrix, and let $r_1, \dots, r_n$ be real numbers. In [2] we prove necessary and sufficient conditions for the existence of a real matrix B with row sums $r_1, \dots, r_n$ and such that $A = B + B^T$. In this section we apply the results of Section 2 to generalize the results of [2] to the case where only some of the row sums r_i are given, and where B has a partially prescribed sign pattern. The following new result is obtained by applying Theorem 2.15 to a sequence of n^2 numbers, arranged in an $n \times n$ matrix, with $S_1, \dots, S_m$ being the rows of the matrix and $\{T_{\{i,j\}} : i, j \in \langle n \rangle\}$ being the partition of $\langle n \rangle \times \langle n \rangle$ defined by

$$T_{\{i,j\}} = \begin{cases} \{(i,j),(j,i)\}, & i \neq j, \\ \{(i,i)\}, & i = j, \end{cases} \qquad i, j \in \langle n \rangle.$$

The numbers r_i, $i \in R \subseteq \langle n \rangle$, are given real numbers, and the numbers $c_{\{i,j\}}$, $i, j \in \langle n \rangle$, are given by

$$c_{\{i,j\}} = \begin{cases} a_{ij}, & i \neq j, \\ a_{ii}/2, & i = j, \end{cases}$$

where $A = (a_{ij})_1^n$ is a given symmetric $n \times n$ matrix.

THEOREM 4.1. *Let A be a symmetric $n \times n$ matrix, let P be a sign pattern $n \times n$ matrix, and let r_i, $i \in R \subseteq \langle n \rangle$, be real numbers. The following are equivalent.*

(i) *There exists an $n \times n$ matrix B with weak sign pattern P, satisfying*

$$\sum_{j=1}^{n} b_{ij} = r_i, \qquad i \in R,$$

and such that $A = B + B^T$.

(ii) *For any two subsets* α *of* $\langle n\rangle$ *and* β *of* $\{\{i,j\} : i,j \in \langle n\rangle\}$ *for which*

$$p_{ij} \leqslant 0, \qquad i \in \alpha^C, \quad \{i,j\} \in \beta,$$

$$p_{ij} \geqslant 0, \qquad i \in \alpha, \quad \{i,j\} \in \beta^C,$$

we have

$$\begin{aligned} \sum_{i\in\alpha} r_i &\geqslant \frac{1}{2} \sum_{\{i,j\}\in\beta} a_{ij} \qquad \textit{whenever} \quad \alpha \subseteq R, \\ \sum_{i\in\alpha^c} r_i &\leqslant \frac{1}{2} \sum_{\{i,j\}\in\beta^c} a_{ij} \qquad \textit{whenever} \quad \alpha^C \subseteq R. \end{aligned} \tag{4.2}$$

REMARK 4.3. In view of Remark 2.18, we also have a new result asserting the equivalence of two statements similar to those in Theorem 4.1, where in statement (i) "weak sign pattern" is replaced by "strong sign pattern" and in statement (ii) the following condition is added: *Strict inequalities hold in* (4.2) *whenever* $p_{ij} < 0$ *for some* $i \in \alpha^C$, $\{i,j\} \in \beta$, *or* $p_{ij} > 0$ *for some* $i \in \alpha$, $\{i,j\} \in \beta^C$.

In order to state our next results we define

DEFINITION 4.4. Let P be a sign pattern $n \times n$ matrix, and let $\alpha \subseteq \langle n\rangle$. The set α is said to be a *P-loose* subset of $\langle n\rangle$ if for every $i \in \alpha$ and $j \in \alpha^C$ we have $p_{ij} = 0$ or $p_{ji} = 0$. By convention, $\emptyset$ and $\langle n\rangle$ are P-loose sets.

We remark that P-loose sets, associated with a pattern matrix P, are originally defined in [2] as D-loose sets, associated with the digraph of the matrix P.

An application of Theorem 4.1 to the case where $R = \langle n\rangle$ and P is a zero pattern yields the following new theorem. The equivalence of statements (i) and (iii) of this theorem is the assertion of Theorem 2.5 in [2].

THEOREM 4.5. *Let A be a symmetric $n \times n$ matrix, let P be a zero pattern $n \times n$ matrix, and let r_i, $i \in \langle n\rangle$, be real numbers. The following are equivalent.*

(i) *There exists an* $n \times n$ *matrix* B *with weak pattern* P, *with row sums* $r_1, \ldots, r_n$, *and such that* $A = B + B^T$.

(ii) *For every subsets* α *of* $\langle n \rangle$ *and* β *of* $\{\{i, j\} : i, j \in \langle n \rangle\}$ *for which*

(4.6)

$$p_{ij} = 0 \qquad \textit{whenever} \quad i \in \alpha^C, \{i,j\} \in \beta, \ \textit{or} \ i \in \alpha, \{i,j\} \in \beta^C,$$

we have

$$\sum_{i \in \alpha} r_i = \frac{1}{2} \sum_{\{i,j\} \in \beta} a_{ij}. \tag{4.7}$$

(iii) *We have*

$$a_{ij} = 0 \qquad \textit{whenever} \quad p_{ij} = p_{ji} = 0, \ i, j \in \langle n \rangle, \tag{4.8}$$

and

(4.9)

$$\sum_{i \in \alpha} r_i = \frac{1}{2} \sum_{i, j \in \alpha} a_{ij} + \sum_{\substack{(i,j) \in \alpha \times \alpha^c \\ p_{ij} \neq 0}} a_{ij} \qquad \textit{for every } P\textit{-loose subset } \alpha \textit{ of } \langle n \rangle.$$

Proof. (i) $\Rightarrow$ (ii): Since all the nonzero entries of P are $(-\infty, \infty)$, it follows by Theorem 4.1 that (i) implies that for any subsets α of $\langle n \rangle$ and β of $\{\{i, j\} : i, j \in \langle n \rangle\}$ for which (4.6) holds we have

$$\sum_{i \in \alpha} r_i \geqslant \frac{1}{2} \sum_{\{i,j\} \in \beta} a_{ij} \tag{4.10}$$

and

$$\sum_{i \in \alpha^c} r_i \leqslant \frac{1}{2} \sum_{\{i,j\} \in \beta^c} a_{ij}.$$

Since (4.6) is invariant under replacement of α by α^C and of β by β^C, the latter inequality would turn, under such a replacement, into

$$\sum_{i\in\alpha} r_i \leqslant \frac{1}{2} \sum_{\{i,j\}\in\beta} a_{ij}. \tag{4.11}$$

The equality (4.7) follows from (4.10) and (4.11).

(ii) $\Rightarrow$ (i): Since (4.6) is invariant under replacement of α by α^C and of β by β^C, it follows from (ii) that for any subsets α of $\langle n\rangle$ and β of $\{\{i,j\} : i,j \in \langle n\rangle\}$ for which (4.6) holds we have both (4.7) and

$$\sum_{i\in\alpha^C} r_i = \frac{1}{2} \sum_{\{i,j\}\in\beta^C} a_{ij}.$$

Our claim now follows by Theorem 4.1.

(ii) $\Rightarrow$ (iii): Let $i,j \in \langle n\rangle$ be such that $p_{ij} = p_{ji} = 0$. The sets $\alpha = \varnothing$ and $\beta = \{\{i,j\}\}$ satisfy (4.6), and it follows from (4.7) that $a_{ij} = 0$, proving (4.8). Now, let α be a P-loose set, let

$$\beta = \{\{i,j\} : i,j \in \alpha\} \cup \left\{\{i,j\} : (i,j) \in \alpha\times\alpha^C \quad \text{and} \quad p_{ij} \neq 0\right\},$$

and let $i,j \in \langle n\rangle$. If $i \in \alpha$ and $\{i,j\} \in \beta^C$ then, by the definition of β, we have $p_{ij} = 0$. If $i \in \alpha^C$ and $\{i,j\} \in \beta$ then, by the definition of β, we have $j \in \alpha$ and $p_{ji} \neq 0$, and since α is a P-loose set, it follows that $p_{ij} = 0$. Therefore, we have $p_{ij} = 0$ whenever $i \in \alpha^C$, $\{i,j\} \in \beta$, or $i \in \alpha$, $\{i,j\} \in \beta^C$. By (ii) we now have (4.7), which, in our case, is exactly (4.9).

(iii) $\Rightarrow$ (ii): Let α be a subset of $\langle n\rangle$, and let β be a subset of $\{\{i,j\} : i,j \in \langle n\rangle\}$ for which (4.6) holds, and assume that $p_{ij} \neq 0$ for some $(i,j) \in \alpha\times\alpha^C$. Since $i \in \alpha$ and $p_{ij} \neq 0$, it follows by (4.6) that $\{i,j\} \in \beta$. Since $\{i,j\} \in \beta$ and $j \in \alpha^C$, it follows by (4.6) that $p_{ji} = 0$. Therefore, α is a P-loose subset of $\langle n\rangle$, and so (4.9) holds. Let

$$\gamma = \{\{i,j\} : i,j \in \alpha\} \cup \left\{\{i,j\} : (i,j) \in \alpha\times\alpha^C \quad \text{and} \quad p_{ij} \neq 0\right\}.$$

By (4.9) we have

$$\sum_{i\in\alpha} r_i = \frac{1}{2} \sum_{i,j\in\alpha} a_{ij} + \sum_{\substack{(i,j)\in\alpha\times\alpha^C \\ p_{ij}\neq 0}} a_{ij} = \frac{1}{2} \sum_{\{i,j\}\in\gamma} a_{ij}. \tag{4.12}$$

Let $i, j \in \langle n \rangle$ be such that $a_{ij} \neq 0$. Distinguish three cases:

(1) $(i,j) \in \alpha \times \alpha$. By definition of γ we have $\{i,j\} \in \gamma$. Also, since $i, j \in \alpha$, and since by (4.8) we have $p_{ij} \neq 0$ or $p_{ji} \neq 0$, it follows by (4.6) that $\{i,j\} \in \beta$.

(2) $(i,j) \in \alpha \times \alpha^C$. By (4.8) we have $p_{ij} \neq 0$ or $p_{ji} \neq 0$. If $p_{ij} \neq 0$ then, by definition of γ, we have $\{i,j\} \in \gamma$. Also, by (4.6) we have $\{i,j\} \in \beta$. If $p_{ij} = 0$ then $\{i,j\} \notin \gamma$, and also, since $j \in \alpha^C$ and $p_{ji} \neq 0$, by (4.6) we have $\{i,j\} \notin \beta$.

(3) $(i,j) \in \alpha^C \times \alpha^C$. By the definition of γ, we have $\{i,j\} \notin \gamma$. Also, since by (4.8) we have $p_{ij} \neq 0$ or $p_{ji} \neq 0$, it follows from (4.6) that $\{i,j\} \notin \beta$.

It follows that whenever $a_{ij} \neq 0$ we have $\{i,j\} \in \beta \Leftrightarrow \{i,j\} \in \gamma$. Hence

$$\frac{1}{2} \sum_{\{i,j\} \in \beta} a_{ij} = \frac{1}{2} \sum_{\{i,j\} \in \gamma} a_{ij},$$

and, in view of (4.12), (ii) follows. ∎

An application of Theorem 4.1 to the case where $R = \langle n \rangle$ and the matrix A and the sign pattern matrix P are nonnegative yields the following new theorem. The equivalence of statements (i) and (iii) of this theorem is asserted in Theorem 3.3 in [2].

THEOREM 4.13. *Let A be a nonnegative symmetric $n \times n$ matrix, let P be a nonnegative sign pattern $n \times n$ matrix, and let r_i, $i \in \langle n \rangle$, be nonnegative numbers. The following are equivalent.*

(i) *There exists an $n \times n$ matrix B with weak sign pattern P, with row sums $r_1, \dots, r_n$, and such that $A = B + B^T$.*

(ii) *For any two subsets α of $\langle n \rangle$ and β of $\{\{i,j\} : i, j \in \langle n \rangle\}$ for which*

$$p_{ij} = 0, \qquad i \in \alpha^C, \qquad \{i,j\} \in \beta,$$

we have

$$\sum_{i \in \alpha} r_i \geqslant \frac{1}{2} \sum_{\{i,j\} \in \beta} a_{ij},$$

$$\sum_{i \in \alpha^C} r_i \leqslant \frac{1}{2} \sum_{\{i,j\} \in \beta^C} a_{ij}. \tag{4.14}$$

(iii) *We have* (4.8), (4.9) *and*

$$(4.15)\quad \sum_{i\in\alpha} r_i \geqslant \frac{1}{2}\sum_{i,j\in\alpha} a_{ij} + \sum_{\substack{(i,j)\in\alpha\times\alpha^C\\ p_{ji}=0}} a_{ij} \qquad \text{for all} \quad \alpha\subseteq\langle n\rangle.$$

Proof. (i) ⇒ (ii) by Theorem 4.1.

(ii) ⇒ (iii): In view of Theorem 4.5, all we have to prove is that (ii) implies (4.15). Let $\alpha\subseteq\langle n\rangle$ and let

$$\beta = \{\{i,j\} : i,j\in\alpha\} \cup \Big\{\{i,j\} : (i,j)\in\alpha^C\times\alpha \quad \text{and} \quad p_{ij}=0\Big\}.$$

By (ii) we have

$$\sum_{i\in\alpha} r_i \geqslant \frac{1}{2}\sum_{\{i,j\}\in\beta} a_{ij},$$

which, in our case, is exactly (4.15).

(iii) ⇒ (ii): Let α be a subset of $\langle n\rangle$, and let β be a subset of $\{\{i,j\} : i,j\in\langle n\rangle\}$ for which $p_{ij}=0$ whenever $i\in\alpha^C$, $\{i,j\}\in\beta$. Let

$$\gamma = \{\{i,j\} : i,j\in\alpha\} \cup \Big\{\{i,j\} : (i,j)\in\alpha^C\times\alpha \quad \text{and} \quad p_{ij}=0\Big\}.$$

Let $\{i,j\}\in\beta\setminus\gamma$. Then $i,j\in\alpha^C$ and $p_{ij}=p_{ji}=0$. By (4.8) we have $a_{ij}=0$, and thus

$$a_{ij}\neq 0,\quad \{i,j\}\in\beta \quad\Rightarrow\quad \{i,j\}\in\gamma.$$

Therefore, since A is nonnegative we have

$$\frac{1}{2}\sum_{\{i,j\}\in\gamma} a_{ij} \geqslant \frac{1}{2}\sum_{\{i,j\}\in\beta} a_{ij},$$

and it now follows from (iii) that

(4.16)

$$\sum_{i\in\alpha} r_i \geqslant \frac{1}{2}\sum_{i,j\in\alpha} a_{ij} + \sum_{\substack{(i,j)\in\alpha\times\alpha^C\\ p_{ji}=0}} a_{ij} = \frac{1}{2}\sum_{(i,j)\in\gamma} a_{ij} \geqslant \frac{1}{2}\sum_{(i,j)\in\beta} a_{ij}.$$

Since $\langle n\rangle$ is clearly a P-loose set, it follows by (4.9) that

$$\sum_{i\in\langle n\rangle} r_i = \frac{1}{2}\sum_{i,j\in\langle n\rangle} a_{ij}. \tag{4.17}$$

By subtracting (4.16) from (4.17) we obtain

$$\sum_{i\in\alpha^c} r_i \leqslant \frac{1}{2}\sum_{(i,j)\in\beta^c} a_{ij},$$

and (ii) follows. ∎

REMARK 4.18. In view of Remark 4.3, we have another new result asserting the equivalence of two statements similar to those in Theorem 4.13, where in statement (i) "weak sign pattern" is replaced by "strong sign pattern"; in statement (ii) the following condition is added: *Strict inequalities hold in* (4.14) *whenever* $p_{ij} > 0$ *for some* $i \in \alpha$, $\{i, j\} \in \beta^C$; and in statement (iii) the following condition is added: *Strict inequality holds in* (4.15) *whenever* $p_{ij}, p_{ji} > 0$ *for some* $i \in \alpha$, $j \in \alpha^C$.

5. HADAMARD ADJUSTMENTS

The discussion in this section is independent of the previous section, and is motivated by the fact that the results of [10] and [11] that are generalized in Section 3 also contain a statement on the existence of row and column scalings that scale a nonnegative matrix A to have the same row and column sums as a matrix B having the same pattern as A. We introduce Hadamard adjustments, by means of which we generalize that scaling result to apply to our main results. In contrast to the previous sections, where known results on nonnegative sign patterns are derived as corollaries of our independently proven theorems, in this section we derive the generalization using known scaling results on nonnegative sign patterns.

DEFINITION 5.1. Let P be a sign pattern p-vector, and let $\mathscr{U} = \{U_1, \dots, U_r\}$ be a partition of $\langle p \rangle$. A p-vector X is said to be a $(P, \mathscr{U})$*-adjustment vector* if:

(i) $X_i = 0$ whenever $P_i = 0$, and $X_i > 0$ whenever $P_i \neq 0$, $i \in \langle p \rangle$.

(ii) For ever $i \in \langle r \rangle$ all the entries $\{X_j : j \in U_i,\ P_j > 0\}$ are the same and all the entries $\{X_j : j \in U_i,\ P_j < 0\}$ are the same.

DEFINITION 5.2. Let A and B be p-vectors. The *Hadamard product* $A \circ B$ is defined to be the p-vector C satisfying $C_i = A_i B_i$, $i \in \langle p \rangle$.

THEOREM 5.3. *Let A and B be p-sequences with the same strong sign pattern P and let $\mathscr{S} = \{S_1, \dots, S_s\}$ and $\mathscr{T} = \{T_1, \dots, T_t\}$ be orthogonal partitions of $\langle p \rangle$. Then there exist a $(P, \mathscr{S})$-adjustment vector Y and a $(P, \mathscr{T})$-adjustment vector X such that the Hadamard product $C = A \circ X \circ Y$ satisfies*

$$\sum_{u \in S_i} C_u = \sum_{u \in S_i} B_u, \qquad i \in \langle s \rangle,$$

$$\sum_{u \in T_j} C_u = \sum_{u \in T_j} B_u, \qquad j \in \langle n \rangle.$$

Proof. We define two nonnegative $p \times q$ matrices $\tilde{B}^+$ and $\tilde{B}^-$ as follows: for every $i \in \langle s \rangle$ and $j \in \langle t \rangle$, if $S_i \cap T_j = \emptyset$ then $\tilde{b}_{ij}^+ = \tilde{b}_{ij}^- = 0$. Otherwise, we have $S_i \cap T_j = \{u\}$ and we define

$$\tilde{b}_{ij}^+ = \begin{cases} B_u, & B_u \geqslant 0, \\ 0, & B_u < 0, \end{cases} \qquad \tilde{b}_{ij}^- = \begin{cases} 0, & B_u \geqslant 0, \\ -B_u, & B_u < 0. \end{cases}$$

Similarly, we define the nonnegative matrices $\tilde{A}^+$ and $\tilde{A}^-$. Observe that the row sums $r_1^+, \dots, r_s^+$ and $r_1^-, \dots, r_s^-$, and the column sums $c_1^+, \dots, c_s^+$ and

$c_1^-, \dots, c_s^-$, of $\tilde{B}^+$ and $\tilde{B}^-$ respectively are given by

$$r_i^+ = \sum_{\substack{u \in S_i \\ B_u > 0}} B_u, \quad r_i^- = -\sum_{\substack{u \in S_i \\ B_u < 0}} B_u, \qquad i \in \langle s \rangle,$$

and

$$c_j^+ = \sum_{\substack{u \in T_j \\ B_u > 0}} B_u, \quad c_j^- = -\sum_{\substack{u \in T_j \\ B_u < 0}} B_u, \qquad j \in \langle t \rangle.$$

Since the nonnegative matrices $\tilde{A}^+$ and $\tilde{A}^-$ have the same pattern as $\tilde{B}^+$ and $\tilde{B}^-$ respectively, it follows by Theorem 2 in [8] (see also Corollary 4.3 in [9]) that there exist diagonal matrices X^1, Y^1, X^2, and Y^2, with positive diagonal elements, such that $Y^1\tilde{A}^+X^1$ has row sums $r_1^+, \dots, r_s^+$ and column sums $c_1^+, \dots, c_s^+$, and such that $Y^2\tilde{A}^-X^2$ has row sums $r_1^-, \dots, r_s^-$ and column sums $c_1^-, \dots, c_s^-$. We define a $(P, \mathscr{S})$-adjustment vector Y as follows: For every $u \in \langle p \rangle$ let $i \in \langle s \rangle$ be such that $u \in S_i$. Then let

$$Y_u = \begin{cases} y_{ii}^1, & P_u > 0, \\ y_{ii}^2, & P_u < 0. \end{cases}$$

Similarly, we define a $(P, \mathscr{T})$-adjustment vector X as follows: For every $u \in \langle p \rangle$ let $j \in \langle t \rangle$ be such that $u \in T_j$. Then let

$$X_u = \begin{cases} x_{jj}^1, & P_u > 0, \\ X_{jj}^2, & P_u < 0. \end{cases}$$

Let C be the Hadamard product $A \circ X \circ Y$ and let $i \in \langle s \rangle$, $j \in \langle t \rangle$. Observe that the sum $\sum_{u \in S_i} C_u$ is equal to the ith row sum of $Y^1\tilde{A}^+X^1$ minus the ith row sum of $Y^2\tilde{A}^-X^2$, that is, to $r_i^+ - r_i^- = \sum_{u \in S_i} B_u$. Similarly, the sum $\sum_{u \in T_j} C_u$ is equal to the jth column sum of $Y^1\tilde{A}^+X^1$ minus the jth column sum of $Y^2\tilde{A}^-X^2$, that is, to $c_j^+ - c_j^- = \sum_{u \in T_j} B_u$. ■

It is natural to ask whether the adjustment vectors X and Y in Theorem 5.3 can be chosen such that for every $i \in \langle s \rangle$ all the entries $\{Y_u : u \in S_i\}$ are the same and for every $j \in \langle t \rangle$ all the entries $\{X_u : u \in T_j\}$ are the same, without the distinction between the cases of $P_u > 0$ and $P_u < 0$. The answer to this question is negative, as is demonstrated by the following example.

EXAMPLE 5.4. The vectors $A = (1, -2, -1, 1)$ and $B = (1, -1, -1, 1)$ have the same strong sign pattern P. Let $S_1 = \{1, 2\}$ and $S_2 = \{3, 4\}$, and let $T_1 = \{1, 3\}$ and $T_2 = \{2, 4\}$. Let Y be a $(P, \mathscr{S})$-adjustment vector of the type $Y = (y_1, y_1, y_2, y_2)$, and let X be a $(P, \mathscr{T})$-adjustment vector of the type $X = (x_1, x_2, x_1, x_2)$. Note that y_1, y_2, x_1, x_2 are all positive numbers. Now, observe that if

$$\sum_{u \in S_1} (A \circ X \circ Y)_u = y_1(x_1 - 2x_2) = \sum_{u \in S_1} B_u = 0$$

then $x_1 = 2x_2$. But then

$$\sum_{u \in S_2} (A \circ X \circ Y)_u = y_2(x_2 - x_1) \neq 0 = \sum_{u \in S_2} B_u = 0.$$

We remark that the technique used in the proof of Theorem 5.3, that is, translating the problem into the problem of existence of a matrix with preassigned row sums and column sums, does not seem to be applicable in the previous sections, mainly because there we prescribe the element sums corresponding only to some of the sets of the partitions, while in the result we use here all row sums and column sums are prescribed.

Finally, it is easy to check that the application of Theorem 5.3 to the setting under discussion in Theorem 4.13 yields

THEOREM 5.5. *Let A and B be nonnegative $n \times n$ matrices of the same strong sign pattern. Then there exists a diagonal matrix Y with positive diagonal elements and a positive symmetric matrix X such that the matrix $C = A \circ (YX)$ has the same row sums as B and also $C + C^T = B + B^T$.*

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Received 7 October 1996; final manuscript accepted 9 October 1996