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Resolvents of Minus M-Matrices and Splittings of M-Matrices

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ABSTRACT

This paper investigates the relationship of the combinatorial properties of a minus M-matrix to the growth rate and decay rate of the individual elements in its resolvent. These results are then used to develop properties of splittings of an M-matrix. Resolvent compatible splittings, distance dominated splittings, and weak graph compatible splittings are introduced here, and their relationships to graph compatible and G-compatible splittings are investigated. Some of the results of this paper are summarized in an implications diagram.

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1. INTRODUCTION

We begin by giving some motivation for our work and some historical references.

As is well known (see Berman and Plemmons [2] and Varga [13]), with any splitting $A = M - N$ one may associate an iterative scheme for solving a system of linear equations $Ax = b$. Interest has centered on the case where A is an M-matrix, and in recent years on the case where A is singular. In particular the convergence of the iteration depends on the value of the spectral radius of $M^{-1}N$ and on the index of 1 when the spectral radius is 1 (or equivalently the index of 0 of $M^{-1}A$). Several papers have dealt with these issues, for example Neumann and Plemmons [7], Schneider [11], Kavanagh [4], Marek and Szyld [6], Barker and Plemmons [1], and Rose [9], and we expand on the results of these papers here.

Some of the papers mentioned above relate the combinatorial structure of a matrix to the spectral properties of the splitting. In this paper we obtain results of a similar type by investigating the growth rate and the decay rate of the elements of the resolvent $(\beta I - A)^{-1}$, where A is a minus M-matrix. A number of our results here are based on properties of principal components of minus M-matrices proved by Neumann and Schneider in [8]. Our results on resolvents are independent of the results on splittings and may be useful in other areas. For example, they may be applicable in the computation of the spectral radius, a corresponding eigenvector, or the entire Perron eigenspace by means of inverse iterations when the rate of convergence is accelerated by means of repeated applications of the resolvents; see for example Stoer and Bulirsch [12] or Golub and van Loan [3].

We now describe our paper in greater detail:

Section 2 contains a list of definitions and some preliminaries. In particular, we introduce the concepts of a *weak graph compatible* splitting, of a *distance dominated* splitting, and of a *resolvent compatible* splitting.

In Section 3, we investigate the resolvents of minus M-matrices. Let A be a minus M-matrix. Let $\beta > 0$. If there is no path in $G(A)$, the graph of A, from i to j, then the (i, j)th entry of $(\beta I - A)^{-1}$ is 0. If there is a path from i to j, let $d = d_{i,j}(A)$ be the (singular) distance from i to j, and let $s = \mathrm{sp}_{i,j}(A)$ be the shortest path length from i to j. We show that as β approaches 0, the rate of growth of the (i, j)th entry of $(\beta I - A)^{-1}$ is as β^{-d}, and that as β approaches $+\infty$, the rate of decay of the (i, j)th entry of $(\beta I - A)^{-1}$ is as $\beta^{-(s+1)}$. Let $Z^{(0)}$ be the projector onto the generalized null space of A along the sum of the generalized eigenspaces associated with all the eigenvalues of A different from 0. Let $Z^{(d-1)} = A^{(d-1)}Z^{(0)}$. Then

$$\lim_{\beta \to 0} \beta^{d}((\beta I - A)^{-1})_{i,j} = (Z^{(d-1)})_{i,j}.$$

Now let $A = \rho I - P$, where P is nonnegative and $\rho > 0$. We show that

$$\lim_{\beta\to\infty} \beta^{s+1}\left((\beta I - A)^{-1}\right)_{i,j} = (P^s)_{i,j}.$$

This may be of particular interest when P represents a Markov process. In Section 3 we work entirely with minus M-matrices, since for such matrices the resolvent is always nonnegative.

In Section 4, we apply the results of Section 3 to splittings, and here we consider M-matrices as is classically done. In [6] Marek and Szyld introduced the notion of G-compatibility, and this has motivated some of our work. We generalize the definition of G-compatibility to get resolvent compatibility. We show that the relationship between resolvent compatibility and various combinatorial conditions on splittings is quite complex. One chief purpose of this paper is to introduce new combinatorial types of splittings, such as a distance dominated splitting and a weak graph compatible splitting, and to investigate the relationship of such splittings to graph compatibility, on the one hand, and to G-compatibility, resolvent compatibility, and other analytic conditions on splittings, involving resolvents, on the other. At the end of Section 4, we provide a diagram illustrating the relationships between four main types of splittings investigated in this paper.

2. DEFINITIONS

We begin with some standard definitions. Suppose $X \in \mathbb{R}^{nn}$.

We let $\rho(X)$ denote the *spectral radius*, $\text{mult}_\lambda(X)$ the degree of λ as a root of the characteristic polynomial, and $\text{index}_\lambda(X)$ the degree of λ as a root of the minimal polynomial. We Let $\lambda_1, \ldots, \lambda_t$ represent the distinct eigenvalues of X, and $\mu_k = \text{index}_{\lambda_k}(X)$. We will write $\langle n \rangle = \{1, \ldots, n\}$.

X is called:

positive $(X \gg 0)$ if $X_{ij} > 0$ for all $i, j \in \langle n \rangle$;
semipositive $(X > 0)$ if $X_{i,j} \geqslant 0$ for all $i, j \in \langle n \rangle$ and $X \neq 0$; and
nonnegative $(X \geqslant 0)$ if $X_{i,j} \geqslant 0$ for all $i, j \in \langle n \rangle$.

X is called a *Z-matrix* if $X = \alpha I - P$ for some $\alpha \in \mathbb{R}$ with P nonnegative. If in addition, $\alpha \geqslant \rho(P)$, then we say X is an *M-matrix*. If $-X$ is a M-matrix, we say X is a *minus M-matrix*.

Let $J, K \subseteq \langle n \rangle$. We will write $X[J, K]$ to represent the submatrix of X whose rows are indexed by the elements of J and whose columns are indexed by the elements of K.

Let $\Gamma = (V, E)$, where V is a finite vertex set and E is an edge set. A *path* from j to k in Γ is a sequence of vertices $j = r_1, r_2, \dots, r_t = k$, with $(r_i, r_{i+1}) \in E$, for $i = 1, \dots, t-1$. A path for which the vertices are pairwise distinct is called a *simple path*. The empty path will be considered a simple path linking every vertex to itself. If there is a path from j to k, we say that j has *access* to k. If j has access to k and k has access to j, we say j and k *communicate*. The communication relation is an equivalence relation; hence we may partition V into equivalence classes, which we will refer to as the *classes* of Γ.

We define the *graph* of X by $G(X) = (V, E)$, where $V = \langle n \rangle$ and $E = \{(i,j) \mid X_{ij} \neq 0\}$. We define the closure of the graph of X by $\overline{G(X)} = (V, E')$, where $V = \langle n \rangle$ and $E' = \{(i,j) \mid i$ has access to j in $G(X)\}$.

It is well known that the indices of X can be ordered so that X is in block lower triangular Frobenius normal form, with each diagonal block irreducible. The irreducible blocks of X correspond to the classes of $G(X)$. If an irreducible block is singular, we call the corresponding class a *singular class*. Similarly, if an irreducible block is nonsingular, we call the corresponding class a *nonsingular class*. Capital letters will be used to represent classes of the various matrices involved, and small letters will be used when referring to their individual elements.

We define the *reduced graph* of X by $R(X) = (V, E)$, where $V = \{J \mid J$ is a class of $G(X)\}$ and $E = \{(J, K) \mid$ there exist $j \in J$ and $k \in K$ with $X_{jk} \neq 0\}$. A vertex J in $R(X)$ is called *singular* or *nonsingular* depending on whether the corresponding class is singular or nonsingular. The (*singular*) *length* of a simple path in $R(X)$ is the sum of the indices of zero of the singular vertices lying on it, and in the case of an M-matrix this is just the number of singular vertices on the path. If there is a path from J to K, define the (*singular*) *distance*, $d(J, K)(X)$, from IJ to K to be the maximal length of a simple path connecting J and K in IR(X). If there is no path from J to K, we set $d(J, K)(X) = -1$.

$A = M - N$ is called a *splitting* if M is nonsingular. A splitting is called:

an *M-splitting* if M is an M-matrix and $N \geqslant 0$;
regular if $M^{-1} \geqslant 0$ and $N \geqslant 0$;
weak regular if $M^{-1} \geqslant 0$ and $M^{-1}N \geqslant 0$;
weak if $M^{-1}N \geqslant 0$; and
a *Z-splitting* if $M^{-1} \geqslant 0$ and $M^{-1}A$ is a Z-matrix.

A splitting $A = M - N$ is called *graph compatible* if $G(M) \subseteq \overline{G(A)}$.

The following definition is from [6]. We include a version specific to the finite dimensional case:

DEFINITION 2.1. Let $A \in \mathbb{R}^{nn}$ be an M-matrix. We can write $A = \alpha I - P$, where $\alpha > 0$ and $P \geqslant 0$. A weak splitting $A = M - N$ is called

G-compatible if there is a positive quantity τ such that for all positive real β,

$$[(1+\beta)I - M^{-1}N]^{-1} \leqslant \tau\left((1+\beta)I - \frac{1}{\alpha}P\right)^{-1}. \tag{1}$$

We would like to point out that this last equation is equivalent to saying there is a positive quantity τ such that for all positive real β,

$$(\beta I + M^{-1}A)^{-1} \leqslant \tau\left(\beta I + \frac{1}{\alpha}(A)\right)^{-1}. \tag{2}$$

We extend this idea to an arbitrary splitting $A = M - N$ of an arbitrary matrix A, and construct our definition so that it is independent of α.

DEFINITION 2.2. Let $A \in \mathbb{R}^{nn}$. A splitting $A = M - N$ is said to be *resolvent compatible* if there exists $\tau > 0$ such that for all positive real β,

$$(\beta I + M^{-1}A)^{-1} \leqslant \tau(\beta I + A)^{-1}. \tag{3}$$

Notice that

$$(\beta I + A)^{-1} = [\beta I - (-A)]^{-1}$$

is the resolvent of $-A$.

Equation (3) is obtained from Equation (2) by replacing $(1/\alpha)A$ on the right hand side of the equation with A. It follows from Corollary 3.8 that there exists a $\tau > 0$ such that for all positive real β, Equation (3) holds if and only if for any positive α there exists a $\tau > 0$ such that for all positive real β, Equation (2) holds. This establishes that a weak splitting of an M-matrix is G-compatible if and only if it is resolvent compatible.

REMARK 2.3. Implicit in the definition of G-compatibility is that $\rho(M^{-1}N) \leqslant 1$; otherwise $\beta I - (-M^{-1}A) = (1+\beta)I - M^{-1}N$ will not be invertible for all $\beta > 0$. The following example, due to Neumann (see [11]), shows that the maximal eigenvalue of $M^{-1}N$ can be greater than 1 even for a regular splitting:

$$A = \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}, \qquad M = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad N = \begin{pmatrix} 0 & 1 \\ 2 & 0 \end{pmatrix},$$

$$M^{-1}N = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}, \qquad M^{-1}A = \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix}.$$

We introduce the following other new definitions:

DEFINITION 2.4. Let $A \in \mathbb{R}^{nn}$. A splitting $A = M - N$ is called *weak graph compatible* if $G(M^{-1}N) \subseteq \overline{G(A)}$.

REMARK 2.5 [11]. Corollary 2.6 shows that for a graph compatible splitting, there are no access relations in $G(M^{-1}A)$ which are not present in $G(A)$; thus graph compatibility implies weak graph compatibility. In Remark 2.3 we provide a weak graph compatible regular splitting which is not graph compatible. Thus weak graph compatibility is indeed weaker than graph compatibility.

We extend the definition of (singular) distance to the vertices of $G(X)$.

DEFINITION 2.6. Let $X \in \mathbb{R}^{nn}$. For $j, k \in \langle n \rangle$ define

$$d_{j,k}(X) = d(J, K)(X), \qquad \text{where} \quad j \in J \text{ and } k \in K,$$

with J, K classes of X.

DEFINITION 2.7. Let $A \in \mathbb{R}^{nn}$. A splitting $A = M - N$ is called a *distance dominated splitting* if $d_{i,j}(M^{-1}A) \leqslant d_{i,j}(A)$ for every $i, j \in \langle n \rangle$.

REMARK 2.8. By the Rothblum index theorem [10], a distance dominated splitting $A = M - N$ of an M-matrix such that $M^{-1}A$ is also an M-matrix will satisfy

$$\text{index}_0(M^{-1}A) \leqslant \text{index}_0(A).$$

DEFINITION 2.9. Let $X \in \mathbb{R}^{nn}$. For each $i, j \in \langle n \rangle$, we define here the *shortest path length*, $\text{sp}_{i,j}(X)$. If there is no path in $G(A)$ from i to j, set $\text{sp}_{i,j}(X) = \infty$. If there is a path from i to j, set $\text{sp}_{i,j}(X) = s$, where s is the number of edges in the shortest path from i to j.

Since the empty path from any vertex to itself has no edges, we take $\text{sp}_{i,i}(X) = 0$ for all $i \in \langle n \rangle$.

3. RESOLVENTS OF MINUS M-MATRICES

Let A be a minus M-matrix. The function $(\beta I - A)^{-1}$ is referred to as the resolvent of A. In this section we establish the orders of the growth rates (as β approaches zero from the right) and the decay rates (as β approaches infinity) of the elements of the resolvent in terms of combinatorial properties of the matrix.

We begin our analysis of the resolvent of a minus M-matrix by establishing the growth rate of an element in terms of the singular distance between the element's indices. This then allows us to compare the resolvents of two minus M-matrices for positive β near zero.

From [5, p. 321] we know that for any β not contained in the spectrum of X, $(\beta I - X)^{-1}$ admits the expansion

$$(\beta I - X)^{-1} = \sum_{k=1}^{t} \sum_{j=0}^{\mu_k - 1} \frac{Z^{k,j}}{(\beta - \lambda_k)^{j+1}}, \tag{4}$$

where $Z^{k,0}$ is a projector onto the generalized eigenspace of λ_k along the sum of the generalized eigenspaces associated with all eigenvalues of X different from λ_k, and $Z^{k,j} = (X - \lambda_k I)^j Z^{k,0}$ for all $0 < j \leqslant \mu_k$. In the special case where $\lambda_k = 0$, to simplify notation, we will write $Z^{k,j} = Z^{(j)}$. Our $Z^{(j)}$ agree with those of [8], and our $Z^{k,j}$ differ from the $Z_{k,j}$ in [5] by a factor of $j!$.

THEOREM 3.1. *Let A be a minus M-matrix. Let $i, j \in \langle n \rangle$. Let $d = d_{i,j}(A)$ and $Z^{(j)}$ be as described above.*

(i) *If $d = -1$, then $((\beta I - A)^{-1})_{i,j} = 0$ for all positive real β.*
(ii) *If $d \geqslant 0$, then $((\beta I - A)^{-1})_{i,j} > 0$ for all positive real β.*
(iii) *If $d > 0$, then for $\beta > 0$,*

$$\lim_{\beta \to 0} \beta^d \big((\beta I - A)^{-1}\big)_{i,j} = (Z^{(d-1)})_{i,j} > 0.$$

(iv) *If $d \geqslant 0$, then for any real $\eta > 0$, $\beta^d((\beta I - A)^{-1})_{i,j}$ can be extended to a continuous function on the interval $[0, \eta]$, which is bounded away from 0 and bounded from above on this interval.*

Proof. (i) and (ii): Since $\beta I - A$ is a nonsingular M-matrix, by [11, Lemma 2.2], we have

$$G\big((\beta I - A)^{-1}\big) = \overline{G(\beta I - A)} = \overline{G(A)}\,.$$

From this it follows that if $d = -1$, then $((\beta I - A)^{-1})_{i,j} = 0$, and if $d \geqslant 0$, then $((\beta I - A)^{-1})_{i,j} \neq 0$. In the latter case, since $((\beta I - A)^{-1})_{i,j}$ is a nonzero element in the inverse of an M-matrix, it must be positive.

(iii): If $d > 0$, then by [8, Section 2, Lemma 2], $(Z^{(k)})_{i,j} = 0$ for all $k \geqslant d$, and by [8, Section 3, Theorem 1], $(Z^{(d-1)})_{i,j} > 0$. We will label the eigenvalue zero as λ_1. Combining this with Equation (4), we see that

$$\beta^d\big((\beta I - A)^{-1}\big)_{i,j} = (Z^{(d-1)})_{i,j} + \beta \sum_{k=1}^{d-2} \beta^{d-k-2}(Z^{(k)})_{i,j}$$

$$+ \beta^d \sum_{m=2}^{t} \sum_{k=0}^{\mu_m - 1} \frac{(Z^{m,k})_{i,j}}{(\beta - \lambda_m)^{k+1}}.$$

Since $d \geqslant 1$ and $\lambda_m \neq 0$ for all $m \in \{2, \dots, t\}$, the last two terms must converge to zero as β approaches zero.

(iv): If $d > 0$, let $F(\beta) = \beta^d((\beta I - A)^{-1})_{i,j}$ for all $\beta > 0$. Then by (iii), $F(\beta)$ can be extended to a continuous function on $[0, \infty)$ by defining $F(0) = (Z^{(d-1)})_{i,j}$. Combining this with (ii), we see that $F(\beta)$ is a positive, continuous, real valued function on the interval $[0, \eta]$. Hence it achieves its maximum and minimum, and its minimum must be positive.

If $d = 0$, let $\Psi = \{k \mid i$ has access to k and k has access to j in $G(A)\}$. Let $B = A[\Psi, \Psi]$, and assume that B is in Frobenius normal form. Since $d = 0$, the blocks on the diagonal of B are nonsingular; thus B is a nonsingular minus M-matrix, and $\beta I - B$ is a nonsingular M-matrix for all $\beta \geqslant 0$. By [8, Section 2, Lemma 1], $(\beta I - B)^{-1} = (\beta I - A)^{-1}[\Psi, \Psi]$. By [11, Lemma 2.2], $G((\beta I - B)^{-1}) = \overline{G(B)}$ for all β in $[0, \infty)$. Thus $((\beta I - A)^{-1})_{i,j} = ((\beta I - B)^{-1})_{i,j}$ is a continuous, positive, real valued function on the interval $[0, \eta]$. Hence it achieves its maximum and minimum, and its minimum will be positive. ■

COROLLARY 3.2. *Let A, B be minus M-matrices. Let $i, j \in \langle n \rangle$. For any $\eta > 0$ the following are equivalent*:

(i) *There exists $\tau_{i,j} > 0$, such that for all positive real $\beta < \eta$,*

$$\big((\beta I - B)^{-1}\big)_{i,j} \leqslant \tau_{i,j}\big((\beta I - A)^{-1}\big)_{i,j}.$$

(ii) $d_{i,j}(B) \leqslant d_{i,j}(A)$.

Proof. Follows from Theorem 3.1 by considering each possible case for $d_{i,j}(A)$ and $d_{i,j}(B)$. ■

REMARK 3.3. We provide here an example which shows that even if Corollary 3.2(ii) is satisfied, we cannot always choose a $\tau_{i,j}$ such that the inequality in (i) will hold for all $\beta > 0$.

Let

$$A = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & -1 \end{pmatrix}, \qquad B = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 1 & 0 & -1 \end{pmatrix}.$$

Then

$$(\beta I - A)^{-1} = \begin{pmatrix} \dfrac{1}{\beta} & 0 & 0 \\ \dfrac{1}{\beta^2} & \dfrac{1}{\beta} & 0 \\ \dfrac{1}{\beta^2(\beta+1)} & \dfrac{1}{\beta(\beta+1)} & \dfrac{1}{\beta+1} \end{pmatrix}$$

and

$$(\beta I - B)^{-1} = \begin{pmatrix} \dfrac{1}{\beta} & 0 & 0 \\ 0 & \dfrac{1}{\beta+1} & 0 \\ \dfrac{1}{\beta(\beta+1)} & 0 & \dfrac{1}{\beta+1} \end{pmatrix}.$$

Notice that $d_{3,1}(A) = 2$, $d_{3,1}(B) = 1$, but if we allow β to approach infinity, there does not exist a $\tau_{3,1}$ independent of β such that $((\beta I - B)^{-1})_{3,1} \leqslant \tau_{3,1}((\beta I - A)^{-1})_{3,1}$.

Next we consider the decay rate (as β approaches ∞) of an element in the resolvent. We show that this rate is determined by the length of the shortest path between the vertices. This then allows us to compare the resolvents of two minus M-matrices in the case where β is allowed to grow without bound. Combining these results with those above, we can predict the behavior of the resolvent for all positive real β.

THEOREM 3.4. *Let A be a minus M-matrix. Write $A = P - \alpha I$, with $\alpha \geqslant 0$ and P nonnegative. Let $i, j \in \langle n \rangle$. Set $s = \mathrm{sp}_{i,j}(A)$.*

(i) *If $s = \infty$, then $((\beta I - A)^{-1})_{i,j} = 0$ for all $\beta > 0$.*
(ii) *If $s < \infty$, then*

$$\lim_{\beta \to \infty} \beta^{s+1}\big((\beta I - A)^{-1}\big)_{i,j} = (P^s)_{i,j} > 0.$$

(iii) *If $s < \infty$, then for any real $\eta > 0$, $\beta^{s+1}((\beta I - A)^{-1})_{i,j}$ is bounded away from 0 and bounded from above on the interval $[\eta, \infty)$.*

Proof. (i): $s = \infty$ if and only if $d_{i,j}(A) = -1$. Now apply Theorem 3.1.

(ii): For $k \geqslant 0$, let $(P^k)_{i,j} = p_k$. Since $(P^k)_{i,j}$ is nonzero if and only if there is a path of length k from i to j in $G(A)$, we know that $p_k = 0$ for all $k < s$, and $p_s > 0$. Then for any $\beta > 0$, let

$$F(\beta) = \beta^{s+1}\big((\beta I - A)^{-1}\big)_{i,j} = \frac{\beta^{s+1}}{\beta + \alpha}\left(\left(I - \frac{1}{\beta + \alpha}P\right)^{-1}\right)_{i,j}.$$

Define

$$G(\zeta) = p_s + \zeta p_{s+1} + \zeta^2 p_{s+2} + \cdots.$$

Then for all $|\zeta| < 1/\rho(P)$,

$$G(\zeta) = \zeta^{-s}\big((I - \zeta P)^{-1}\big)_{i,j}.$$

Let $\zeta = 1/(\beta + \alpha)$. Then $\beta = (1 - \zeta\alpha)/\zeta$ and, since $\alpha \geqslant \rho(P)$, we have $|\zeta| < 1/\rho(P)$ for all $\beta > 0$. Moreover

$$F(\beta) = \left(\frac{1 - \zeta\alpha}{\zeta}\right)^{s+1} \zeta\big((I - \zeta P)^{-1}\big)_{i,j} = (1 - \zeta\alpha)^{s+1} G(\zeta).$$

Thus

$$\lim_{\beta\to\infty} F(\beta) = \lim_{\zeta\to 0}(1-\zeta\alpha)^{s+1}G(\zeta) = G(0) = p_s.$$

(iii): By (ii), we can choose $\vartheta > 0$ such that for all $\beta > \vartheta$,

$$\frac{p_s}{2} < F(\beta) < 2p_s.$$

Moreover, $\beta^{s+1}((\beta I - A)^{-1})_{i,j}$ is a continuous, positive, real valued function on the compact interval $[\eta, \vartheta]$ and so achieves its maximum and minimum on this interval, and its minimum will be positive. ∎

REMARK 3.5. It should be noted that the diagonal elements of P are not uniquely defined in Theorem 3.4. However, since we are dealing with the shortest path from i to j, if $i \neq j$, the element $(P^s)_{i,j}$ is independent of the diagonal elements. If $i = j$, then $s = 0$ and $(P^0)_{i,i} = 1$.

REMARK 3.6. Let $P \in \mathbb{R}^{nn}$ be nonnegative. Let $s = \mathrm{sp}_{ij}(P)$. If s is finite, then s is the smallest nonnegative integer such that $(P^s)_{ij} > 0$. Whenever $\beta \geqslant \rho(P)$, $P - \beta I$ is a minus M-matrix; hence by part (ii) of Theorem 3.4, s is also the smallest nonnegative integer such that

$$\lim_{\beta\to\infty} \beta^{s+1}((\beta I - P)^{-1})_{i,j} > 0,$$

and this limit provides an asymptotic approximation to $(P^s)_{ij}$.

Let P represent the transition matrix associated with a finite homogeneous Markov process with states $\mathscr{S}_1, \dots, \mathscr{S}_n$. For any positive integer $k \geqslant 1$ and any indices $i, j \in \langle n\rangle$, $(P^k)_{i,j}$ is the probability that if the system is initially in state $\mathscr{S}_i$, then after k time units it will be in state $\mathscr{S}_j$. Thus part (ii) of the above theorem provides us with asymptotic means of approximating the probability that given that the system is initial state $\mathscr{S}_i$, it will be in state $\mathscr{S}_j$ for the first time after $\mathrm{sp}_{i,j}(A)$ time steps.

COROLLARY 3.7. *Let A, B be minus M-matrices. Let $i, j \in \langle n\rangle$. For any $\eta > 0$, the following are equivalent:*

(i) $sp_{i,j}(B) \geqslant \mathrm{sp}_{i,j}(A)$.
(ii) *There exists $\tau_{i,j} > 0$ such that for all positive real $\beta > \eta$,*

$$((\beta I - B)^{-1})_{i,j} \leqslant \tau_{i,j}((\beta I - A)^{-1})_{i,j}.$$

Proof. Follows from Theorem 3.4. ∎

COROLLARY 3.8. *Let A, B be minus M-matrices. Then the following are equivalent*:

(i) $d_{i,j}(B) \leqslant d_{i,j}(A)$ *and* $\mathrm{sp}_{i,j}(B) \geqslant \mathrm{sp}_{i,j}(A)$ *for all* $i, j \in \langle n \rangle$.
(ii) *There exists* $\tau > 0$ *such that for all positive real* β,

$$(\beta I - B)^{-1} \leqslant \tau(\beta I - A)^{-1}.$$

Proof. Follows from Corollary 3.2 and Corollary 3.7. ∎

4. RELATIONSHIPS BETWEEN SPLITTINGS

In this section we discuss splittings of M-matrices and resolvents of their negatives. Let $A = M - N$ be a splitting of an M-matrix. We begin our discussion by relating the relative growth rates (for positive β near zero) of the resolvents of the negative of A and the negative of $M^{-1}A$ to the corresponding combinatorial property.

THEOREM 4.1. *Let $A = M - N$ be a splitting of an M-matrix such that $M^{-1}A$ is also an M-matrix. Then for any $\eta > 0$, the following are equivalent*:

(i) *There exists* $\tau > 0$ *such that for all positive real* $\beta < \eta$,

$$(\beta I + M^{-1}A)^{-1} \leqslant \tau(\beta I + A)^{-1}.$$

(ii) *The splitting is distance dominated.*

Proof. Follows from Corollary 3.2. ∎

Next we show how distance dominated splittings relate to two other combinatorial types of splittings: graph compatible splittings and weak graph compatible splittings.

THEOREM 4.2. *Let $A = M - N$ be a graph compatible splitting of an M-matrix such that $M^{-1}A$ is also an M-matrix. Then the splitting is distance dominated.*

Proof. Since the splitting is graph compatible, if we view A in block lower triangular Frobenius normal form, then M and M^{-1} will also be block lower triangular conformable with the blocks of A. For any class K of A, $(M^{-1}A)[K, K] = (M[K, K])^{-1}A[K, K]$, and by the assumption that $M^{-1}A$ is an M-matrix, $(M^{-1}A)[K, K]$ must also be an M-matrix. If $A[K, K]$ is nonsingular, then $(M^{-1}A)[K, K]$ is also nonsingular. If $A[K, K]$ is singular, then it is an irreducible singular M-matrix and hence has "property c." Moreover, there exists $x \gg 0$ such that $A[K, K]x = 0$. Then $(M[K, K])^{-1}A[K, K]x = 0$, and it follows from [2, p. 155] that $(M^{-1}A)[K, K]$ is also an M-matrix with "property c," and hence $\text{index}_0((M^{-1}A)[K, K]) = 1$.

Since by [11, Corollary 2.6] there are no access relationships in $G(M^{-1}A)$ which are not present in $G(A)$, the singular distance cannot grow. ∎

REMARK 4.3. We provide here an example of a graph compatible splitting of an M-matrix which is not distance dominated. This shows that the condition that $M^{-1}A$ is an M-matrix cannot be omitted in Theorem 4.2:

$$A = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ -1 & 0 & 1 \end{pmatrix}, \qquad M = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 1 \\ -2 & 1 & 0 \end{pmatrix},$$

$$N = \begin{pmatrix} 0 & 1 & 0 \\ 0 & -2 & 2 \\ -1 & 1 & -1 \end{pmatrix}, \quad M^{-1}A = \begin{pmatrix} 1 & -1 & 0 \\ 1 & -2 & 1 \\ 1 & -1 & 0 \end{pmatrix}.$$

Notice that $\text{index}_0(A) = 1$ while $\text{index}_0(M^{-1}A) = 2$. Thus $d_{1,2}(A) = 1$, but $d_{1,2}(M^{-1}A) = 2$.

REMARK 4.4. As a consequence of Theorem 4.2 and Schneider [11, Theorem 4.4 and Lemma 4.1], it is clear that every graph compatible, weak regular splitting of an M-matrix is distance dominated. On the other hand, in Remark 2.3 we have given an example of a distance dominated regular splitting of an M-matrix which is not graph compatible.

THEOREM 4.5. *Let $A = M - N$ be a distance dominated splitting of an M-matrix such that $M^{-1}A$ is an M-matrix. Then the splitting is weak graph compatible.*

Proof. By Theorem 4.1, for any $\eta > 0$ there exists $\tau > 0$ such that for all positive real $\beta < \eta$, $(\beta I + M^{-1}A)^{-1} \leqslant \tau(\beta I + A)^{-1}$. Then since we are dealing with M-matrices, by [11, Lemma 2.2] we have

$$G\left((\beta I + M^{-1}A)^{-1}\right) = \overline{G(M^{-1}A)} = \overline{G(M^{-1}N)} \quad \text{and}$$

$$G\left((\beta I + A)^{-1}\right) = \overline{G(A)}.$$

Thus if $i, j \in \langle n \rangle$, $i \neq j$, such that i does not have access to j in $G(A)$, then $((\beta I + A)^{-1})_{i,j} = 0$, which forces $((\beta I + M^{-1}A)^{-1})_{i,j} \leqslant 0$. But $\beta I + M^{-1}A$ is an M-matrix and so has a nonnegative inverse; thus $((\beta I + M^{-1}A)^{-1})_{i,j} = 0$, and thus i does not have access to j in $G(M^{-1}A)$. Hence $G(M^{-1}N) \subseteq \overline{G(A)}$. ■

REMARK 4.6. We prove here an example of a weak graph compatible, weak regular splitting of an M-matrix which is not distance dominated. This example was taken from [4]:

$$A = \begin{pmatrix} 0 & -1 \\ 0 & 1 \end{pmatrix}, \qquad M = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix},$$

$$N = \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}, \qquad M^{-1}A = \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix}.$$

Notice that $d_{1,2}(A) = 1$ while $d_{1,2}(M^{-1}A) = 2$.

Distance dominated splittings relate the resolvents of the appropriate matrices only for β near zero. Resolvent compatibility compares the resolvents of the appropriate matrices for all positive β, and hence compares their relative growth rates as β approaches zero and their relative decay rates as β approaches ∞. We give a combinatorial characterization of resolvent compatibility. We also show that G-compatibility (and hence resolvent compatibility) is not equivalent to graph compatibility.

THEOREM 4.7. *Let $A = M - N$ be a splitting of an M-matrix such that $M^{-1}A$ is an M-matrix. Then the following are equivalent*:

(i) *$A = M - N$ is a resolvent compatible splitting.*
(ii) *$A = M - N$ is a splitting such that for every $i, j \in \langle n \rangle$,*

$$d_{i,j}(M^{-1}A) \leqslant d_{i,j}(A) \quad \textit{and} \quad \mathrm{sp}_{i,j}(M^{-1}A) \geqslant \mathrm{sp}_{i,j}(A).$$

Proof. The result follows from applying Corollary 3.8 to $-A$ and $-M^{-1}A$. ■

REMARK 4.8. The example in Remark 2.3 shows that the condition that $M^{-1}A$ is an M-matrix cannot be omitted in Theorem 4.7. This example satisfies the combinatorial conditions of Theorem 4.7(ii) (and hence is distance dominated), but $M^{-1}A$ is not an M-matrix, and the splitting is not resolvent compatible.

REMARK 4.9. We provide here an example of a G-compatible (and hence resolvent compatible) regular splitting of an M-matrix which is not graph compatible. Unfortunately this contradicts [6, Lemma 5.7]:

$$A = \begin{pmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad M = \begin{pmatrix} 1 & -1 & 0 \\ -1 & 1 & 1 \\ 0 & 1 & 0 \end{pmatrix},$$

$$N = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \qquad M^{-1}A = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

REMARK 4.10. We prove here an example of a (graph compatible) M-splitting for which $\operatorname{index}_1(M^{-1}N) > 1$. This splitting cannot be G-compatible (nor resolvent compatible). This answers [6, Open Question 5.9] and contradicts [6, Lemma 5.8]:

$$A = \begin{pmatrix} 0 & 0 & 0 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix}, \qquad M = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix},$$

$$N = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad M^{-1}A = \begin{pmatrix} 0 & 0 & 0 \\ -1 & 0 & 0 \\ -1 & -1 & 0 \end{pmatrix},$$

$$M^{-1}N = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}.$$

Notice that $\operatorname{sp}_{3,1}(A) = 2$, while $\operatorname{sp}_{3,1}(M^{-1}A) = 1$. Now apply Theorem 4.7.

As pointed out in Remark 2.5, there exists an example of a weak graph compatible regular splitting of an M-matrix which is not graph compatible. In our next theorem we show that weak graph compatibility, with an additional hypothesis, implies graph compatibility.

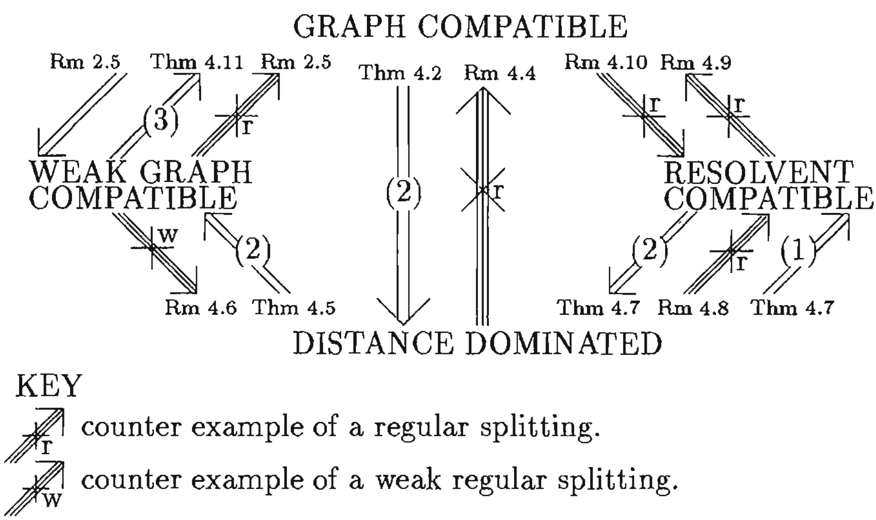
THEOREM 4.11. *Let $A = M - N$ be a weak graph compatible, regular splitting. If, for every class K of A, $(M^{-1})[K, K]$ has a positive element in every column, then the splitting is also graph compatible.*

Proof. Suppose not. Then there exists $j, k \in \langle n \rangle$ such that $M_{j,k} \neq 0$ and j does not have access to k in $G(A)$. Let $j \in J$ and $k \in K$, with J and K classes of A. Since, by assumption, j does not have access to k in $G(A)$, it must be that $A[J, K] = 0$. Since the splitting is regular, it follows that $N_{j,k} > 0$. But then since $(M^{-1})[J, J]$ has a nonzero element in every column, $(M^{-1}[J, J]N[J, K] > 0$. Since M^{-1} and N are nonnegative matrices, this implies that $(M^{-1}N)[J, K] > 0$. This contradicts that $G(M^{-1}N) \subseteq \overline{G(A)}$. ∎

A special case of the above theorem is:

COROLLARY 4.12. *Let $A = M - N$ be a weak graph compatible, regular splitting. If for every $i \in \langle n \rangle$ one has $(M^{-1})_{ii} > 0$, then the splitting is also graph compatible.*

In Figure 1 we provide a diagram showing the relationships between four types of splittings which have been considered in this paper. Notice that we can now establish other relationships by following various arrows around the diagram. Note that a G-compatible splitting is just a resolvent compatible weak splitting of an M-matrix.



(1) for a splitting such that $M^{-1}A$ is an M–matrix and for every $i, j \in \langle n \rangle, sp_{i,j}(M^{-1}A) \geq sp_{i,j}(A)$.
(2) for a splitting such that $M^{-1}A$ is an M–matrix.
(3) for a regular splitting such that $(M^{-1})[K, K]$ has a nonzero element in every column, for all classes K of A.

FIG. 1. Relationships established for splittings of an M-matrix.

OPEN QUESTION 4.13. In Remark 4.6 we give an example of a weak graph compatible, *weak* regular splitting which is not a distance dominated splitting. Does there exist a weak graph compatible regular splitting of an M-matrix which is not a distance dominated splitting?

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