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Sum Decompositions of Symmetric Matrices

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Dedicated to Chandler Davis

Submitted by Shmuel Friedland

ABSTRACT

Given a symmetric $n \times n$ matrix A and n numbers $r_1, \ldots, r_n$, necessary and sufficient conditions for the existence of a matrix B, with a given zero pattern, with row sums $r_1, \ldots, r_n$, and such that $A = B + B^T$ are proven. If the pattern restriction

*The research of all three authors was supported by the Fundação Calouste Gulbenkian, Lisboa.

[†]The research of this author was carried out within the activity of the Centro de Álgebra da Universidade de Lisboa.

[‡]The research of these authors was supported by their joint grant 90-00434 from the United States–Israel Binational Science Foundation, Jerusalem, Israel.

§The research of these authors was supported in part by NSF grant DMS-9123318.

655 Avenue of the Americas, New York, NY 10010 0024-3795/94/$7.00

is relaxed, then such a matrix B exists if and only if the sum $r_1 + \cdots + r_n$ is equal to half the sum of the elements of A. The case where A and B are nonnegative matrices is solved as well.

1. INTRODUCTION

The question of the existence of an entrywise nonnegative $m \times n$ matrix B with row sums $r_1, \dots, r_m$ and column sums $c_1, \dots, c_n$ is of long standing, e.g. [1], [4], and [5]. In particular, it follows from [1] that such a matrix B exists if and only if $r_1 + \cdots + r_m = c_1 + \dots + c_n$. In fact, the author in [1] goes further and studies the existence of such a matrix with a given zero pattern. Some obviously necessary conditions turn to be also sufficient. In this paper we study the case where the matrix B is not necessarily nonnegative and where the matrix $B + B^T$ is given, namely, given a symmetric $n \times n$ matrix A and n numbers $r_1, \dots, r_n$, we find necessary and sufficient conditions for the existence of a matrix B satisfying

$$A = B + B^T; \qquad R_i(B) = r_i, \quad i = 1, 2, \dots, n, \tag{1.1}$$

where $R_i(B)$ denotes the row sum of the ith row of B. There are two versions of this problem. In one case we also prescribe the zero pattern of the required matrix B. In the other case, called the *graph-free* case, we have no restrictions on the zero pattern. Another interesting problem is where A is nonnegative and B is required to be nonnegative.

In the next section we discuss the general problem. Of course, a necessary condition for the existence of a matrix B satisfying (1.1) is that the sum $r_1 + \cdots + r_n$ is equal to half the sum $R_1(A) + \cdots + R_n(A)$, that is,

$$\sum_{i=1}^{n} r_i = \frac{1}{2} \sum_{i,j=1}^{n} a_{ij}. \tag{1.2}$$

Among other results, we show that in the graph-free case the condition (1.2) is also sufficient for the existence of a general matrix B, over an arbitrary field, satisfying (1.1). Under pattern restrictions, an extra graph theoretic condition is needed. The condition (1.2) is not sufficient also in the case where A is a nonnegative matrix and we require B to be a nonnegative matrix. This harder case, which has an interpretation in the theory of network flows, is solved in Section 3.

2. THE GENERAL CASE

For a positive integer n we denote by $\langle n\rangle$ the set $\{1,\dots,n\}$. For a subset S of $\langle n\rangle$ we denote by S^C the complement of S in $\langle n\rangle$. Finally, for a digraph D we denote by $E(D)$ and $V(D)$ the arc set and the vertex set of D respectively.

We start with a few graph theoretic definitions.

DEFINITION 2.1. Let $D=(V,E)$ be a digraph. A digraph $D'=(V',E')$ is said to be a *subdigraph* of D if $V'\subseteq V$ and $E'\subseteq E$. We write $D'\subseteq D$ to indicate that D' is a subdigraph of D.

DEFINITION 2.2. A set S of vertices in a digraph D is said to be *D-loose* if for every $i\in S$ and every $j\in V(D)\setminus S$ at least one of the arcs (i,j) and (j,i) is not present in D. By convention, $\varnothing$ and $V(D)$ are D-loose sets.

DEFINITION 2.3. The *symmetric closure* $\overline{D}$ of a digraph D is the digraph with $V(\overline{D})=V(D)$, and where (i,j) is an arc in $\overline{D}$ whenever (i,j) and/or (j,i) is an arc in D.

DEFINTION 2.4. Let A be an $n\times n$ matrix. The *digraph* $D(A)$ *of* A is defined as the digraph with vertex set $\langle n\rangle$, and where (i,j) is an arc in $D(A)$ if and only if $a_{ij}\neq 0$.

We can now state the main theorem of this section.

THEOREM 2.5. *Let A be a symmetric $n\times n$ matrix over an arbitrary field $\mathbf{F}$ with characteristic different from 2, let $r_1,\dots,r_n$ be n numbers in $\mathbf{F}$, and let D be a digraph satisfying $D(A)\subseteq\overline{D}$. The following are equivalent*:

(i) *There exists an $n\times n$ matrix B over $\mathbf{F}$, with $D(B)\subseteq D$, with row sums $r_1,\dots,r_n$, and such that $A=B+B^T$.*

(ii) *We have*

$$\sum_{i\in S} r_i=\frac{1}{2}\sum_{i,j\in S}a_{ij}+\sum_{(i,j)\in S\times S^C\cap E(D)}a_{ij}\qquad \text{for every } D\text{-loose set } S. \tag{2.6}$$

Proof. (i) $\Rightarrow$ (ii): It follows from (i) that for every subset S of $\langle n\rangle$ we have

$$\sum_{i\in S} r_i=\frac{1}{2}\sum_{i,j\in S}a_{ij}+\sum_{(i,j)\in S\times S^C\cap E(D)}b_{ij}.$$

Let S be a D-loose set, and let $(i,j) \in S \times S^C \cap E(D)$. Since S is a D-loose set, we have $(j,i) \notin E(D)$, and since $A = B + B^T$, it follows that $b_{ij} = a_{ij}$. Therefore, we have

$$\sum_{i \in S} r_i = \frac{1}{2} \sum_{i,j \in S} a_{ij} + \sum_{(i,j) \in S \times S^C \cap E(D)} b_{ij}$$

$$= \frac{1}{2} \sum_{i,j \in S} a_{ij} + \sum_{(i,j) \in S \times S^C \cap E(D)} a_{ij}.$$

(ii) ⇒ (i): Define a set of numbers $\{X_S : S \subseteq \langle n \rangle\}$ by

$$X_\varnothing = 0,$$

$$X_{\{i\}} = r_i - \tfrac{1}{2} a_{ii}, \qquad i \in \langle n \rangle, \tag{2.7}$$

$$X_{\{i,j\}} = X_{\{i\}} + X_{\{j\}} - a_{ij}, \qquad i,j \in \langle n \rangle, \quad i \neq j, \tag{2.8}$$

$$X_S = \sum_{i \in S} X_{\{i\}} - \sum_{\substack{i,j \in S \\ i<j}} \left(X_{\{i\}} + X_{\{j\}} - X_{\{i,j\}} \right), \qquad S \subseteq \langle n \rangle, \quad |S| > 2. \tag{2.9}$$

Assume that $(i,j),(j,i) \notin E(D)$. Since $D(A) \subseteq \overline{D}$, it follows that $(i,j) \notin E(D(A))$, and by (2.8) we have

$$X_{\{i,j\}} = X_{\{i\}} + X_{\{j\}} \qquad \text{whenever} \quad (i,j),(j,i) \notin E(D). \tag{2.10}$$

Let S be a D-loose set with $|S| > 2$. By (2.7) and (2.9) we have

$$X_S = \sum_{i \in S} X_{\{i\}} - \sum_{\substack{i,j \in S \\ i<j}} \left(X_{\{i\}} + X_{\{j\}} - X_{\{i,j\}} \right)$$

$$= \sum_{i \in S} r_i - \frac{1}{2} \sum_{i \in S} a_{ii} - \sum_{\substack{i,j \in S \\ i<j}} \left(X_{\{i\}} + X_{\{j\}} - X_{\{i,j\}} \right).$$

By (2.6) we now obtain

$$X_S = \frac{1}{2}\sum_{i,j\in S} a_{ij} + \sum_{(i,j)\in S\times S^C\cap E(D)} a_{ij} - \sum_{\substack{i,j\in S\\ i<j}} \left(X_{\{i\}} + X_{\{j\}} - X_{\{i,j\}}\right)$$

$$= \sum_{\substack{i,j\in S\\ i<j}} a_{ij} + \sum_{(i,j)\in S\times S^C\cap E(D)} a_{ij} - \sum_{\substack{i,j\in S\\ i<j}} \left(X_{\{i\}} + X_{\{j\}} - X_{\{i,j\}}\right),$$

which, in view of (2.8), yields that

$$X_S = \sum_{(i,j)\in S\times S^C\cap E(D)} \left(X_{\{i\}} + X_{\{j\}} - X_{\{i,j\}}\right) \qquad \text{for every } D\text{-loose set } S. \tag{2.11}$$

In particular, it follows from (2.11) that $X_{\langle n\rangle} = 0$. By Theorem (3.6) of [2], it follows from (2.9), (2.10), and (2.11) that there exists an $n\times n$ matrix B, with $D(B)\subseteq D$, such that

$$\sum_{(i,j)\in S\times S^C} b_{ij} = X_S, \qquad S\subseteq \langle n\rangle. \tag{2.12}$$

Define b_{ii} to be $\frac{1}{2}a_{ii}$, $i\in\langle n\rangle$. It then follows from (2.12) that the ith row sum of B is $b_{ii} + X_{\{i\}}$, which, by (2.7), is equal to r_i. Also, it follows from (2.12) that

$$b_{ij} + b_{ji} = X_{\{i\}} + X_{\{j\}} - X_{\{i,j\}}, \qquad i,j\in\langle n\rangle, \quad i\neq j.$$

In view of (2.8), we have $b_{ij} + b_{ji} = a_{ij}$, $i,j\in\langle n\rangle$, $i\neq j$, and so $A = B + B^T$. ■

REMARK 2.13. A similar result holds for Hermitian complex matrices A and a matrix B that is required to satisfy $A = B + B^*$. In this case, (2.6) should be replaced by

$$\operatorname{Re}\left(\sum_{i\in S} r_i\right) = \frac{1}{2}\sum_{i,j\in S} a_{ij} + \sum_{(i,j)\in S\times S^C\cap E(D)} a_{ij}$$

whenever S is a D-loose set.

If we choose D to be complete digraph with n vertices, then the only D-loose sets are $\varnothing\ \langle n\rangle$, and we obtain the following graph-free version of Theorem 2.5.

THEOREM 2.14. *Let A be a symmetric $n \times n$ matrix over an arbitrary field $\mathbf{F}$ with characteristic different from 2, and let $r_1, \dots, r_n$ be n numbers in $\mathbf{F}$. The following are equivalent:*

(i) *There exists an $n \times n$ matrix B over $\mathbf{F}$ with row sums $r_1, \dots, r_n$ and such that $A = B + B^T$.*

(ii) *We have*

$$\sum_{i=1}^{n} r_i = \frac{1}{2} \sum_{i,j=1}^{n} a_{ij}.$$

As an interesting corollary of Theorem 2.14 we can obtain the following.

THEOREM 2.15. *Let $r_1, \dots, r_n$ and $c_1, \dots, c_n$ be real numbers. The following are equivalent:*

(i) *There exists a real $n \times n$ matrix B, with row sums $r_1, \dots, r_n$ and column sums $c_1, \dots, c_n$, such that $B + B^T$ is nonnegative entrywise.*

(ii) *We have $r_1 + \cdots + r_n = c_1 + \cdots + c_n$ and $r_i + c_i \geqslant 0$, $i \in \langle n\rangle$.*

Proof. (i) $\Rightarrow$ (ii): Clearly, since $r_1, \dots, r_n$ are row sums and $c_1, \dots, c_n$ are column sums of a matrix B, we have $r_1 + \cdots + r_n = c_1 + \cdots + c_n$. Since $r_i + c_i$ is the ith row sum of the nonnegative matrix $B + B^T$, it follows that $r_i + c_i \geqslant 0$.

(ii) $\Rightarrow$ (i): Let $r_i = r_i + c_i$, $i \in \langle n\rangle$. By [1], there exists a nonnegative matrix C with row sums *and* column sums $s_1, \dots, s_n$. Hence, $A = \frac{1}{2}(C + C^T)$ is a nonnegative symmetric matrix with row sums $s_1, \dots, s_n$. We have

$$\sum_{i=1}^{n} r_i = \frac{1}{2}\left(\sum_{i=1}^{n} r_i + \sum_{i=1}^{n} c_i\right) = \frac{1}{2}\sum_{i=1}^{n} s_i = \frac{1}{2}\sum_{i,j} a_{ij}.$$

By Theorem 2.14, there exists a real matrix B with row sums $r_1, \dots, r_n$ and such that $B + B^T = A$. Note that the ith column sum of B is $s_i - r_i = c_i$. ■

COROLLARY 2.16. *Let $r_1, \dots, r_n$ be real numbers. The following are equivalent:*

(i) *There exists a real $n \times n$ matrix B with row sums $r_1, \dots, r_n$ and such that $B + B^T$ is nonnegative.*

(ii) *We have $r_1 + \cdots + r_n \geqslant 0$.*

Proof. (i) $\Rightarrow$ (ii) follows because $r_1 + \cdots + r_n$ is equal to half the sum of the elements of $B + B^T$.

(ii) $\Rightarrow$ (i): Let $r = r_1 + \cdots + r_n$, and define $c_i = -r_i + 2r/n$. Observe that we have both $r_1 + \cdots + r_n = c_1 + \cdots + c_n$ and $r_i + c_i \geqslant 0$, $i \in \langle n \rangle$. By Theorem 2.15, there exists a real $n \times n$ matrix B, with row sums $r_1, \dots, r_n$ and column sums $c_1, \dots, c_n$, such that $B + B^T$ nonnegative. ■

3. THE NONNEGATIVE CASE

Theorems 2.5 and 2.14 do not hold in general if A is a nonnegative matrix and B is required to be a nonnegative matrix, as is demonstrated by the following example.

EXAMPLE 3.1. Let

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 5 \end{pmatrix},$$

and let $r_1 = 3$ and $r_2 = 1$. Observe that for every nonnegative matrix B such that $A = B + B^T$ we must have $b_{22} = \dfrac{1}{2} a_{22} = 2.5$ and $b_{21} \geqslant 0$. Hence, $R_2(B) \geqslant 2.5$, and so, although the condition (1.2) is satisfied, there exists no nonnegative matrix B with row sums r_1, r_2 and such that $A = B + B^T$.

As is observed in Example 3.1, a necessary condition in the nonnegative case is

$$r_i \geqslant \tfrac{1}{2} a_{ii}, \qquad i \in \langle n \rangle. \tag{3.2}$$

In the sequel we shall show that for $n \leqslant 2$ the conditions (1.2) and (3.2) form a necessary and sufficient condition for the existence of a nonnegative matrix B, with no pattern restrictions, satisfying (1.1); see Corollary 3.23. However, we shall show that for $n > 2$ the conditions (1.2) and (3.2) are not sufficient. In the latter case we need a generalized version of (3.2), that is, (3.4) in the case that the pattern of B is prescribed, and (3.20) in the graph-free case.

Our main result here is the following.

THEOREM 3.3. *Let A be a nonnegative symmetric $n \times n$ matrix, let D be a digraph, with loops on all vertices, satisfying $D(A) \subseteq \overline{D}$, and let $r_1, \dots, r_n$ be nonnegative numbers. The following are equivalent:*

(i) *There exists a nonnegative $n \times n$ matrix B, with $D(B) \subseteq D$, with row sums $r_1, \dots, r_n$, and such that $A = B + B^T$.*

(ii) *We have* (2.6) *and*

$$\sum_{i\in S} r_i \geqslant \frac{1}{2}\sum_{i,j\in S} a_{ij} + \sum_{\substack{(i,j)\in S\times S^C\\ (j,i)\notin E(D)}} a_{ij} \qquad \textit{for every} \quad S\subseteq\langle n\rangle. \tag{3.4}$$

Proof. (i) $\Rightarrow$ (ii): By Theorem 2.5, (i) implies (2.6). Let S be a subset of $\langle n\rangle$. We have

$$\sum_{i\in S} r_i = \sum_{(i,j)\in S\times\langle n\rangle} b_{ij} = \sum_{i,j\in S} b_{ij} + \sum_{(i,j)\in S\times S^C} b_{ij}$$

$$\geqslant \frac{1}{2}\sum_{i,j\in S}\left(b_{ij}+b_{ji}\right) + \sum_{\substack{(i,j)\in S\times S^C\\ (j,i)\notin E(D)}} b_{ij}. \tag{3.5}$$

Since $b_{ij}+b_{ji}=a_{ij}$, $i,j\in\langle n\rangle$, and since $D(B)\subseteq D$, it follows that whenever $(j,i)\notin E(D)$ we have $b_{ij}=a_{ij}$, and so (3.4) follows from (3.5).

(ii) $\Rightarrow$ (i): Define an $n\times n$ matrix $\tilde{A}$ by

$$\tilde{a}_{ij} = \begin{cases} a_{ij}, & (i,j),(j,i)\in E(D),\\ 0 & \text{otherwise,}\end{cases}$$

and define the numbers $\tilde{r}_1,\dots,\tilde{r}_n$ by

$$\tilde{r}_i = r_i - \sum_{\substack{j=1\\ (j,i)\notin E(D)}}^{n} a_{ij}.$$

First, observe that by (3.4) we have

$$\sum_{i\in S} r_i \geqslant \frac{1}{2}\sum_{i,j\in S} a_{ij} + \sum_{\substack{(i,j)\in S\times S^C\\ (j,i)\notin E(D)}} a_{ij}$$

$$= \frac{1}{2}\sum_{\substack{i,j\in S\\ (i,j),(j,i)\in E(D)}} a_{ij} + \sum_{\substack{i,j\in S\\ (i,j)\notin E(D)}} a_{ij} + \sum_{\substack{(i,j)\in S\times S^C\\ (j,i)\notin E(D)}} a_{ij}$$

$$= \frac{1}{2}\sum_{i,j\in S}\tilde{a}_{ij} + \sum_{\substack{(i,j)\in S\times\langle n\rangle\\ i\neq j,\,(j,i)\notin E(D)}} a_{ij},$$

which implies

$$\sum_{i\in S} \tilde{r}_i \geqslant \frac{1}{2} \sum_{i,j\in S} \tilde{a}_{ij}. \tag{3.6}$$

We shall now prove that the following linear system of $m = (n^2 + 3n)/2$ equations with n^2 variables x_{ij}, $i, j \in \langle n \rangle$, has a nonnegative solution:

$$\begin{aligned} x_{ij} + x_{ji} &= \tilde{a}_{ij} \qquad i, j \in \langle n \rangle, \quad i \leqslant j, \\ \sum_{j=1}^{n} x_{ij} &= \tilde{r}_i \qquad\qquad i \in \langle n \rangle. \end{aligned} \tag{3.7}$$

Let C be the coefficient matrix of the system (3.7), and let E be the m-vector whose elements are the right hand sides of the equations in (3.7). By the Farkas lemma, e.g. Corollary 7.1d in [6, p. 89], the system (3.7) has a nonnegative solution if and only if for every m-vector F satisfying $F^TC \geqslant 0$ we have $F^TE \geqslant 0$. Denote the element of F that corresponds to (that is, is in the same place as) $\tilde{a}_{ij}$ by f_{ij}, and the element of F that corresponds to $\tilde{r}_i$ by f_i. Since each variable x_{ij} appears in exactly two equations (in the equation $x_{ij} + x_{ji} = \tilde{a}_{ij}$ and in the ith row sum equation), it follows that each column of C contains exctly two nonzero elements (two 1's if it corresponds to x_{ij} where $i \neq j$, and a 2 and a 1 if it corresponds to x_{ii}) and the rest zeros, and that F satisfies $F^TC \geqslant 0$ if and only if

$$f_i + 2f_{ii} \geqslant 0, \quad f_i + f_{ij} \geqslant 0, \quad f_j + f_{ij} \geqslant 0, \qquad i, j \in \langle n \rangle. \tag{3.8}$$

Let F be a vector satisfying (3.8). If F is a nonnegative vector, then clearly we have $F^TE \geqslant 0$. If F has some negative elements, then we shall show that $F^TE \geqslant \bar{F}^TE$ for some vector $\bar{F}$ satisfying

$$\bar{f}_i + 2\bar{f}_{ii} \geqslant 0, \quad \bar{f}_i + \bar{f}_i\bar{f}_{ij} \geqslant 0, \quad \bar{f}_j + \bar{f}_{ij} \geqslant 0, \qquad i, j \in \langle n \rangle \tag{3.9}$$

and such that $\bar{F}$ has less negative elements than F has. Repeating this procedure, we end up with a nonnegative vector $\tilde{F}$ such that $F^TE \geqslant \tilde{F}^TE$. Since we have $\tilde{F}^TE \geqslant 0$, it follows that $F^TE \geqslant 0$, and our claim follows by the Farkas lemma.

Assume first that some of the f_i's are negative, and let $T = \{i : f_i < 0\}$. Let $f = \min_{i\in T}|f_i|$. By (3.8) we have

$$f_{ij} \geqslant f \qquad \forall i, j, \quad \{i, j\} \cap T \neq \varnothing, \tag{3.10}$$

and

$$f_{ii} \geqslant \frac{1}{2} f, \qquad i \in T. \tag{3.11}$$

Since $\langle n \rangle$ is a D-loose set, by (2.6) we have

$$\sum_{i \in \langle n \rangle} r_i = \frac{1}{2} \sum_{i,j \in \langle n \rangle} a_{ij}.$$

Hence,

$$\sum_{i \in \langle n \rangle} r_i = \frac{1}{2} \sum_{i,j \in \langle n \rangle} \tilde{a}_{ij} + \sum_{\substack{i,j \in \langle n \rangle \\ i \neq j, (j,i) \notin E}} a_{ij},$$

which implies

$$\sum_{i \in \langle n \rangle} \tilde{r}_i = \frac{1}{2} \sum_{i,j \in \langle n \rangle} \tilde{a}_{ij}. \tag{3.12}$$

By (3.6) we have

$$\sum_{i \in T^C} \tilde{r}_i \geqslant \frac{1}{2} \sum_{i,j \in T^C} \tilde{a}_{ij}. \tag{3.13}$$

Subtracting (3.13) from (3.12) yields

$$\sum_{i \in T} \tilde{r}_i \leqslant \frac{1}{2} \sum_{i \in T \text{ and/or } j \in T} \tilde{a}_{ij}.$$

Therefore,

$$\tfrac{1}{2} f \sum_{i \in T \text{ and/or } j \in T} \tilde{a}_{ij} - f \sum_{i \in T} \tilde{r}_i \geqslant 0,$$

and hence

$$F^T E \geqslant F^T E - \left(\tfrac{1}{2} f \sum_{i \in T \text{ and/or } j \in T} \tilde{a}_{ij} - f \sum_{i \in T} \tilde{r}_i \right) = \overline{F}^T E. \tag{3.14}$$

In order to find the elements of $\bar{F}$, observe that if $i \in T$, $i \neq j$, then both $\tilde{a}_{ij}$ and $\tilde{a}_{ji}$ appear in the corresponding sum in (3.14), and so $\tilde{a}_{ij}$ actually appears twice. Therefore, the elements of $\bar{F}$ are

$$\bar{f}_i = \begin{cases} f_i, & i \notin T, \\ f_i + f, & i \in T, \end{cases}$$

and

$$\bar{f}_{ij} = \begin{cases} f_{ij}, & \{i,j\} \cap T = \varnothing, \\ f_{ij} - f, & \{i,j\} \cap T \neq \varnothing, \quad i \neq j, \\ f_{ii} - \frac{1}{2}f, & i \in T, \quad i = j. \end{cases}$$

Obviously, we have $f_i \geqslant 0 \Rightarrow \bar{f}_i \geqslant 0$. Also, it follows by (3.10) and (3.11) that $f_{ij} \geqslant 0 \Rightarrow \bar{f}_{ij} \geqslant 0$. Furthermore, for at least one $i \in T$ we have $f_i = -f$ and now $\bar{f}_i = 0$. Therefore, $\bar{F}$ has less negative elements than F has.

Now, for $i \neq j$ we have

$$\bar{f}_i + \bar{f}_{ij} = \begin{cases} f_i + f_{ij}, & i \in T, \\ f_i + f_{ij} - f, & i \notin T, \quad j \in T, \\ f_i + f_{ij}, & \{i,j\} \cap T = \varnothing. \end{cases} \tag{3.15}$$

By (3.8) we have $f_i + f_{ij} \geqslant 0$. By definition of T, whenever $i \notin T$ we have $f_i \geqslant 0$. Also, by (3.10) we have $f_{ij} \geqslant f$ whenever $j \in T$. Hence it follows from (3.15) that $\bar{f}_i + \bar{f}_{ij} \geqslant 0$. Quite similarly we show that $\bar{f}_j + \bar{f}_{ij} \geqslant 0$. Also, for every i we have $\bar{f}_i + 2\bar{f}_{ii} = f_i + 2f_{ii}$, and so the vector $\bar{F}$ satisfies (3.9).

Now assume that the f_i's are all nonnegative but some of the f_{ij}'s are negative. Let $T = \{i : f_{ij} < 0 \text{ or } f_{ji} < 0 \text{ for some } j\}$, and let

$$f = \min\Big\{\{|f_{ij}| : f_{ij} < 0,\, i \neq j\},\, \{2|f_{ii}| : f_{ii} < 0\}\Big\}.$$

By (3.8) we have

$$f_i \geqslant f \qquad \forall i \in T. \tag{3.16}$$

By (3.6) we have

$$f \sum_{i \in T} \tilde{r}_i - \tfrac{1}{2} f \sum_{i,j \in T} \tilde{a}_{ij} \geqslant 0.$$

Thus,

$$F^T E \geqslant F^T E - \left(f \sum_{i \in T} \tilde{r}_i - \tfrac{1}{2} f \sum_{i,j \in T} \tilde{a}_{ij} \right) = \bar{F}^T E,$$

where the elements of $\bar{F}$ are

$$\bar{f}_i = \begin{cases} f_i, & i \notin T, \\ f_i - f, & i \in T, \end{cases}$$

and

$$\bar{f}_{ij} = \begin{cases} f_{ij} + f, & i,j \in T, \quad i \neq j, \\ f_{ii} + \tfrac{1}{2} f, & i \in T, \quad i = j, \\ f_{ij}, & \text{otherwise}. \end{cases}$$

Obviously, we have $f_{ij} \geqslant 0 \Rightarrow \bar{f}_{ij} \geqslant 0$. Also, it follows by (3.16) that $f_i \geqslant 0 \Rightarrow \bar{f}_i \geqslant 0$. Furthermore, either we have at least one pair of indices i, j, $i \neq j$, such that $f_{ij} = -f$ and now $\bar{f}_{ij} = 0$, or we have at least one index i such that $f_{ii} = -\frac{1}{2} f$ and now $\bar{f}_{ii} = 0$. Therefore, $\bar{F}$ has less negative elements than F has.

Now, for $i \neq j$ we have

$$\bar{f}_i + \bar{f}_{ij} = \begin{cases} f_i + f_{ij}, & i,j \in T, \\ f_i - f + f_{ij}, & i \in T, \quad j \notin T, \\ f_i + f_{ij}, & i \notin T. \end{cases} \tag{3.17}$$

By (3.8) we have $f_i + f_{ij} \geqslant 0$. By (3.16), whenever $i \in T$ we have $f_i \geqslant f$. Also, by the definition of T we have $f_{ij} \geqslant 0$ whenever $j \notin T$. Hence it follows from (3.17) that $\bar{f}_i + \bar{f}_{ij} \geqslant 0$. Quite similarly we show that $\bar{f}_j + \bar{f}_{ij} \geqslant 0$. Also, for every i we have $\bar{f}_i + 2\bar{f}_{ii} = f_i + 2 f_{ii}$, and so the vector $\bar{F}$ satisfies (3.9).

As is explained above, it now follows by the Farkas lemma that the system (3.7) has a nonnegative solution $\{x_{ij} : i, j \in \langle n \rangle\}$. Let B be the $n \times n$ matrix defined by

$$b_{ij} = \begin{cases} x_{ij}, & (i,j),(j,i) \in E(D), \\ a_{ij}, & (j,i) \notin E(D), \\ 0, & (i,j) \notin E(D). \end{cases} \tag{3.18}$$

Observe that since $D(A) \subseteq \overline{D}$, if $(i,j),(j,i) \notin E(D)$ then $(i,j) \notin D(A)$ and so $a_{ij} = 0$. Therefore, the matrix B is well defined by (3.18). Let $i,j \in \langle n \rangle$. If $(i,j),(j,i) \in E(D)$ then, by (3.18), $b_{ij} + b_{ji} = x_{ij} + x_{ji} = \tilde{a}_{ij} = a_{ij}$. If $(i,j) \in E(D)$ and $(j,i) \notin E(D)$ then, by (3.18), $b_{ij} + b_{ji} = a_{ij} + 0 = a_{ij}$. If $(i,j) \notin E(D)$ and $(j,i) \in E(D)$ then, by (3.18), $b_{ij} + b_{ji} = 0 + a_{ji} = a_{ij}$. If $(i,j),(j,i) \notin E(D)$ then, by (3.18), $b_{ij} + b_{ji} = 0 = a_{ij}$. Thus, in any case, $b_{ij} + b_{ji} = a_{ij}$, and so $A = B + B^T$. Also, if $(i,j) \notin E(D)$ then, by (3.18), $b_{ij} = 0$, and so $(i,j) \notin D(B)$, implying that $D(B) \subseteq D$. Finally, we have

$$\begin{aligned} R_i(B) &= \sum_{j=1}^{n} b_{ij} = \sum_{\substack{j \in \langle n \rangle \\ (i,j),(j,i) \in E(D)}} x_{ij} + \sum_{\substack{j=1 \\ (j,i) \notin E(D)}}^{n} a_{ij} \\ &= \tilde{r}_i + \sum_{\substack{j=1 \\ (j,i) \notin E(D)}}^{n} a_{ij} = r_i, \qquad i \in \langle n \rangle, \end{aligned}$$

and hence B is the required matrix. ■

The solution of the graph-free case is obtained when D is chosen to be the complete digraph with n vertices.

THEOREM 3.19. *Let A be a nonnegative symmetric $n \times n$ matrix, and let $r_1, \ldots, r_n$ be nonnegative numbers. The following are equivalent:*

(i) *There exists a nonnegative $n \times n$ matrix B with row sums $r_1, \ldots, r_n$ and such that $A = B + B^T$.*

(ii) *We have* (1.2), *and for every $S \subseteq \langle n \rangle$ we have*

$$\sum_{i \in S} r_i \geqslant \frac{1}{2} \sum_{i,j \in S} a_{ij}. \tag{3.20}$$

REMARK 3.21. Let S be a subset of $\langle n \rangle$ with $|S| > k$. Then the complement S^C of S satisfies $|S^C| \leqslant n - k - 1$. By subtracting the inequality (3.2) from (1.2) we obtain

$$\sum_{i \in S^C} r_i \leqslant \frac{1}{2} \sum_{i \in S^C \text{ and/or } j \in S^C} a_{ij}.$$

Since $|S^C| = n - |S|$, condition (ii) in Theorem 3.19 is equivalent to each of the following conditions:

(iii) The equality (1.2) holds, and for some nonnegative integer k, $k < n$, the inequality (3.20) holds for every subset S of $\langle n \rangle$ with $1 \leqslant |S| \leqslant k$, and the inequality

$$\sum_{i \in S} r_i \leqslant \frac{1}{2} \sum_{i \in S \text{ and/or } j \in S} a_{ij} \tag{3.22}$$

holds for every subset S of $\langle n \rangle$ with $1 \leqslant |S| \leqslant n - k - 1$.

(iv) The equality (1.2) holds, and the inequalities (3.20) and (3.22) hold for every subset S of $\langle n \rangle$.

In the special case of $n = 2$ we have the following immediate corollary of Theorem (3.19).

COROLLARY 3.23. *Let A be a nonnegative symmetric 2×2 matrix, and let r_1, r_2 be nonnegative numbers. The following are equivalent:*

(i) *There exists a nonnegative 2×2 matrix B with row sums r_1, r_2 and such that $A = B + B^T$.*

(ii) *We have* (1.2) *and* (3.2).

It is natural to ask whether (1.2) and (3.2) form a necessary and sufficient condition for the existence of a nonnegative $n \times n$ matrix B satisfying (1.1) also in the case $n \geqslant 3$. The answer to this question is negative, as demonstrated by the following example.

EXAMPLE 3.24. Let

$$A = \begin{pmatrix} 0 & 8 & 2 \\ 8 & 4 & 4 \\ 2 & 4 & 4 \end{pmatrix},$$

and let $r_1 = 4$, $r_2 = 5$, and $r_3 = 9$. It is easy to verify that we have both (1.2) and (3.2). Nevertheless, since (3.20) is not satisfied for the set $S = \{1, 2\}$, it follows from Theorem 3.19 that there exists no nonnegative 3×3 matrix B with row sums r_1, r_2, r_3 and such that $A = B + B^T$.

We conclude the paper with a characterization of the positive decomposition case.

THEOREM 3.25. *Let A be a positive (entrywise) symmetric $n \times n$ matrix, and let $r_1, \ldots, r_n$ be positive numbers. The following are equivalents:*

(i) *There exists a positive* $n \times n$ *matrix* B *with row sums* $r_1, \dots, r_n$ *and such that* $A = B + B^T$.

(ii) *We have* (1.2), *as well as a strict inequality in* (3.20) *for every subset* S *of* $\langle n \rangle$.

Proof. (i) $\Rightarrow$ (ii): Since $b_{ij} > 0$ for all $i, j \in \langle n \rangle$, using the very same proof as for the corresponding implication in Theorem 3.19, we obtain strict inequalities in (3.20).

(ii) $\Rightarrow$ (i): Define a new symmetric matrix $\bar{A}$ by $\bar{a}_{ij} = a_{ij} - \epsilon$, $i, j \in \langle n \rangle$, and n new numbers $\bar{r}_1, \dots, \bar{r}_n$ by $\bar{r}_i = r_i - n\epsilon/2$, $i \in \langle n \rangle$. If we choose $\epsilon > 0$ sufficiently small, then the matrix $\bar{A}$ is positive and the numbers $\bar{r}_1, \dots, \bar{r}_n$ are positive. Furthermore, since we have strict inequalities in (3.20), the inequalities there still hold if we replace a_{ij} by $\bar{a}_{ij}$ and r_i by $\bar{r}_i$. By Theorem 3.19, there exists a nonnegative matrix $\bar{B}$ with row sums $\bar{r}_1, \dots, \bar{r}_n$ and such that $\bar{A} = \bar{B} + \bar{B}^T$. The matrix B defined by $b_{ij} = \bar{b}_{ij} + \epsilon/2$, $i, j \in \langle n \rangle$, is a positive matrix with row sums $r_1, \dots, r_n$ and such that $A = B + B^T$. ■

Professor Alan Hoffman and a referee have pointed out to us that network flow techniques which were used to solve a related problem on tournaments in [7] and [3] could be used to derive the results of Section 3. We hope to exploit network flow techniques in some generalizations in a forthcoming paper.

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Received 8 March 1993; final manuscript accepted 8 November 1993