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Existence of Matrices with Prescribed Off–Diagonal Block Element Sums

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Necessary and sufficient conditions are proven for the existence of a square matrix, over an arbitrary field, such that for every principal submatrix the sum of the elements in the row complement of the submatrix is prescribed. The problem is solved in the cases where the positions of the nonzero elements of A are contained in a given set of positions, and where there is no restriction on the positions of the nonzero elements of A. The uniqueness of the solution is studied as well. The results are used to solve the cases where the matrix is required to be symmetric and/or nonnegative entrywise.

1. INTRODUCTION

Problems concerning the existence of nonnegative matrices with prescribed row and column sums, often related to scaling problems, have been long studied, e.g. [1], [4], [5] and many other paper. It is clear that the conditions for existence determined in these papers force certain relations between the sums of the elements in the off–diagonal submatrices. In this paper we solve a related problem, namely we find necessary and sufficient conditions for the existence of a real matrix and of a nonnegative matrix with the sum of the elements in each off–diagonal submatrix prescribed.

We now describe the paper in some more detail.

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Our notation and definitions are given in Section 2. In Section 3 we study our problem over an arbitrary field, and where the positions of the nonzero elements of A are contained in a given set of positions. As a corollary we also solve the problem in the cases that we have no restriction on the positions of the nonzero elements of A (the "graph-free" case). The proof of our main theorem yields a graph theoretic characterization of those cases in which the existence problem has a unique solution. In the case that the solution is not unique, we describe all possible solutions.

The results of Section 3 are applied in Section 4 to obtain a solution for the existence problem of a symmetric matrix or even a nonnegative (entrywise) symmetric matrix with the required properties. The paper is concluded with a solution for the harder case where our matrix is required to be (not necessarily symmetric) nonnegative.

This paper is a companion paper to [3] where we solve the analogous problem for maxima in place of sums.

2. NOTATION AND DEFINITIONS

This Section contains all the notation and definitions used in this paper.

Notation 2.1 For a positive integer n We denote by $\langle n \rangle$ the set $\{1, \ldots, n\}$.

Notation 2.2 For a subset S of $\langle n \rangle$ we denote by S^C the complement of S in $\langle n \rangle$.

Notation 2.3 For a digraph D we denote by $E(D)$ and $V(D)$ the arc set and the vertex set of D respectively.

Definition 2.4 Let $D = (V, E)$ be a digraph. A digraph $D' = (V', E')$ is said to be a *subdigraph* of D if $V' \subseteq V$ and $E' \subseteq E$. We write $D' \subseteq D$ to indicate that D' is a subdigraph of D.

Definition 2.5 Let D be a digraph.

(i) A digraph D is said to be *strongly connected* if for every two vertices i and j of D there is a path from i and j in D.
(ii) A subdigraph D' of D is said to be a *strong component* of D if D' is a maximal strongly connected subdigraph of D.

Definition 2.6 *The symmetric part* of a digraph D is the subdigraph of D whose vertex set is $V(D)$, and whose arc set consists of all arcs (i, j) of D such that (j, i) too is an arc of D. A digraph D is said to be *symmetric* if it is equal to its symmetric part. We denote the symmetric part of D by Sym (D).

Definition 2.7 A set S of vertices in a digraph D is said to be D- *loose* if for every $i \in S$ and every $j \in V(D) \backslash S$ at least one of the arcs (i, j) and (j, i) is not present in D. By convention, $\emptyset$ and $V(D)$ are D-loose sets.

Remark 2.8 It is easy to verify that a set S of vertices in a digraph D is D-loose if and only if S is a union of strong components of Sym (D).

DEFINITION 2.9 The *(undirected) graph $G(D)$ of a digraph D* is the simple graph with vertex set $V(D)$, and where $\{i, j\}$ is an edge of $G(D)$ if and only if (i, j) or (j, i) is an arc of D.

DEFINITION 2.10 A graph G' is said to be a *subgraph* of a graph G if the vertex set and edge set of G' are contained in the vertex set and the edge set of G respectively.

DEFINITION 2.11

i) A graph G is *connected* if for every two vertices i and j of G there is a path between i and j in G.
ii) A subgraph G' of G is said to be a *component* of G if G' is a maximal connected subgraph of G.
iii) A *spanning tree* $\widetilde{G}$ for a connected graph G is connected subgraph of G, with the same vertex set as G, such that $\widetilde{G}$ has no cycles.

Remark 2.12

i) Observe that the graph of a strongly connected digraph is connected.
ii) As is well known, every connected graph G has a spanning tree. If G is not a tree then it has more than one spanning tree.

DEFINITION 2.13 *A leaf* in a graph is a vertex with just one vertex adjacent to it.

We conclude this Section with a definition and notation that involve matrices.

DEFINITION 2.14 Let A be an $n \times n$ matrix. The digraph $D(A)$ of A is defined as the diagraph with vertex set $\langle n \rangle$, and where (i, j) is an arc in $D(A)$ if and only if $a_{ij} \neq 0$.

NOTATION Let A be an $n \times n$ matrix, and let S be a nonempty proper subset of $\langle n \rangle$. We denote

$$R_S(A) = \sum_{\substack{i,j=1 \\ i \in S, j \notin S}}^{n} a_{ij}.$$

By convention, $R_\emptyset = R_{\langle n \rangle} = 0$.

3. THE GENERAL EXISTENCE AND UNIQUENESS PROBLEMS

In this Section we study the existence of a general matrix, over an arbitrary field, with a given digraph, and such that for every principal submatrix the sum of the elements in the row complement of the submatrix is given.

We start with three equivalent conditions on sets of numbers.

PROPOSITION 3.1 *Let $\{X_S : S \subseteq \langle n \rangle\}$ be a set of numbers such that $X_\phi = X_{\langle n \rangle} = 0$. The following are equivalent*:

(i) *For every subsets S_1, T_1, S_2 and T_2 of $\langle n \rangle$ such that*

$$S_1 \backslash T_1 = S_2 \backslash T_2;\ T_1 \backslash S_1 = T_2 \backslash S_2$$

we have

$$X_{S_1} + X_{T_1} - X_{S_1 \cup T_1} - X_{S_1 \cap T_1} = X_{S_2} + X_{T_2} - X_{S_2 \cup T_2} - X_{S_2 \cap T_2}.$$

(ii) *For every subsets S and T of $\langle n \rangle$ such that*

$$S \backslash T = \{r\}; T \backslash S = \{t\}$$

We have

$$X_S + X_T - X_{S \cup T} - X_{S \cap T} = X_{\{r\}} + X_{\{t\}} - X_{\{r,t\}}.$$

(iii) *For every nonempty subset S of $\langle n \rangle$ we have*

$$X_S = \sum_{i \in S} X_{\{i\}} - \sum_{\substack{i,j \in S \\ i<j}} (X_{\{i\}} + X_{\{j\}} - X_{\{i,j\}}). \tag{3.2}$$

Proof

(i) $\Rightarrow$ (ii) is immediate, by applying (i) to the sets S, T, $\{r\}$ and $\{t\}$.

(ii) $\Rightarrow$ (iii). We prove this implication by induction on $|S|$. It is easy to verify that (3.2) holds for $|S| = 1, 2$. Assume that (3.2) holds for $|S| < k$ where $k \geq 2$, and let $|S| = k$. Let r, $t \in S$. Define $S_1 = S \backslash \{t\}$ and $T_1 = S \backslash \{r\}$. By (ii) we have

$$X_{\{r\}} + X_{\{t\}} - X_{\{r,t\}} = X_{S_1} + X_{T_1} - X_S - X_{S \backslash \{r,t\}},$$

and hence

$$X_S = X_{S_1} + X_{T_1} - X_{S \backslash \{r,t\}} + X_{\{r,t\}} - X_{\{r\}} - X_{\{t\}}.$$

Applying the inductive assumption to X_{S_1}, X_{T_1} and $X_{S \backslash \{r,t\}}$ yields

$$\begin{aligned} X_S &= \sum_{\substack{i \in S \\ i \neq t}} X_{\{i\}} - \sum_{\substack{i,j \in S \\ i,j \neq t \\ i<j}} (X_{\{i\}} + X_{\{j\}} - X_{\{i,j\}}) + \sum_{\substack{i \in S \\ i \neq r}} X_{\{i\}} - \sum_{\substack{i,j \in S \\ i,j \neq r \\ i<j}} (X_{\{i\}} + X_{\{j\}} - X_{\{i,j\}}) \\ &\quad - \sum_{\substack{i \in S \\ i \neq r,t}} X_{\{i\}} + \sum_{\substack{i,j \in S \\ i,j \neq r,t \\ i<j}} (X_{\{i\}} + X_{\{j\}} - X_{\{i,j\}}) + X_{\{r,t\}} - X_{\{r\}} - X_{\{t\}} \\ &= \sum_{i \in S} X_{\{i\}} - \sum_{\substack{i,j \in S \\ i<j}} (X_{\{i\}} + X_{\{j\}} - X_{\{i,j\}}). \end{aligned}$$

(iii) $\Rightarrow$ (ii). Let S and T be subsets of $\langle n \rangle$. By (3.2) we have

$$\begin{aligned} X_S &= \sum_{i \in S \cap T} X_{\{i\}} + \sum_{i \in S \backslash T} X_{\{i\}} - \sum_{\substack{i,j \in S \\ i<j}} (X_{\{i\}} + X_{\{j\}} - X_{\{i,j\}}), \\ X_T &= \sum_{i \in S \cap T} X_{\{i\}} + \sum_{i \in T \backslash S} X_{\{i\}} - \sum_{\substack{i,j \in T \\ i<j}} (X_{\{i\}} + X_{\{j\}} - X_{\{i,j\}}), \\ X_{S \cap T} &= \sum_{i \in S \cap T} X_{\{i\}} - \sum_{\substack{i,j \in S \cap T \\ i<j}} (X_{\{i\}} + X_{\{j\}} - X_{\{i,j\}}), \end{aligned}$$

and

$$X_{S\cup T}=\sum_{i\in S\cap T}X_{\{i\}}+\sum_{i\in S\setminus T}X_{\{i\}}+\sum_{i\in T\setminus S}X_{\{i\}}-\sum_{\substack{i,j\in S\cup T\\ i<j}}(X_{\{i\}}+X_{\{j\}}-X_{\{i,j\}}).$$

It follows that

$$X_S+X_T-X_{S\cap T}-X_{S\cup T}=\sum_{\substack{i\in S\setminus T\\ j\in T\setminus S\\ i<j}}(X_{\{i\}}+X_{\{j\}}-X_{\{i,j\}})+\sum_{\substack{i\in T\setminus S\\ j\in S\setminus T\\ i<j}}(X_{\{i\}}+X_{\{j\}}-X_{\{i,j\}}) \tag{3.3}$$

Since the right hand side of (3.3) depends only on $S\setminus T$ and $T\setminus S$, (i) follows. ■

We continue with a lemma that involves matrices.

LEMMA 3.4 *Let A be an $n\times n$ matrix, let $\{X_S : S\subseteq\langle n\rangle\}$ be a set of numbers such that $X_\phi=X_{\langle n\rangle}=0$, and let T be a subset of $\langle n\rangle$. If*

$$\begin{cases} a_{ij}+a_{ji}=X_{\{i\}}+X_{\{j\}}-X_{\{i,j\}}, & i,j\in T,\ i\neq j\\ R_{\{i\}}(A)=X_{\{i\}}, & i\in T,\\ X_T=\sum_{i\in T}X_{\{i\}}-\sum_{\substack{i,j\in T\\ i<j}}(X_{\{i\}}+X_{\{j\}}-X_{\{i,j\}}), & \end{cases} \tag{3.5}$$

then

$$R_T(A)=X_T.$$

Proof $R_T(A)$ is equal to the sum of all off-diagonal elements in the rows of A indexed by T, minus the sum of the off-diagonal elements in the principal submatrix of A whose rows and columns are indexed by T. Thus, in view of (3.5), we obtain

$$R_T(A)=\sum_{i\in T}R_{\{i\}}(A)-\sum_{\substack{i,j\in T\\ i<j}}(a_{ij}+a_{ji})=\sum_{i\in T}X_{\{i\}}-\sum_{\substack{i,j\in T\\ i<j}}(X_{\{i\}}+X_{\{j\}}-X_{\{i,j\}})=X_T$$

proving our assertion. ■

We now prove our main result.

THEOREM 3.6 *Let $\{X_S : S\subseteq<n>\}$ be a set of numbers such that $X_\emptyset=X_{\langle n\rangle}=0$, and let $D=(\langle n\rangle,E)$ be a digraph. The following are equivalent:*

(i) *There exists an $n\times n$ matrix A, with $D(A)\subseteq D$, such that*

$$R_S(A)=X_S,\quad S\subseteq\langle n\rangle. \tag{3.7}$$

(ii) *Any of the equivalent conditions of Proposition* (3.1) *holds, as well as*

$$X_{\{i,j\}}=X_{\{i\}}+X_{\{j\}},\quad \textit{whenever}\quad (i,j),(j,i)\notin E, \tag{3.8}$$

and

$$X_S=\sum_{(i,j)\in S\times S^c\cap E}(X_{\{i\}}+X_{\{j\}}-X_{\{i,j\}}),\quad \textit{for every } D-\textit{loose set } S. \tag{3.9}$$

(iii) *Any of the equivalent conditions of Propostion* (3.1) *holds, as well as* (3.8) *and*

$$X_S = \sum_{(i,j)\in S\times S^C\cap E} (X_{\{i\}} + X_{\{j\}} - X_{\{i,j\}}), \quad \begin{array}{l}\textit{whenever } S \textit{ is the vertex set of}\\ \textit{a component of } G\,(\mathrm{Sym}(D))\end{array} \tag{3.10}$$

Proof (i) $\Rightarrow$ (ii). We prove that (i) implies Condition (i) of Proposition (3.1), as well as (3.8) and (3.9). Let S and T be subsets of $\langle n\rangle$. It is easy to verify that

$$R_S(A) + R_T(A) - R_{S\cup T}(A) - R_{S\cap T}(A) = \sum_{i\in S\setminus T, j\in T\setminus S} a_{ij} + \sum_{i\in T\setminus S, j\in S\setminus T} a_{ij}.$$

In view of (i) we have

$$X_S + X_T - X_{S\cup T} - X_{S\cap T} = \sum_{i\in S\setminus T, j\in T\setminus S} a_{ij} + \sum_{i\in T\setminus S, j\in S\setminus T} a_{ij}. \tag{3.11}$$

Since the right hand side of (3.11) depends only on $S\setminus T$ and $T\setminus S$, Condition (i) of Proposition (3.1) follows. Next, observe that

$$a_{ij} + a_{ji} = R_{\{i\}}(A) + R_{\{j\}}(A) - R_{\{i,j\}}(A), \quad i,j\in\langle n\rangle, i\neq j.$$

Therefore, if $(i,j),(j,i)\neq E$ then it follows that

$$0 = R_{\{i\}}(A) + R_{\{j\}}(A) - R_{\{i,j\}}(A).$$

In view of (i), (3.8) follows. Finally, let S be a D-loose set, and let $(i,j)\in S\times S^C\cap E$. By Definition (2.7), we have $(j,i)\notin E$. Therefore, it follows that

$$a_{ij} = a_{ij} + a_{ji} = R_{\{i\}}(A) + R_{\{j\}}(A) - R_{\{i,j\}}(A).$$

In view of (i), (3.9) follows.
(ii) $\Rightarrow$ (iii) is immediate, in view of Remark (2.8).
(iii) $\Rightarrow$ (i). Here we prove that Condition (iii) of Proposition (3.1), together with (3.8) and (3.10), implies (i). We construct a matrix satisfying (i), using the following algorithm.

ALGORITHM 3.12

Step 0 We assign arbitrary values to the diagonal elements of A, and we let

$$a_{ij} = \begin{cases} 0, & (i,j)\notin E\\ X_{\{i\}} + X_{\{j\}} - X_{\{i,j\}}, & (i,j)\in E, (j,i)\notin E.\end{cases} \tag{3.13}$$

Observe that in view of (3.8), (3.13) does not yield a contradiction in the case that $(i,j),(j,i)\notin E$.

Let $C_1,\ldots,C_m$ be the strong components of the symmetric part of D.

Step $k, k=1,\ldots,m$
We choose a spanning tree G^1 for $G(C_k)$. We assign arbitrary values to the elements in the positions $\{(i,j)\in E(C_k) : \{i,j\}\notin E(G^1), i<j\}$, and we assign the symmetrically located elements values using the formula $a_{ji} = X_{\{i\}} + X_{\{j\}} - X_{\{i,j\}} - a_{ij}$. Let $p_k = |V(C_k)| - 1$ be the number of edges in G^1.

Step k.t, $t = 1, \dots, p_k$
Take a leaf u in G^t, with the vertex v adjacent to u in G^t. Observe that a_{uv} is the only element in the u-th row of A that has not yet been determined. We define a_{uv} to be

$$a_{uv} = X_{\{u\}} - \sum_{\substack{j=1 \\ j \neq u,v}}^{n} a_{uj} \tag{3.14}$$

and we let $a_{vu} = X_{\{u\}} + X_{\{v\}} - X_{\{u,v\}} - a_{uv}$. Finally we define G^{t+1} to be the graph obtained by removing the edge $\{u, v\}$ from G^t.

Step m + 1 : STOP
First, note that Algorithm (3.12) constructs a matrix A satisfying

$$a_{ij} + a_{ji} = X_{\{i\}} + X_{\{j\}} - X_{\{i,j\}}, \quad i, j \in n, \ i \neq j. \tag{3.15}$$

Let $i \in \langle n \rangle$. Observe that the i-th row of A is completely determined in Step 0 of Algorithm (3.12) if and only if $\{i\}$ is a D-loose set, that is if and only if the singleton $\{i\}$ is a component of $G(\mathrm{Sym}(D))$. By (3.10) and (3.13) we have

$$R_{\{i\}}(A) = \sum_{\substack{j=1 \\ j \neq i, (i,j) \in E}}^{n} (X_{\{i\}} + X_{\{j\}} - X_{\{i,j\}} = X_{\{i\}}.$$

We complete the assignment of each of the other rows of A in one of the Steps $k.t$, $t = 1, \dots, p_k, k = 1, \dots, m$. In each such step we determine two elements: a_{uv} and a_{vu}. The element a_{uv} is the last to be determined in the u-th row, and is determined so that $R_{\{u\}}(A) = X_{\{u\}}$. The element a_{vu} is the last to be determined in the v-th row if and only if $t = p_k$, and it is the last element to be determined in the rows indexed by the set $S = V(C_k)$. So, let us assume that $t = p_k$. It follows that

$$R_{\{i\}}(A) = X_{\{i\}}, i \in S \backslash \{v\}. \tag{3.16}$$

Since we have (3.2), (3.15) and (3.16), it follows from Lemma (3.4) that

$$R_{S \backslash \{v\}}(A) = X_{S \backslash \{v\}}. \tag{3.17}$$

Since C_k is a strong component of Sym (D), it follows from (3.10) that

$$X_S = \sum_{(i,j) \in S \times S^C \cap E} (X_{\{i\}} + X_{\{j\}} - X_{\{i,j\}}).$$

As is observed in Ramark (2.8), S is a D–loose set. Therefore, all the elements in the rows indexed by S and the columns indexed by S^C are determined by (3.13) and so

$$R_S(A) = \sum_{(i,j) \in S \times S^C \cap E} (X_{\{i\}} + X_{\{j\}} - X_{\{i,j\}}) = X_S. \tag{3.18}$$

It follows from (3.15) that

$$\sum_{j \in S \backslash \{v\}} (a_{vj} + a_{jv}) = \sum_{j \in S \backslash \{v\}} (X_{\{v\}} + X_{\{j\}} - X_{\{v,j\}}) = \sum_{\substack{i,j \in S \\ i=v \text{ or } j=v \\ i<j}} (X_{\{v\}} + X_{\{j\}} - X_{\{v,j\}}). \tag{3.19}$$

Observe that

$$R_{S\setminus\{v\}}(A)+R_{\{v\}}(A)-R_S(A)=\sum_{j\in S\setminus\{v\}}(a_{vj}+a_{jv}).$$

It now follows from (3.17), (3.18) and (3.19) that

$$R_{\{v\}}(A)=X_S-X_{S\setminus\{v\}}+\sum_{\substack{i,j\in S\\ i=v \text{ or } j=v\\ i<j}}(X_{\{v\}}+X_{\{j\}}-X_{\{v,j\}}).$$

Using (3.2) we obtain

$$\begin{aligned}R_{\{v\}}(A)&=\sum_{i\in S}X_{\{i\}}-\sum_{\substack{i,j\in S\\ i<j}}(X_{\{i\}}+X_{\{j\}}-X_{\{i,j\}})-\sum_{i\in S\setminus\{v\}}X_{\{i\}}\\&\quad+\sum_{\substack{i,j\in S\setminus\{v\}\\ i<j}}(X_{\{i\}}+X_{\{j\}}-X_{\{i,j\}})\\&\quad+\sum_{\substack{i,j\in S\\ i=v \text{ or } j=v\\ i<j}}(X_{\{v\}}+X_{\{j\}}-X_{\{v,j\}})=X_{\{v\}}.\end{aligned}$$

It now follows that

$$R_{\{i\}}(A)=X_{\{i\}},\quad i\in\langle n\rangle. \tag{3.20}$$

Since we have (3.2) for every subset of S of $\langle n\rangle$, as well as (3.15) and (3.20), it follows from Lemma (3.4) that (3.7) holds. ■

Our proof of Theorem (3.6) also yields an answer to the uniqueness question.

THEOREM 3.21 *Let $\{X_S : S\subseteq\langle n\rangle\}$ be a set of numbers such that $X_\emptyset=X_{\langle n\rangle}=0$, and let $D=(\langle n\rangle,E)$ be a digraph. The following are equivalent:*

(i) *There exists a unique $n\times n$ matrix A, with $D(A)\subseteq D$, such that $R_S(A)=X_S$, $S\subseteq\langle n\rangle$.*
(ii) *Any of the equivalent conditions of Theorem* (3.6) *holds, and the graphs of the strong component of the symmetric part of D are all trees.*

Proof

(i) ⇒(ii). If the graph of some strong component of the symmetric part of D is not a tree then, using Algorithm (3.12), at least one of the elements of A can be assigned an arbitrary value, and hence there exist infinitely many matrices satisfying our requirements.

(ii) ⇒(i). If the graphs of the strong component of the symmetric part of D are all trees then, for every such a component C_k, $G(C_k)$ is the (only) spanning tree G^1 for itself, and the set $\{(i,j)\in E(C_k):\{i,j\}\notin E(G^1), i<j\}$ is empty. In view of (3.15), Algorithm (3.12) constructs the unique matrix A with $D(A)\subseteq D$, satisfying (3.7). ■

Theorem (3.6) provides a necessary and sufficient condition for the existence of a matrix A, with $D(A)\subseteq D$, satisfying (3.7). Theorem (3.21) provides a necessary and sufficient condition for the uniqueness of such a matrix. The following theorem essentially describes all solutions of this problem.

THEOREM 3.22 *Let A and B be $n\times n$ matrices. The following are equivalent:*

(i) *For every subset S of $\langle n\rangle$ we have $R_S(A) = R_S(B)$.*
(ii) *$A - B = C + D$, where D is any diagonal matrix and C is an anti-symmetric matrix satisfying $R_S(C) = 0, S \subseteq \langle n\rangle$.*
(iii) *$A - B = C + D$, where D is any diagonal matrix and C is an anti-symmetric matrix with zero row sums.*

Proof

(i) $\Rightarrow$ (ii). We let D be the diagonal matrix with the same diagonal elements as $A - B$, and let $C = A - B - D$. The matrix C has zero diagonal elements. Also, (i) implies that $R_S(C) = 0$, $S \subseteq \langle n\rangle$. Therefore, it follows that

$$c_{ij} + c_{ji} = R_{\{i\}}(C) + R_{\{j\}}(C) - R_{\{i,j\}}(C) = 0 \quad , i, j \in \langle n\rangle, i \neq j,$$

which means that C is an anti-symmetric matrix.

(ii) $\Rightarrow$ (i). We have $R_S(A) = R_S(B) + R_S(C) + R_S(D), S \subseteq \langle n\rangle$. Since D is a diagonal matrix, we have $R_S(D) = 0, S \subseteq \langle n\rangle$. Thus, it follows from (ii) that $R_S(A) = R_S(B)$, $S \subseteq \langle n\rangle$.

(ii) $\Rightarrow$ (iii) is immediate.

(iii) $\Rightarrow$(ii). Let S be any subset of $\langle n\rangle$. Note that $R_S(C)$ is equal to the sum of all row sums of the rows of C indexed by S, all are given to be 0, minus the sum of the elements in the principle submatrix of C whose rows and columns are indexed by S. Since C is an anti-symmetric matrix, the latter sum is equal to 0, and it follows that $R_S(C) = 0$. ■

We conclude this Section with the following "graph-free" version of the Theorem (3.6), obtained from Theorem (3.6) by choosing D to be the complete digraph with n vertices.

THEOREM 3.23 *Let $\{X_S : S \subseteq \langle n\rangle\}$ be a set of numbers such that $X_\phi = X_{\langle n\rangle} = 0$. The following are equivalent:*

(i) *There exist an $n\times n$ matrix A such that $R_S(A) = X_S, S \subseteq \langle n\rangle$.*
(ii) *Any of the equivalent conditions of Proposition* (3.1) *holds.*

4. THE SYMMETRIC AND THE NONNEGATIVE CASES

Clearly, when looking for symmetric matrices A to satisfy Theorem (3.6.i), we may assume, without loss of generality, that D is a symmetric digraph. The following theorem is a corollary of Theorem (3.6) in the symmetric case.

THEOREM 4.1 *Let $\{X_S : S \subseteq \langle n\rangle\}$ be a set of numbers such that $X_\emptyset = X_{\langle n\rangle} = 0$, and let $D = (\langle n\rangle, E)$ be a digraph. The following are equivalent:*

(i) *There exists a symmetric $n\times n$ matrix A, with $D(A) \subseteq D$, such that $R_S(A) = X_S, S \subseteq \langle n\rangle$.*

(ii) *Any of the equivalent conditions of Theorem* (3.6) *holds, and* $X_S = X_{S^c}, S \subseteq \langle n\rangle$.

Proof

(i) $\Rightarrow$ (ii). Since A is symmetric, we have $R_S(A) = R_S(A^T) = R_{S^c}(A), S \subseteq \langle n\rangle$. Thus, (i) yields that $X_S = X_{S^c}, S \subseteq \langle n\rangle$. The rest of the implication follows from Theorem (3.6).

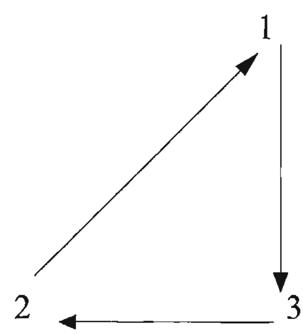
(ii) $\Rightarrow$ (i). By Theorem (3.6), there exists a (not necessarily symmetric) $n \times n$ matrix B, with $D(B) \subseteq D$, such that $R_S(B) = X_S, S \subseteq \langle n\rangle$. The symmetric matrix $A = \frac{1}{2}(B + B^T)$ satisfies

$$R_S(A) = \frac{1}{2}(R_S(B) + R_S(B^T)) = \frac{1}{2}(R_S(B) + R_{S^c}(B) = \frac{1}{2}(X_S + X_{S^c}) = X_S, S \subseteq \langle n\rangle.$$

Also, since D is a symmetric digraph it follows that $D(B^T) \subseteq D$, and hence $D(A) \subseteq D$. ■

We remark that, in general, unless D is given to be a symmetric digraph, Conditions (ii) and (iii) of the Theorem (3.6) together with $X_S = X_{S^c}, S \subseteq \langle n\rangle$, are not sufficient for the existence of a symmetric matrix A, with $(D(A) \subseteq D$, satisfying (3.7). This is demonstrated by the following example.

EXAMPLE 4.2 Let D be the digraph



and let $X_{\{1\}} = X_{\{2\}} = X_{\{3\}} = X_{\{1,2\}} = X_{\{1,3\}} = X_{\{2,3\}} = 1$. The matrix

$$A = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

has digraph D and satisfies (3.7). Furthermore, since Sym(D) has no arcs, it follows by Theorem (3.21) that A is the unique matrix with digraph D satisfying (3.7). Therefore, although $X_S = X_{S^c}, S \subseteq \langle n\rangle$, there exists no symmetric matrix satisfying our requirements.

The graph-free corollary of Theorem (4.1) is the following.

THEOREM 4.3 *Let $\{X_S : S \subseteq \langle n\rangle\}$ be a set of numbers such that $X_\emptyset = X_{\langle n\rangle} = 0$. The Following are equivalent*:

(i) *There exists a symmetric $n \times n$ matrix A such that $R_S(A) = X_S, S \subseteq \langle n\rangle$.*
(ii) *Any of the equivalent conditions of Proposition* (3.1) *holds, and $X_S = X_{S^c}, S \subseteq \langle n\rangle$.*

In the nonnegative symmetric case we obtain

THEOREM 4.4 *Let $\{X_S : S \subseteq \langle n\rangle\}$ be a set of numbers such that $X_\emptyset = X_{\langle n\rangle} = 0$, and let $D = ((\langle n\rangle), E)$ be a digraph. The following are equivalent*:

(i) *There exists a symmetric nonnegative $n \times n$ matrix A, with $D(A) \subseteq D$, such that $R_S(A) = X_S, S \subseteq \langle n\rangle$.*
(ii) *Condition* (ii) *in Theorem* (4.1) *holds, as well as*

$$X_{\{i,j\}} \leq X_{\{i\}} + X_{\{j\}}, \quad i, j \in \langle n\rangle, \ i \neq j. \tag{4.5}$$

Proof

(i) $\Rightarrow$ (ii). Since A is nonnegative, we have $a_{ij} + a_{ji} = R_{\{i\}}(A) + R_{\{j\}}(A) - R_{\{i,j\}}(A) \geq 0$.
Thus, (i) yields (4.5). The rest of the implication follows from Theorem (4.1).
(ii) $\Rightarrow$ (i). By Theorem (4.1), there exists a symmetric $n \times n$ matrix A, with $D(A) \subseteq D$, such that $R_S(A) = X_S, S \subseteq \langle n\rangle$. Since

$$a_{ij} = \frac{1}{2}(R_{\{i\}}(A) + R_{\{j\}}(A) - R_{\{i,j\}}(A)) = \frac{1}{2}(X_{\{i\}} + X_{\{j\}} - X_{\{i,j\}}),$$

it follows from (4.5) that A is nonnegative. ■

The graph-free version of Theorem (4.4) is

THEOREM 4.6 *Let $\{X_S : S \subseteq \langle n\rangle\}$ be a set of numbers such that $X_\emptyset = X_{\langle n\rangle} = 0$. The following are equivalent*:

(i) *There exists a symmetric nonnegative $n \times n$ matrix A such that $R_S(A) = X_S, S \subseteq \langle n\rangle$.*
(ii) *Condition* (ii) *in Theorem* (4.3) *holds, as well as* (4.5).

The relation between Theorems (4.1) and (4.4), that is the fact that the only additional condition needed in the *nonnegative* symmetric case is (4.5), raises the question whether the following satements are true:

(i) Let $\{X_S : S \subseteq \langle n\rangle\}$ be a set of numbers such that $X_\emptyset = X_{\langle n\rangle} = 0$, and let $D = (\langle n\rangle, E)$, be a digraph. Then there exists a nonnegative $n \times n$ matrix A, with $D(A) \subseteq D$ satisfying $R_S(A) = X_S, S \subseteq \langle n\rangle$, if and only if any of the equivalent conditions in Theorem (3.6), as well as (4.5), holds.
(ii) Let $\{X_S : S \subseteq \langle n\rangle\}$ be a set of numbers such that $X_\emptyset = X_{\langle n\rangle} = 0$. Then there exists a nonnegative $n \times n$ matrix A satisfying $R_S(A) = X_S, S \subseteq \langle n\rangle$, if and only if any of the equivalent conditions in Propositoin (3.1), as well as (4.5), holds.

The answer to this question is negative as demonstrated by the following example.

EXAMPLE 4.7 Let $n = 3$, let $X_{\{1\}} = X_{\{2\}} = X_{\{3\}} = 1, X_{\{1,2\}} = X_{\{1,3\}} = 2, X_{\{2,3\}} = -1$, and let D be the complete digraph with three vertices. It is easy to check that this set satisfies the equivalent conditions in Theorem (3.6). Indeed, the matrix

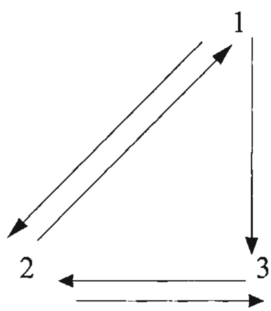
$$A = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 2 \\ 0 & 1 & 0 \end{pmatrix}$$

satisfies $R_S(A) = X_S, S \subseteq \langle n \rangle$. Also, we have (4.5). Nevertheless, there exists no nonnegative matrix A satisfying (3.7), since $X_{\{2,3\}} < 0$.
Example (4.7) suggests that another necessary condition for the existence of a nonnegative matrix A satisfying (3.7) should be

$$X_S \geq 0, \quad S \subseteq \langle n \rangle. \tag{4.8}$$

Indeed, we shall prove that in the graph-free case, Condition (4.8), together with (4.5) and any of the equivalent conditions in Proposition (3.1), forms a necessary and sufficient condition for the existence of a nonngeative matrix A satisfying (3.7), see Corollary (4.14). However, under pattern restrictions, Condition (4.8), together with (4.5) and any other of the equivalent conditions in Theorem (3.6), does not form a necessary and sufficient condition for the existence of a nonnegative matrix A satisfying (3.7), as demonstrated by the following example.

EXAMPLE 4.9 Let $n = 3$, let $X_\emptyset = 0$, $X_{\{1\}} = 3$, $X_{\{2\}} = X_{\{3\}} = 3$, $X_{\{1,2\}} = 3$, $X_{\{1,3\}} = 2$, $X_{\{2,3\}} = 0$ and $X_{\{1,2,3\}} = 0$ and let D be the digraph.



It is easy to verify that Condition (ii) of Theorem (3.6) holds (for this purpose check Condition (iii) of Proposition (3.1)). Also, we have (4.5) and (4.8). Indeed, the matrix

$$A = \begin{pmatrix} 0 & 1 & 2 \\ 1 & 0 & 1 \\ 2 & 1 & 0 \end{pmatrix}$$

satisfies (3.7). Nevertheless, there exists no 3×3 matrix A, with $D(A) \subseteq D$, satisfying (3.7). For such a matrix we would have $a_{13} + a_{31} = X_{\{1\}} + X_{\{3\}} - X_{\{1,3\}} = 4$. Since $(3,1) \notin E(D)$, it would follow that $a_{13} = 4$, implying that $X_{\{1\}} \geq 4$, while $X_{\{1\}} = 3$. Under pattern restrictions we therefore need a modified version of (4.8), that is Condition (4.11) below.

THEOREM 4.10 *$\{X_S : S \subseteq \langle n\rangle\}$ be a set of nonnegative numbers such that $X_\phi = X_{\langle n\rangle} = 0$ and let $D = (\langle n\rangle, E)$ be a digraph. The following are equivalent:*

(i) *There exists a nonnegative $n \times n$ matrix A, with $D(A) \subseteq D$, such that $R_S(A) = X_S, S \subseteq \langle n\rangle$.*

(ii) *Any of the equivalent conditions in Theorem (3.6) holds, as well as (4.5) and*

$$X_S \geq \sum_{\substack{(i,j)\in S\times S^C \\ (j,i)\notin E(D)}} (X_{\{i\}} + X_{\{j\}} - X_{\{i,j\}}, \quad S \subseteq \langle n\rangle). \tag{4.11}$$

Proof (i) $\Rightarrow$ (ii). Since A is nonnegative, we have $a_{ij} + a_{ji} = R_{\{i\}}(A) + R_{\{j\}}(A) - R_{\{i,j\}}(A) \geq 0$.Thus, (i) yields (4.5). Also, if $(j,i) \notin E(D)$ then $a_{ij} = a_{ij} + a_{ji} = R_{\{i\}}(A) + R_{\{j\}} - R_{\{i,j\}}(A) = X_{\{i\}} + X_{\{j\}} - X_{\{i,j\}}$. It now follows that for every subset S of $\langle n\rangle$ we have

$$X_S = \sum_{(i,j)\in S\times S^C} a_{ij} \geq \sum_{\substack{(i,j)\in S\times S^C \\ (j,i)\notin E(D)}} a_{ij} = \sum_{\substack{(i,j)\in S\times S^C \\ (j,i)\notin E(D)}} (X_{\{i\}} + X_{\{j\}} - X_{\{i,j\}})$$

(ii) $\Rightarrow$ (i). Define a nonnegative symmetric matrix C, with zero diagonal elements, by

$$c_{ij} = \begin{cases} X_{\{i\}} + X_{\{j\}} - X_{\{i,j\}}, & i,j \in \langle n\rangle, i \neq j \\ 0, & i,j \in \langle n\rangle, i = j \end{cases}$$

It follows by (3.2) and (4.11) that for every subset S of $\langle n\rangle$ we have

$$\begin{aligned} \sum_{i\in S} X_{\{i\}} &\geq \sum_{\substack{i,j\in S \\ i<j}} (X_{\{i\}} + X_{\{j\}} - X_{\{i,j\}}) + \sum_{\substack{(i,j)\in S\times S^C \\ (j,i)\notin E(D)}} (X_{\{i\}} + X_{\{j\}} - X_{\{i,j\}}) \\ &= \sum_{\substack{i,j\in S \\ i<j}} c_{ij} + \sum_{\substack{(i,j)\in S\times S^C \\ (j,i)\notin E(D)}} c_{ij} = \frac{1}{2}\sum_{i,j\in S} c_{ij} + \sum_{\substack{(i,j)\in S\times S^C \\ (j,i)\notin E(D)}} c_{ij}. \end{aligned} \tag{4.12}$$

By (3.2) and (3.9), for every D–loose set S we have

$$\begin{aligned} \sum_{i\in S} X_{\{i\}} &= \sum_{\substack{i,j\in S \\ i<j}} (X_{\{i\}} + X_{\{j\}} - X_{\{i,j\}}) + \sum_{(i,j)\in S\times S^C\cap E} (X_{\{i\}} + X_{\{j\}} - X_{\{i,j\}}) \\ &= \frac{1}{2}\sum_{i,j\in S} c_{i,j} + \sum_{(i,j)\in S\times S^C\cap E} c_{ij}. \end{aligned} \tag{4.13}$$

In view of (4.12) and (4.13), it follows from Theorem (3.3) in [2] that there exists a nonnegative $n \times n$ matrix A, with $D(A) \subseteq D$, with row sums $X_{\{1\}} \ldots, X_{\{n\}}$, and such that $A + A^T = C$. We have

$$a_{ij} + a_{ji} = c_{ij} = X_{\{i\}} + X_{\{j\}} - X_{\{i,j\}}, \quad i,j \in \langle n\rangle, i \neq j.$$

Also, since A has zero diagonal elements it follows that $R_{\{i\}}(A) = X_{\{i\}}, i \in \langle n\rangle$. Since we have (3.2) for every subset S of $\langle n\rangle$, it now follows from Lemma (3.4) that the nonnegative matrix A satisfies (3.7). ∎

If we choose D to be the complete digraph with n vertices then we obtain the following graph-free corollary of Theorem (4.10)

COROLLARY 4.14 *Let* $\{X_S : S \subseteq \langle n \rangle\}$ *be a set of nonnegative numbers such that* $X_\phi = X_{\langle n \rangle} = 0$. *The following are equivalent*:

(i) *There exists a nonnegative* $n \times n$ *matrix* A *such that* $R_S(A) = X_S$, $S \subseteq \langle n \rangle$.
(ii) *Any of the equivalent conditions in Theorem* (3.6) *it holds, as well as* (4.5) *and* (4.8).

Finally, one may ask whether we can drop Condition (4.5) from Theorem (4.10.ii) and Corollary (4.14.ii). The answer to this question is negative, as demonstrated by the following example.

EXAMPLE 4.15 Let $n = 3$, let $X_\emptyset = 0$, $X_{\{1\}} = 2$, $X_{\{2\}} = 4$, $X_{\{3\}} = 4$, $X_{\{1,2\}} = 1$, $X_{\{1,3\}} = 5$, $X_{\{2,3\}} = 2$, and $X_{\{1,2,3\}} = 0$, and let D be the complete digraph with three vertices. It is easy to verify that Condition (ii) of Theorem (3.6) holds (for this purpose check Condition (iii) of Proposition (3.1)). Also, we have (4.11), which is equivalent to (4.8) in this case. Indeed, the matrix

$$A = \begin{pmatrix} 0 & 4 & -2 \\ 1 & 0 & 3 \\ 1 & 1 & 0 \end{pmatrix}$$

satisfies (3.7). However, by Theorem (4.10) there exists no nonnegative matrix satisfying (3.7), since $X_{\{1,3\}} = 5 > 4 = X_{\{1\}} + X_{\{3\}}$.

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