Existence of Matrices With Prescribed Off-Diagonal Block Element Maxima

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ABSTRACT

Necessary and sufficient conditions are proven for the existence of a real square matrix such that for every principal submatrix the maximal (or minimal) value of an element in the row complement of the submatrix is prescribed. The problem is solved in the cases where the positions of the nonzero elements of A are contained

in a given set of positions, where the positions of the nonzero elements of A are all given, and where there is no restriction on the positions of the nonzero elements of A. The uniqueness of the solution is studied as well.

1. INTRODUCTION

In analogy with well-studied problems concerning the existence of nonnegative matrices with a prescribed pattern and with prescribed row and columns sums (e.g. [7, 1, 3, 4] and many other papers), Rothblum, Schneider, and Schneider [5] have recently studied the existence of matrices with a prescribed pattern and prescribed row and column maxima. Further, it is clear that the conditions for existence determined in [5] force certain relations on the maxima of elements in the off-diagonal submatrices. In this paper, we find necessary and sufficient conditions for the existence of a real matrix with prescribed pattern and prescribed maxima of the elements in all off-diagonal blocks.

In most of the papers referenced above the existence problems studied were motivated by scaling problems. Another such scaling problem, namely the max-balancing problem, was considered in [6]. There it is pointed out that, in contrast with the corresponding problem for sums, max-balancing the rows and columns of a matrix does not imply max-balancing symmetrically placed off-diagonal blocks. Thus the authors were led to prove a condition for max-balancing off-diagonal blocks. This is an additional motivation for our study.

We now describe the paper in some more detail.

In Section 2 we solve three levels of the existence problem:

(i) The case in which the positions of the nonzero elements of A are contained in a given set of positions.

(ii) The case in which the positions of the nonzero elements of A are all given.

(iii) The case in which we have no restriction on the positions of the nonzero elements of A (the "graph-free" case).

Both the maxima and the minima versions of all three cases are fully characterized.

The existence problems do not always have a unique solution. In Sec-

tion 3 we study the uniqueness of the solutions for the problems discussed in Section 2. We define the *ROWB* algorithm for removal of certain arcs in a digraph, and then show that a given pattern yields a unique solution for *every* set of given maxima if and only if the *ROWB* algorithm terminates after one iteration at most. A given pattern yields a unique solution for *some* set of given maxima if and only if the *ROWB* algorithm terminates after a finite number of iterations.

Motivated by the results of Section 4, Section 5 is devoted to a short discussion of digraphs for which the *ROWB* algorithm terminates after one iteration at most.

This paper is a companion paper to [2], where we solve the analogous problem with sums in place of maxima.

2. THE EXISTENCE PROBLEM

In this section we discuss the existence of a matrix such that for every principal submatrix we prescribe the maximal value of an element in the row complement of the submatrix.

We start with a few definitions.

NOTATION 2.1.   We denote by $\langle n \rangle$ the set $\{1, \ldots, n\}$.

DEFINITION 2.2.

  (i) Let $D = (V, E)$ be a digraph, and let S and T be disjoint subsets of V. We denote by $\widetilde{S}_T$ the set $\{i \in S : (i,j) \in E \text{ for some } j \in T\}$.
 (ii) We denote by $\widetilde{S}$ the set $\widetilde{S}_{S^c}$, where S^c denotes the complement of S in V. We say that S is a *relevant* subset of V if $\widetilde{S}$ is nonempty.
(iii) We denote by $S^{\perp}$ the maximal subset T of S^c such that $\widetilde{S}_T = \emptyset$.

DEFINITION 2.3.   Let A be an $n \times n$ matrix. The *digraph* $D(A)$ *of* A is defined as the digraph with vertex set $\langle n \rangle$, and where (i,j) is an arc in $D(A)$ if and only if $a_{ij} \neq 0$.

DEFINITION 2.4.   Let $D = (V, E)$ be a digraph. A digraph $D' = (V', E')$ is said to be a *subdigraph* of D if $V' \subseteq V$ and $E' \subseteq E$. We write $D' \subseteq D$ to indicate that D' is a subdigraph of D.

CONVENTION 2.5.   The maximum of the elements of the empty subset of the reals will be considered to be $-\infty$.

The main result in this section is

THEOREM 2.6. *Let $\{X_S : S \subseteq \langle n \rangle\}$ be a set of real numbers, and let $D = (\langle n \rangle, E)$ be a digraph. The following are equivalent:*

(i) *There exists an $n \times n$ matrix A, with $D(A) \subseteq D$, such that*

$$\max_{(i,j) \in S \times S^c \cap E} \{a_{i,j}\} = X_S, \qquad S \subseteq \langle n \rangle. \tag{2.7}$$

(ii) *We have*

$$
\begin{aligned}
X_\emptyset &= X_{\langle n \rangle} = -\infty, & &\tag{2.8}\\
X_S &\leq \max\{X_{S \cup T}, X_{S'}\}, & S, T \subseteq \langle n \rangle, \quad & S \cap T = \emptyset,\\
& & \widetilde{S}_T \subseteq S' \subseteq S \quad &\tag{2.9}\\
X_{S \cup T} &\leq \max\{X_S, X_T\}, & S, T \subseteq \langle n \rangle, &\tag{2.10}\\
X_{S \cap T} &\leq \max\{X_S, X_T\}, & S, T \subseteq \langle n \rangle. &\tag{2.11}
\end{aligned}
$$

Proof. (i) $\Rightarrow$ (ii): In view of Convention 2.5, (2.8) follows immediately from (i). Let S and T be disjoint subsets of $\langle n \rangle$. By Definition 2.2, it follows that

$$
\begin{aligned}
(S \times S^c) \cap E &= [(S \times T) \cap E] \cup ([S \times (S \cup T)^c] \cap E)\\
&= [(S' \times T) \cap E] \cup ([S \times (S \cup T)^c] \cap E)\\
&\subseteq [(S' \times S'^c) \cap E] \cup ([(S \cup T) \times (S \cup T)^c] \cap E).
\end{aligned}
$$

Therefore, clearly (i) implies (2.9). Now let S and T be subsets of $\langle n \rangle$. It follows from (i) that

$$
\begin{aligned}
X_{S \cup T} &= \max_{(i,j) \in [S \cup T \times (S \cup T)^c] \cap E} \{a_{ij}\}\\
&= \max \left\{ \max_{(i,j) \in [S \times (S \cup T)^c] \cap E} \{a_{ij}\}, \max_{(i,j) \in [T \times (S \cup T)^c] \cap E} \{a_{ij}\} \right\}\\
&\leq \max \left\{ \max_{(i,j) \in S \times S^c \cap E} \{a_{ij}\}, \max_{(i,j) \in T \times T^c \cap E} \{a_{ij}\} \right\}\\
&= \max\{X_S, X_T\}.
\end{aligned}
$$

Similarly, we have

$$
X_{S \cap T} = \max_{(i,j) \in [S \cap T \times (S \cap T)^c] \cap E} \{a_{ij}\} = \max_{(i,j) \in [S \cap T \times (S^c \cup T^c)] \cap E} \{a_{ij}\}
$$

$$= \max\left\{\max_{(i,j)\in[S\cap T\times S^c)\cap E}\{a_{ij}\}, \max_{(i,j)\in[S\cap T\times T^c)\cap E}\{a_{ij}\}\right\}$$

$$\leq \max\left\{\max_{(i,j)\in S\times S^c\cap E}\{a_{ij}\}, \max_{(i,j)\in T\times T^c\cap E}\{a_{ij}\}\right\}$$

$$= \max\{X_S, X_T\}.$$

(ii) $\Rightarrow$ (i): We construct an $n \times n$ matrix A by

$$a_{ij} = \begin{cases} \min_{\substack{T\subseteq\langle n\rangle \\ (i,j)\in T\times T^c}} \{X_T\}, & (i,j)\in E, \\ 0, & (i,j)\notin E. \end{cases} \tag{2.12}$$

We shall prove that A satisfies (2.7). For $(i,j) \in E$ we denote by T_{ij} a subset of $\langle n\rangle$ such that $(i,j) \in T_{ij} \times T_{ij}^c$ and $a_{ij} = X_{T_{ij}}$. Let S be a subset of $\langle n\rangle$. If $S = \emptyset$ or $S = \langle n\rangle$, then (2.7) follows from (2.8). So assume that $S \neq \emptyset, \langle n\rangle$. We have

$$\max_{(i,j)\in S\times S^c\cap E}\{a_{ij}\} = \max_{(i,j)\in S\times S^c\cap E}\left\{\min_{\substack{T\subseteq\langle n\rangle \\ (i,j)\in T\times T^c}}\{X_T\}\right\} \tag{2.13}$$

$$\leq \max_{(i,j)\in S\times S^c\cap E}\{X_S\} = X_S.$$

Let

$$W_i = \bigcap_{\substack{j\in S^c \\ (i,j)\in E}} T_{ij}, \qquad i \in \widetilde{S}.$$

For every element $i \in \langle n\rangle$ we denote by $\mathrm{supp}(i)$ the set $\{k \in \widetilde{S} : i \in W_k\}$. Note that $\mathrm{supp}(i)$ may be empty. We define the equivalence relation in $\langle n\rangle$ by $i \sim j \Leftrightarrow \mathrm{supp}(i) = \mathrm{supp}(j)$. We denote by C_i the equivalence class of i, and we denote $\mathrm{supp}(C_i) = \mathrm{supp}(i)$. Observe that $i \in W_i$ whenever $i \in \widetilde{S}$, and hence

$$C_i \cap \widetilde{S} \subseteq \mathrm{supp}(C_i) \subseteq \bigcap_{j\in\mathrm{supp}(C_i)} W_j. \tag{2.14}$$

We define the sets

$$T_i = \left(\bigcap_{j\in\mathrm{supp}(C_i)} W_j\right)\bigcap S, \qquad i \in \widetilde{S},$$

and we now claim that

$$X_{T_i} \leq \max_{(j,k)\in(S\times S^c)\cap E} \left\{ \min_{\substack{T\subseteq\langle n\rangle \\ (j,k)\in T\times T^c}} \{X_T\} \right\}, \qquad i \in \widetilde{S}. \qquad (2.15)$$

We prove this claim by a decreasing induction on the cardinality $|\text{supp}(C_i)|$ of C_i. Assume first that $\text{supp}(C_i)$ is of maximal cardinality m. By (2.14) we have

$$C_i \cap \widetilde{S} \subseteq \left(\bigcap_{j\in\text{supp}(C_i)} W_j \right) \bigcap \widetilde{S}. \qquad (2.16)$$

Assume that there exists

$$v \in \left[\left(\bigcap_{j\in\text{supp}(C_i)} W_j \right) \bigcap \widetilde{S} \right] \setminus C_i. \qquad (2.17)$$

Since $v \in \bigcap_{j\in\text{supp}(C_i)} W_j$, we have $\text{supp}(C_i) \subseteq \text{supp}(v)$. Furthermore, it follows from (2.17) that $\text{supp}(C_i) \neq \text{supp}(v)$, and hence $|\text{supp}(v)| > |\text{supp}(C_i)|$, in contradiction to the maximality of $|\text{supp}(C_i)|$. Therefore, our assumption that there exists v satisfying (2.17) is false, and it follows from (2.16) that

$$C_i \cap \widetilde{S} = \left(\bigcap_{j\in\text{supp}(C_i)} W_j \right) \bigcap \widetilde{S}. \qquad (2.18)$$

Now, let $u \in T_i$ and let $v \in S^c$. If $(u,v) \in E$ then $u \in \widetilde{S}$. By (2.18) we have $u \in C_i$. By the construction of our sets T_{ij} we have $u \in T_{uv}$ and $v \notin T_{uv}$, and so $u \in W_u$ while $v \notin W_u$. Since $u \in C_i$ we have $\text{supp}(u) = \text{supp}(C_i)$, and hence, since $v \notin W_u$, it follows that $v \notin \bigcap_{j\in\text{supp}(C_i)} W_j$. Thus, we have

$$\left(\left[\left(\bigcap_{j\in\text{supp}(C_i)} W_j \right) \bigcap S \right] \times \left[\left(\bigcap_{j\in\text{supp}(C_i)} W_j \right) \bigcap S^c \right] \right) \bigcap E = \emptyset.$$
$$(2.19)$$

In view of (2.19), substitution of the sets $(\bigcap_{j\in\text{supp}(C_i)} W_j)\bigcap S$, $(\bigcap_{j\in\text{supp}(C_i)} W_j)\bigcap S^c$, and $\emptyset$ for the sets S, T, and S' in (2.9) respectively

yields $X_{T_i} \leq X_{\bigcap_{j \in \mathrm{supp}(C_i)} W_j}$, and by (2.11) we have

$$
\begin{aligned}
X_{T_i} &\leq \max_{j \in \mathrm{supp}(C_i)} \{X_{W_j}\} \leq \max_{\substack{j \in \mathrm{supp}(C_i) \\ k \in S^c \\ (j,k) \in E}} \{X_{T_{jk}}\} \\[2mm]
&= \max_{\substack{j \in \mathrm{supp}(C_i) \\ k \in S^c \\ (j,k) \in E}} \left\{ \min_{\substack{T \subseteq \langle n \rangle \\ (j,k) \in T \times T^c}} \{X_T\} \right\} \\[2mm]
&\leq \max_{(j,k) \in (S \times S^c) \cap E} \left\{ \min_{\substack{T \subseteq \langle n \rangle \\ (j,k) \in T \times T^c}} \{X_T\} \right\},
\end{aligned}
$$

proving (2.15) in the case that $\mathrm{supp}(C_i)$ is of maximal cardinality m. Assume now that (2.15) holds whenever $|\mathrm{supp}(C_i)| > r$, where $r < m$, and let i be such that $|\mathrm{supp}(C_i)| = r$. Let $u \in C_i \cap S$ and let $v \in S^c$. If $(u, v) \in E$ then $u \in \widetilde{S}$. By the construction of our sets T_{ij} we have $u \in T_{uv}$ and $v \notin T_{uv}$, and so $u \in W_u$ while $v \notin W_u$. Since $u \in C_i$, we have $\mathrm{supp}(u) = \mathrm{supp}(C_i)$, and hence, since $v \notin W_u$, it follows that $v \notin \bigcap_{j \in \mathrm{supp}(C_i)} W_j$. Thus, we have

$$
\left((C_i \cap S) \times \left[\left(\bigcap_{j \in \mathrm{supp}(C_i)} W_j \right) \cap S^c \right] \right) \cap E = \emptyset. \qquad (2.20)
$$

In view of (2.20), substitution of the sets T_i, $(\bigcap_{j \in \mathrm{supp}(C_i)} W_j) \cap S^c$, and $T_i \backslash C_i$ for the sets S, T, and S' in (2.9) respectively yields

$$
X_{T_i} \leq \max \left\{ X_{\bigcap_{j \in \mathrm{supp}(C_i)} W_j}, X_{T_i \backslash C_i} \right\},
$$

and by (2.11) and the definition of W_j we have

$$
X_{T_i} \leq \max \left\{ \max_{(j,k) \in [\mathrm{supp}(C_i) \times S^c] \cap E} \left\{ \min_{\substack{T \subseteq \langle n \rangle \\ (j,k) \in T \times T^c}} \{X_T\} \right\}, X_{T_i \backslash C_i} \right\}. \qquad (2.21)
$$

Let $u \in T_i \backslash C_i$, First, observe that by the definition of T_i we have $\mathrm{supp}(i) \subseteq \mathrm{supp}(u)$. Furthermore, since $u \notin C_i$, we have $\mathrm{supp}(i) \neq \mathrm{supp}(u)$, and hence $|\mathrm{supp}(C_u)| > |\mathrm{supp}(C_i)|$. Now, let $v \in T_u$. Since $\mathrm{supp}(C_i) \subseteq$

$\operatorname{supp}(C_u)$, it follows by the definition of T_i that $v \in T_i$. Also, $\operatorname{supp}(C_i) \subseteq \operatorname{supp}(C_u) \subseteq \operatorname{supp}(v)$, and since $\operatorname{supp}(C_i) \neq \operatorname{supp}(C_u)$, it follows that $v \notin C_i$. Therefore, $v \in T_i \backslash C_i$. It follows that there exist $s_1, \dots, s_p \in \widetilde{S}$ such that $T_i \backslash C_i = \bigcup_{k=1}^{p} T_{s_k}$, and where $|\operatorname{supp}(C_{s_j})| > |\operatorname{supp}(C_i)|, j \in \langle p \rangle$. By (2.10) and the inductive assumption we obtain

$$X_{T_i \backslash C_i} \le \max_{k \in \langle p \rangle} \{ X_{T_{s_k}} \} \le \max_{(j,k) \in (S \times S^c) \cap E} \left\{ \min_{\substack{T \subseteq \langle n \rangle \\ (j,k) \in T \times T^c}} \{ X_T \} \right\},$$

and by (2.21) we obtain (2.15).

Observe that $\widetilde{S} \subseteq \widehat{S} = \bigcup_{i \in S} T_i \subseteq S$. Substitution of the sets S, S^c, and $\widehat{S}$ for the sets S, T, and S' in (2.9) respectively yields $X_S \le X_{\widehat{S}}$. By (2.10) we obtain $X_S \le \max_{i \in \widetilde{S}} \{ X_{T_i} \}$, and by (2.15) we have

$$X_S \le \max_{(j,k) \in (S \times S^c) \cap E} \left\{ \min_{\substack{T \subseteq \langle n \rangle \\ (j,k) \in T \times T^c}} \{ X_T \} \right\}. \tag{2.22}$$

This proves our claim. It now follows from (2.13) and (2.22) that $\max_{(i,j) \in S \times S^c \cap E} \{ a_{ij} \} = X_S, S \subseteq \langle n \rangle$.   ∎

Let $S \subseteq \langle n \rangle$. If we choose T in (2.9) to be S^c, then we get

$$X_S \le X_{S'}, \qquad \widetilde{S} \subseteq S' \subseteq S. \tag{2.23}$$

Another weaker form of (2.9) is

$$X_S \le \max \left\{ X_{S \cup T}, X_{\widetilde{S}_T} \right\}, \qquad S, T \subseteq \langle n \rangle, \quad S \cap T = \emptyset. \tag{2.24}$$

It is natural to ask whether the condition (2.9) in Theorem 2.6 can be replaced by (2.23) and/or (2.24). The answer to this question is negative, as demonstrated by the following example.

EXAMPLE 2.25. Let $n = 4$, and let D be the digraph with four vertices and all possible arcs except for $(2, 4)$ and $(3, 4)$. For a subset S of $\langle 4 \rangle$ define

$$X_S = \begin{cases} -\infty, & S = \emptyset, \langle 4 \rangle, \\ 1, & S = \{1, 3\}, \\ 2, & \text{otherwise}. \end{cases}$$

The only $S \subseteq \langle 4 \rangle$ for which $S \neq \widetilde{S}$ is $S = \{1,2,3\}$, where $\widetilde{S} = \{1\}$. Since $X_{\{1,2,3\}} = X_{\{1\}} = 2$, (2.23) is satisfied. Now, let $S, T \subseteq \langle 4 \rangle, S \cap T = \emptyset$. Observe that we have $S \neq \widetilde{S}_T$ if and only if $T = \{4\}$ and $\{2,3\} \cap S \neq \emptyset$. In these cases we have

$$
\begin{aligned}
S &= \{2\}, \{3\}, \{1,2\}, \{2,3\}, \{1,3\} \quad \Rightarrow \quad X_S = 2 = X_{S \cup T}, \\
S &= \{1,2,3\} \quad \Rightarrow \quad \widetilde{S}_T = \{1\} \quad \Rightarrow \quad X_S = 2 = X_{\widetilde{S}_T},
\end{aligned}
$$

and hence (2.24) is satisfied. Finally, it is immediate to check that (2.10) and (2.11) are satisfied. Nevertheless, if we choose $S = \{1,2,3\}$, $T = \{4\}$, and $S' = \{1,3\}$, then $\widetilde{S}_T = \{1\} \subseteq S' \subseteq S$ and $X_S = 2 > 1 = \max\{X_{S \cup T}, X_{\widetilde{S}_T}\}$. Therefore, (2.9) is not satisfied, and by Theorem 2.6 there exists no $n \times n$ matrix A, with $D(A) \subseteq D$, satisfied (2.7).

With the convention that the minimum of the elements of the empty subset of the reals will be considered as ∞, we obtain the following corollary of Theorem 2.6.

THEOREM 2.26. *Let* $\{X_S : S \subseteq \langle n \rangle\}$ *be a set of numbers, and let* $D = (\langle n \rangle, E)$ *be a digraph. The following are equivalent:*

(i) *There exists an $n \times n$ matrix A, with $D(A) \subseteq D$, such that*

$$
\min_{(i,j) \in S \times S^c \cap E} \{a_{ij}\} = X_S, \qquad S \subseteq \langle n \rangle.
$$

(ii) *We have*

$$
X_\emptyset = X_{\langle n \rangle} = \infty,
$$

$$
\begin{aligned}
X_S &\geq \min\{X_{S \cup T}, X_S\}, \qquad S, T \subseteq \langle n \rangle, \ S \cap T = \emptyset, \ \widetilde{S}_T \subseteq S' \subseteq S, \\
X_{S \cup T} &\geq \min\{X_S, X_T\}, \qquad S, T \subseteq \langle n \rangle, \\
X_{S \cap T} &\geq \min\{X_S, X_T\}, \qquad S, T \subseteq \langle n \rangle.
\end{aligned}
$$

Proof. A matrix A satisfies $\min_{(i,j) \in S \times S^c \cap E}\{a_{ij}\} = X_S, S \subseteq \langle n \rangle$, if and only if the matrix $B = -A$ satisfies $\max_{(i,j) \in S \times S^c \cap E}\{b_{ij}\} = -X_S$ whenever $S \subseteq \langle n \rangle$. Our claim now follows from Theorem 2.6. ∎

Theorems 2.6 and 2.26 provide a necessary and sufficient condition for the existence of a matrix A, with $D(A) \subseteq D$, with prescribed off-diagonal block element maxima or minima. The case $D(A) = D$ is covered by the following corollaries.

COROLLARY 2.27.   *Let $\{X_S : S \subseteq \langle n \rangle\}$ be a set of real numbers, and let $D = (\langle n \rangle, E)$ be a digraph. The following are equivalent:*

(i) *There exists an $n \times n$ matrix A, with $D(A) = D$, such that*

$$\max_{(i,j) \in S \times S^c \cap E} \{a_{ij}\} = X_S, \qquad S \subseteq \langle n \rangle.$$

(ii) *We have (2.8), (2.9), (2.10), (2.11), and*

$$X_S \neq 0 \qquad \text{for every relevant subset } S \text{ of } \langle n \rangle. \qquad (2.28)$$

Proof.   (i) $\Rightarrow$ (ii): (2.8), (2.9), (2.10), and (2.11) follow from (i) by Theorem 2.6. Since $D(A) = D, \max_{(i,j) \in S \times S^c \cap E} \{a_{ij}\} \neq 0$ whenever S is relevant. (2.28) now follows from (i).

(ii) $\Rightarrow$ (i): In view of (2.28), the matrix defined by (2.12), constructed in the proof of the corresponding implication in Theorem 2.5, satisfies $D(A) = D$. ∎

Similarly, we obtain

COROLLARY 2.29.   *Let $\{X_S : S \subseteq \langle n \rangle\}$ be a set of real numbers, and let $D = (\langle n \rangle, E)$ be a digraph. The following are equivalent:*

(i) *There exists an $n \times n$ matrix A, with $D(A) = D$, such that*

$$\min_{(i,j) \in S \times S^c \cap E} \{a_{ij}\} = X_S, \qquad S \subseteq \langle n \rangle.$$

(ii) *We have (2.8), (2.9), (2.10), (2.11), and (2.28).*

The following two corollaries are the "graph-free" versions of our results.

COROLLARY 2.30.   *Let $\{X_S : S \subseteq \langle n \rangle, S \neq \emptyset, \langle n \rangle\}$ be a set of numbers. The following are equivalent:*

(i) *There exists an $n \times n$ matrix A such that*

$$\max_{(i,j) \in S \times S^c} \{a_{ij}\} = X_S, \qquad S \subseteq \langle n \rangle, \quad S \neq \emptyset, \langle n \rangle.$$

(ii) *For every nonempty proper subsets S and T of $\langle n \rangle$ we have $X_{S \cup T} \leq \max\{X_S, X_T\}$ whenever $S \cup T \neq \langle n \rangle$, and $X_{S \cap T} \leq \max\{X_S, X_T\}$ whenever $S \cap T \neq \emptyset$.*

COROLLARY 2.31.   *Let* $\{X_S : S \subseteq \langle n \rangle, S \neq \emptyset, \langle n \rangle\}$ *be a set of numbers. The following are equivalent:*

(i) *There exists an* $n \times n$ *matrix* A *such that*

$$\min_{(i,j) \in S \times S^c} \{a_{ij}\} = X_S, \qquad S \subseteq \langle n \rangle, \quad S \neq \emptyset, \langle n \rangle.$$

(ii) *For every nonempty proper subsets* S *and* T *of* $\langle n \rangle$ *we have* $X_{S \cup T} \geq \min\{X_S, X_T\}$ *whenever* $S \cup T \neq \langle n \rangle$, *and* $X_{S \cap T} \geq \min\{X_S, X_T\}$ *whenever* $S \cap T \neq \emptyset$.

REMARK 2.32.   The graph-free case could be proven directly rather easily. The implication (i) $\Rightarrow$ (ii) in Corollary 2.30 follows as before. As to the implication (ii) $\Rightarrow$ (i), we define a_{ij} as in (2.12) and derive (2.13). Now suppose that for some nonempty proper subset S of $\langle n \rangle$ we have

$$\max_{(i,j) \in S \times S^c} \{a_{ij}\} < X_S.$$

It follows by (2.12) and (2.13) that

$$\max_{(i,j) \in S \times S^c} \{T_{ij}\} < X_S. \tag{2.33}$$

Note that

$$\bigcup_{i \in S} \left(\bigcap_{j \in S^c} T_{ij} \right) = S. \tag{2.34}$$

Therefore, it follows from (2.33) and (2.34) by (2.10) and (2.11) that

$$X_S \leq \max_{(i,j) \in S \times S^c} \{T_{ij}\} < X_S,$$

which is a contradiction.

However, in general we cannot employ such a simple proof, as we do not have an analogue of (2.34).

REMARK 2.35.   Observe that in the case of nonnegative X_S's, the matrix A defined by (2.12) is nonnegative. Therefore, if the numbers $\{X_S : S \subseteq \langle n \rangle\}$ are given to be nonnegative, then our results provide necessary and sufficient conditions for the existence of a *nonnegative* matrix A satisfying the required conditions.

3.   THE UNIQUENESS PROBLEM

The existence problems discussed in the previous section do not neces-
sarily yield a unique solution, as demonstrated by the following example.

EXAMPLE 3.1.   Let $n = 4$, let D be the complete (loopless) digraph
with four vertices, and let

$$X_\emptyset = X_{\langle 4 \rangle} = -\infty,$$
$$X_{\{1\}} = 3, \qquad X_{\{2\}} = 7, \qquad X_{\{3\}} = 9, \qquad X_{\{4\}} = 12,$$
$$X_{\{1,2\}} = 5, X_{\{1,3\}} = 9, X_{\{2,3\}} = 8, X_{\{1,4\}} = 12, X_{\{2,4\}} = 12, X_{\{3,4\}} = 11,$$
$$X_{\{1,2,3\}} = 6, \qquad X_{\{1,2,4\}} = 12, X_{\{1,3,4\}} = 11, X_{\{2,3,4\}} = 10.$$

The numbers $\{X_S : S \subseteq \langle 4 \rangle\}$ satisfy (2.8), (2.9), (2.10), and (2.11). The
matrix

$$A = \begin{pmatrix} 0 & 3 & 3 & 3 \\ 7 & 0 & 5 & 5 \\ 8 & 9 & 0 & 6 \\ 10 & 11 & 12 & 0 \end{pmatrix},$$

defined by (2.12), satisfies (2.7). However, (2.7) is satisfied also by

$$A = \begin{pmatrix} 0 & 1 & 2 & 3 \\ 7 & 0 & 4 & 5 \\ 8 & 9 & 0 & 6 \\ 10 & 11 & 12 & 0 \end{pmatrix}.$$

In this section we study the uniqueness of the solutions for the problems
discussed in the previous section. For a digraph D we denote by $E(D)$ and
$V(D)$ the arc set and the vertex set of D respectively. Our problem can be
stated as follows:

Given an $n \times n$ matrix A. Does there exist another $n \times n$ matrix B,
with $D(B) \subseteq D(A)$, such that

$$\max_{(i,j) \in S \times S^c \cap E(D(A))} \{b_{ij}\} = \max_{(i,j) \in S \times S^c \cap E(D(A))} \{a_{ij}\} \qquad (3.2)$$

for every relevant subset S of $V(D(A))$?
    We shall define the $ROWB$ algorithm for removal of certain arcs in
a digraph, and we shall use it to characterize those graphs D such that
for *every* matrix A with $D(A) = D$, there exists no other matrix B with
$D(B) \subseteq D$ satisfying (3.2), as well as those graphs D such that for *some*

matrix A with $D(A) = D$, there exists no other matrix B with $D(B) \subseteq D$ satisfying (3.2).

DEFINITION 3.3. An arc (i, j) in a digraph $D = (V, E)$ is said to be a *one-way bridge* if there exists a set S of vertices in D such that (i, j) is the only arc from S to S^c.

The following easy proposition can be regarded as an alternative definition for one-way bridges.

PROPOSITION 3.4. *An arc (i, j) in a digraph D is a one-way bridge if and only if the digraph obtained by removing (i, j) from D has no path from i to j.*

Proof. If (i, j) is a one-way bridge, then by Definition 3.3 there exists a set S of vertices in D such that (i, j) is the only arc from S to S^c. It follows that the digraph obtained by removing (i, j) from D has no path from i to j. Conversely, let D' be the digraph obtained by removing (i, j), and let S be the set of vertices k such that there exists a path from i to k in D' (S includes i). If D' has no path from i to j, then j belongs to S^c. Clearly, there is no arc in D' from S to S^c. Therefore, (i, j) is the only arc from S to S^c. ∎

DEFINITION 3.5. The $ROWB$ algorithm (removal of one-way bridges):

Initialization: We let D_1 be the digraph D.

Step $i, i = 1, \ldots$: We remove all the one-way bridges from D_i, and we denote the resulting digraph by D_{i+1}.

The $ROWB$ algorithm terminates after $k - 1$ steps if D_k has no one-way bridges. We then denote D_k by $\check{D}$. The $ROWB$ *index* of D is defined to be equal to $k - 1$ if $\check{D}$ has no arcs, and is ∞ otherwise. Clearly, $\check{D}$ has no one-way bridges.

EXAMPLE 3.6.

(i) It is easy to verify that any digraph consisting of one cycle has $ROWB$ index 1. Also, the $ROWB$ index of any digraph D, such that the (undirected) graph obtained by removing the directions from the arcs of D is a tree, is equal to 1. Clearly, any digraph consisting of disjoint digraphs of $ROWB$ index 1 has $ROWB$ index 1.

(ii) The one-way bridges in the graph
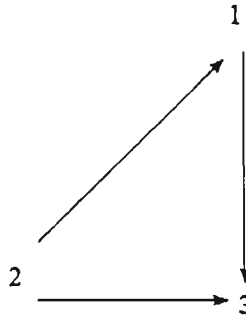
are $(1, 3)$ and $(2, 1)$. Therefore, the only arc in D_2 is $(2, 3)$, and, being a one-way bridge in D_2, it gets removed in the second step. Thus, the $ROWB$ index of D is 2.

(iii) It is easy to verify that for any complete digraph D with more than two vertices we have $\check{D} = D$, and so D has $ROWB$ index ∞. Also, every digraph with multiple arcs has $ROWB$ index ∞.

We can now characterize those patterns that yield a unique solution for *every* set of given maxima.

THEOREM 3.7. *Let* $D = (\langle n \rangle, E)$ *be a digraph. The following are equivalent:*

(i) *For every* $n \times n$ *matrix* A *with* $D(A) \subseteq D$, *there exists no other* $n \times n$ *matrix* B *with* $D(B) \subseteq D$ *satisfying* (3.2).

(ii) *For every* $n \times n$ *matrix* A *with* $D(A) = D$, *there exists no other* $n \times n$ *matrix* B *with* $D(B) \subseteq D$ *satisfying* (3.2)

(iii) *The* $ROWB$ *index of* D *is less than or equal to* 1.

Proof. (i) $\Rightarrow$ (ii) is trivial.

(ii) $\Rightarrow$ (iii): Assume that (iii) does not hold, that is, some arc (k, l) in $D(A)$ is not a one-way bridge in $D(A)$. Define a matrix A_ϵ, of the same size as A, by

$$(A_\epsilon)_{ij} = \begin{cases} 1, & (i,j) \in E(D), \quad (i,j) \neq (k,l), \\ \epsilon, & (i,j) = (k,l), \\ 0 & \text{otherwise}, \end{cases} \tag{3.8}$$

where $\epsilon \leq 1$. Let S be a nonempty proper subset of vertices of D. If there are no arcs from S to S^c, then

$$\max_{(i,j)\in S\times S^c \cap E(D(A))}\{(A_\epsilon)_{ij}\} = 0.$$

If there are arcs from S to S^c, then, since (i,j) cannot be the only arc from S to S^c and since $\epsilon \leq 1$, it follows from (3.8) that

$$\max_{(i,j)\in S\times S^c \cap E(D(A))}\{(A_\epsilon)_{ij}\} = 1.$$

Thus, all A_ϵ share the same set

$$\left\{ \max_{(i,j)\in S\times S^c \cap E(D(A))}\{(A_\epsilon)_{ij}\} : S \subseteq \langle n \rangle \right\},$$

in contradiction to (ii).

(iii) $\Rightarrow$ (i): Let A and B be $n \times n$ matrices, with $D(A) \subseteq D$ and $D(B) \subseteq D$, satisfying (3.2). If the $ROWB$ index of D is less than or equal to 1, then every arc (k,l) in D is a one-way bridge, that is, there exists a set S of vertices in D such that (k,l) is the only arc from S to S^c. Therefore, from (3.2) we obtain

$$\max_{(i,j)\in S\times S^c \cap E(D(B))}\{b_{ij}\} = b_{kl} = \max_{(i,j)\in S\times S^c \cap E(D(A))}\{a_{ij}\} = a_{kl},$$

and so $A = B$.   ∎

The characterizations of the patterns that yield a unique solution of *some* set of given maxima is as follows.

THEOREM 3.9. *Let $D = (\langle n \rangle, E)$ be a digraph. The following are equivalent:*

(i) *For some $n \times n$ matrix A with $D(A) = D$, there exists no other $n \times n$ matrix B with $D(B) \subseteq D$ satisfying (3.2)*
(ii) *The $ROWB$ index of D is finite.*

Proof. (i) $\Rightarrow$ (ii): Assume that the $ROWB$ index of D is ∞, and let A be any matrix with $D(A) = D$. As $\check{D}$ has arcs, let (k,l) be an arc in $\check{D}$ such that a_{kl} is minimal. For $\epsilon > 0$ we define a matrix A_ϵ, of the same size as A, by

$$(A_\epsilon)_{ij} = \begin{cases} a_{ij}, & (i,j) \neq (k,l), \\ a_{kl} - \epsilon, & (i,j) = (k,l). \end{cases} \tag{3.10}$$

Let S be a set of vertices such that $(k, l) \in S \times S^c$. Since $\check{D}$ has no one-way bridges, it follows that there exists at least one more arc in $\check{D}$ from S to S^c. It now follows from (3.10), by the minimality of a_{kl}, that

$$
\max_{(i,j)\in S \times S^c \cap E(D(B))} \{(A_\epsilon)_{ij}\}
$$
$$
= \max_{(i,j)\in S \times S^c \cap E(D(A))} \{a_{ij}\} \quad \text{for every} \quad S \subseteq \langle n \rangle,
$$

in contradiction to (i).

(ii) $\Rightarrow$ (i): We define a matrix A, with $D(A) = D$, as follows. For each $(k, l) \in E(D)$ we let $a_{kl} = m$, where m is the number of the step of the *ROWB* algorithm in which (k, l) gets removed from D. Now, let B an $n \times n$ matrix, with $D(B) \subseteq D$, satisfying (3.2), and let (k, l) be an arc in D. We prove that $a_{kl} = b_{kl}$ by induction on the number m of the step of the *ROWB* algorithm in which (k, l) gets removed from D. If $m = 1$, then there exists a set S of vertices in D such that (k, l) is the only arc from S to S^c, and from (3.2) we obtain

$$
\max_{(i,j)\in S \times S^c \cap E(D(B))} \{b_{ij}\} = b_{kl} = \max_{(i,j)\in S \times S^c \cap E(D(A))} \{a_{ij}\} = a_{kl}.
$$

Assume that our claim holds for $m < p$ where $p > 1$, and let $m = p$. Since (k, l) gets removed from D in the pth step of the *ROWB* algorithm, it follows that there exists a set S of vertices in D such that (k, l) is an arc from S to S^c, and every other arc from S to S^c gets removed in an earlier step. By the inductive assumption we have

$$
b_{ij} = a_{ij} < p \quad \text{for every} \quad (i, j) \in S \times S^c, \quad (i, j) \neq (k, l). \quad (3.11)
$$

Since by (3.2) we have

$$
\max_{(i,j)\in S \times S^c \cap E(D(B))} \{b_{ij}\} = \max_{(i,j)\in S \times S^c \cap E(D(A))} \{a_{ij}\} = p,
$$

it now follows from (3.11) that we necessarily have $a_{kl} = b_{kl}$. ∎

We conclude this section with a description of all possible solutions to the existence problem of matrices with off-diagonal block element maximum.

DEFINITION 3.12. Let $\mathcal{X} = \{X_S : S \subseteq \langle n \rangle, S \neq \emptyset, \langle n \rangle\}$ be a set of numbers, and let $D = (\langle n \rangle, E)$ be a digraph. Let A be the matrix defined by (2.12). A set $\mathcal{P}$ of arcs in D is said to be $\mathcal{X}$-*complete* if for every relevant subset S of $\langle n \rangle$ there exists $(i, j) \in S \times S^c \cap P$ such that $a_{ij} = X_S$.

THEOREM 3.13. *Let $\mathcal{X} = \{X_S : S \subseteq \langle n \rangle, S \neq \emptyset, \langle n \rangle\}$ be a set of numbers, and let $D = (\langle n \rangle, E)$ be a digraph. Let A be the matrix defined by (2.12). The following are equivalent:*

(i) *An $n \times n$ matrix B satisfies*

$$\max_{(i,j) \in S \times S^c \cap E} \{b_{ij}\} = X_S \qquad \text{for every relevant subset } S \text{ of } \langle n \rangle.$$

(ii) *There exists an $\mathcal{X}$-complete set $\mathcal{P}$ of arcs in D such that $b_{ij} = a_{ij}$ whenever $(i,j) \in \mathcal{P}$ and $b_{ij} \leq a_{ij}$ whenever $(i,j) \in E \backslash \mathcal{P}$.*

Proof. (i) $\Rightarrow$ (ii): Let $(i,j) \in E$, and let S be a subset of $\langle n \rangle$ such that

$$(i,j) \in S \times S^c \quad \text{and} \quad X_S = \min_{\substack{T \subseteq \langle n \rangle \\ (i,j) \in (T \times T^c) \cap E}} \{X_T\}.$$

We have $a_{ij} = X_S = \max_{(k,l) \in S \times S^c \cap E} \{b_{kl}\} \geq b_{ij}$. Also, for every relevant subset S of $\langle n \rangle$ there exists $(i_S, j_S) \in S \times S^c \cap E$ such that $b_{i_S j_S} = X_S$, and since $X_S \geq a_{i_S j_S} \geq b_{i_S j_S} = X_S$, we have $a_{i_S j_S} = b_{i_S j_S}$. Therefore, $P = \{(i_S, j_S) : S \text{ is a relevant subset of } \langle n \rangle\}$ is an x-complete set.

(ii) $\Rightarrow$ (i): Let S be a relevant subset of $\langle n \rangle$. It follows from (ii) that

$$\max_{(i,j) \in S \times S^c \cap E(D(A))} \{b_{ij}\} \leq \max_{(i,j) \in S \times S^c \cap E(D(A))} \{a_{ij}\} = X_S.$$

Furthermore, since $\mathcal{P}$ is an x-complete set, there exists $(i,j) \in S \times S^c \cap \mathcal{P}$ such that $a_{ij} = X_S$. By (ii) we have $b_{ij} = a_{ij} = X_S$, and so $\max_{(i,j) \in S \times S^c \cap E} \{b_{ij}\} = X_S$. ∎

4. DIGRAPHS WITH *ROWB* INDEX 1

Motivated by Theorem 3.7, we include here a short discussion of digraphs with *ROWB* index 1. Note that a digraph has *ROWB* index 1 if and only if every arc in the graph is a one-way bridge.

DEFINITION 4.1. Let D be a digraph.

(i) A digraph D is said to be *strongly connected* if for every two vertices i and j in D there is a path from i to j in D.
(ii) A subdigraph S of D is said to be a *strong component* if S is a maximal strongly connected subdigraph of D.

(iii) A digraph D is said to be *minimal strongly connected* if D is strongly connected and no digraph obtained by removing an arc from D is strongly connected.

DEFINITION 4.2.   The *condensation* D^* of a digraph D is a digraph whose vertices are the strong components $S_1, \ldots, S_m$ of D, and where (S_i, S_j) is an arc in D^* with multiplicity k if there are exactly k arcs from vertices in S_i to vertices in S_j.

THEOREM 4.3.   *Let D be a digraph. The following are equivalent:*

(i) *D has ROWB indeed 1.*
(ii) *The strong components of D, as well as the condensation D^*, have ROWB index 1.*

Proof.   (i) $\Rightarrow$ (ii): Clearly, if D has $ROWB$ index 1,then every subdigraph of D has $ROWB$ index 1, and, in particular, the strong components of D have $ROWB$ index 1. Now, let (S_i, S_j) be an arc in D^*. By Definition 4.2, there exist a vertex u in S_i and a vertex v in S_j such that (u, v) is an arc in D. Since D has $ROWB$ index 1, (u, v) is a one-way bridge, and by Proposition 3.4, the digraph obtained by removing (u, v) from D has no path from u to v. It follows that (S_i, S_j) is a simple arc in D^* and that the digraph obtained by removing (S_i, S_j) from D^* has no path from S_i to S_j. By Proposition 3.4, D^* has $ROWB$ index 1.

(ii) $\Rightarrow$ (i): Let (u, v) be an arc in D. If u and v belong to the same strong component of D, then, since the strong components of D have $ROWB$ index 1, it follows that the digraph obtained by removing (u, v) from D has no path from u to v, and by Proposition 3.4, (u, v) is a one-way bridge in D. Assume now that u and v belong to two different strong components S_i and S_j of D respectively. Since D^* has $ROWB$ index 1, (S_i, S_j) is a one-way bridge in D^*, and it follows that (u, v) is the only arc in D from a vertex in S_i to a vertex in S_j, and that the digraph obtained by removing (S_i, S_j) from D^* has no path from S_i to S_j. It follows that the digraph obtained by removing (u, v) from D has no path from u to v, and by Proposition 3.4, (u, v) is a one-way bridge in D. ∎

Theorem 4.3 reduces the problem of characterizing digraphs of $ROWB$ index 1 to the problem of characterizing strongly connected digraphs with $ROWB$ index 1 (the strong components of D) as well as acyclic digraphs with $ROWB$ index 1 (the condensation D^*). We conclude the paper with a proposition that handles the strongly connected case.

PROPOSITION 4.4. *A strongly connected digraph D has ROWB index 1 if and only if D is minimal strongly connected.*

Proof. Let D be a minimal strongly connected digraph, and let (i,j) be an arc in D. Observe that, since D is strongly connected, if the digraph D' obtained by removing (i,j) from D has a path from i to j, then D' too is strongly connected. Thus, since D is minimal strongly connected, it follows that D' has no path from i to j, and by Proposition 3.4, (i,j) is a one-way bridge in D. Hence, D has $ROWB$ index 1. Conversely, assume that D has $ROWB$ index 1, and let (i,j) be an arc in D. By Proposition 3.4, the digraph obtained by removing (i,j) from D has no path from i to j, and so it is not strongly connected. Therefore, D is minimal strongly connected. ∎

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