Spectral Radii of Certain Iteration Matrices and Cycle Means of Digraphs

Ludwig Elsner
Fakultät für Mathematik
Universität Bielefeld
Postfach 100131
D-4800 Bielefeld 1, Germany

Daniel Hershkowitz*
Mathematics Department
Technion—Israel Institute of Technology
Haifa 32000, Israel

and

Hans Schneider*†
Mathematics Department
University of Wisconsin—Madison
Madison, Wisconsin 53706

ABSTRACT

Motivated by questions arising in the study of asynchronous iterative methods for solving linear systems, we consider the spectral radius of products of certain one cycle matrices. The spectral radius of a matrix in our class is a monotonic increasing function of the length of the cycle of the matrix, but this is known to be false for products of such matrices. The thrust of our investigation is to determine sufficient conditions under which the spectral radius of the product increases (decreases) when the lengths of the cycles of the factors increase (decrease). We also find sufficient

conditions for the spectral radius of the product to be independent of the order of the factors. Our chief tool is an auxiliary directed weighted graph whose cycle means determine the eigenvalues of the matrix product, and our main results are stated in terms of the maximal cycle mean of this graph.

1. INTRODUCTION

The question studied in this paper came up in a more general form in connection with investigations of iterative methods for solving a $k \times k$ linear system

$$(1.1) \qquad\qquad x = Bx + c.$$

Here one usually assumes that the elements of the matrix B are nonnegative and that the spectral radius $\rho(B)$ of B satisfies $\rho(B) < 1$, so the basic iteration $x^{(i+1)} = Bx^{(i)} + c$ converges to the solution $x^* = (I - B)^{-1}c$ of (1.1). Splitting the work to calculate $Bx + c$ between several parallel processors, operating independently one of each other in an asynchronous manner, and where the assignment of subtasks and storage for the current iterate is done by a central processor, leads to iterative processes which can be described by an iterative procedure of the form

$$(1.2) \quad x^{(i)} = \left(I - E_{j_i}\right)x^{(i-1)} + E_{j_i}\left(Bx^{(i-s_i)} + c\right), \qquad i = 1, 2, \ldots,$$

where $x^{(0)} = x^{(-1)} = \cdots = x^{(-n)}$. This depends on a sequence $\{j_i\}_{i=1}^{\infty}$, $1 \leqslant j_i \leqslant m$, where j_i describes the subtask completed at time i, and a sequence $\{s_i\}_{i=1}^{\infty}$, $1 \leqslant s_i \leqslant n$, where s_i is a measure of the "age" of information used in calculating the jth approximation. The nonzero 0–1 $k \times k$ matrices E_i, $i = 1, \ldots, m$, describing the m different subtasks, satisfy $\sum_{i=1}^{m} E_i = I$. For technical details we refer the reader to the papers [1], [2], and [3].

For the procedure (1.2) it has been established in [1] that, independently of $x^{(0)}$,

$$\lim_{i \to \infty} x^{(i)} = x^*,$$

provided that each number in $\{1, \ldots, m\}$ appears infinitely many times in the sequence $\{j_i\}_{i=1}^{\infty}$.

When discussing the dependence of the rate of convergence of (1.2) on the parameters $\{s_i\}$, it is tempting to conjecture that using older information will not decrease the convergence rate. Partial results of this kind were given

in [2], generalizing results in [3]. In particular, if we define the rate of convergence $\tilde{R}(\{s_i\})$ of (1.2) by

$$\tilde{R}(\{s_i\}) = \sup_{x^{(0)} \in \mathbb{R}^k} \limsup_{i \to \infty} \|x^{(i)} - x^*\|^{1/i},$$

where $\| \ \|$ is some vector norm, then it was shown that

(1.3) $$s_i' \leqslant s_i, \qquad i = 1, 2, \ldots,$$

implies

(1.4) $$\tilde{R}(\{s_i'\}) \leqslant \tilde{R}(\{s_i\}),$$

provided that either

(1.5) $$s_{i+1} \leqslant s_i + 1, \qquad i = 1, 2, \ldots,$$

or

(1.6) $$s_{i+1}' \leqslant s_i' + 1, \qquad i = 1, 2, \ldots .$$

However, in general the implication $(1.3) \Rightarrow (1.4)$ does not hold. In [2], a counterexample was given in the simplest case

$$m = k = 1 \quad (\text{i.e.} \quad j_i = 1, \ i = 1, 2, \ldots), \qquad B = (\rho),$$

$$s_i = s_{i+p}, \quad i = 1, 2, \ldots,$$

for a fixed period p. In this case, (1.2) reduces to

(1.7) $$x^{(i)} = \rho x^{(i-s_i)} + c, \qquad i = 1, 2, \ldots,$$

where $x^{(i)} \in \mathbb{R}$. It turns out, on embedding the real valued multilevel iteration (1.7) in a one-level iteration in $\mathbb{R}^n$ where $n \geqslant \max\{s_i, i = 1, \ldots, p\}$, that the convergence rate $\tilde{R}(\{s_i\})$ of (1.7) is given by

(1.8) $$\tilde{R}(\{s_i\}) = R(\{r_i\}) = \rho \left(A_{r_1} A_{r_2} \cdots A_{r_p} \right)^{1/p},$$

where the $n \times n$ matrix A_{r_k} is defined by

$$\left(A_{r_k} \right)_{ij} = \begin{cases} 1, & i = j + 1, \\ \rho, & (i, j) = (1, r_k), \\ 0, & \text{otherwise}, \end{cases}$$

$\rho(A_{r_1} A_{r_2} \cdots A_{r_p})$ denotes the spectral radius of $A_{r_1} A_{r_2} \cdots A_{r_p}$, and $r_i = s_{p+1-i}$, $i = 1, \ldots, p$.

Our goal in this paper is to study conditions for the implication

$$(1.9) \qquad r_i' \leqslant r_i, \quad i = 1, 2, \ldots, \quad \Rightarrow \quad R(\{r_i'\}) \leqslant R(\{r_i\}),$$

where $R(\{r_i\})$ is given by (1.8). Observe that (1.9) is equivalent to (1.3) $\Rightarrow$ (1.4).

We now describe the results in our paper. In Section 2 we define the matrices A_{d_i} whose graph has just one cycle of length d_i, and we associate with each product of p matrices $A_{d_1}, \ldots, A_{d_p}$ an auxiliary directed weighted graph $\Delta(d_1, \ldots, d_p)$ on p vertices. In this auxiliary graph there is just one arc from each vertex, and the length and weight d_i of this arc equals the length of the cycle in the corresponding matrix in the product. These observations are used to describe all the eigenvalues of products $A_{d_1} A_{d_2} \cdots A_{d_p}$.

In Section 3 we show that the eigenvalues of the product $A_{d_1} \cdots A_{d_p}$ are determined by the cycle means of the auxiliary graph. In particular, for fixed p, the spectral radius $\rho(A_{d_1} \cdots A_{d_p})$ is a monotonic increasing function of the maximum cycle mean $\mu(\Delta(\delta))$, where $\delta = (d_1, \ldots, d_p)$. (Actually, in Sections 2 and 3 we consider a slightly more general situation.) Thus, it is possible to state the desired results on $\rho(A_{d_1} \cdots A_{d_p})$ in terms of $\mu(\Delta(\delta))$, and hence in the remaining sections of the paper we consider the maximal cycle mean of the graph $\Delta(\delta)$ as δ is varied.

In Section 4 we define the concept of a downward (upward) optimal sequence: in brief, a sequence δ such that $\mu(\Delta(\delta))$ does not increase (decrease) when the lengths of the arcs δ do not increase (decrease). Equivalently, a sequence is downward optimal if and only if (1.9), with r_i and r_i' replaced by d_i and d_i' respectively, holds; and it is upward optimal if and only if (1.9), with r_i and r_i' replaced by d_i' and d_i respectively, holds. In Theorem 4.15 we prove our most general sufficient condition for δ to be downward optimal. A simple intuitive special case is stated in Theorem 4.17 and Corollary 4.18.

In Section 5 we derive analogous results for upward optimal sequences; see Theorems 5.6 and 5.8. Among others, our results in Sections 4 and 5

imply the result in [2] that if either (1.5) holds or (1.6) holds, then the implication (1.3) $\Rightarrow$ (1.4) holds. We remark that we have found somewhat shorter proofs for the special cases of Theorems 4.17 and 5.8, which however are not given in our paper.

In Section 6 we prove several sufficient conditions for the maximum cycle mean $\mu(\Delta(\delta))$ to be invariant under all permutations of $(d_1, \ldots, d_p)$.

2.   EIGENVALUES OF PRODUCTS OF CERTAIN MATRICES

In this section we describe all the eigenvalues of products of matrices which are slight generalizations of the matrices A_{r_k} defined in the previous section. All the matrices in this paper are $n \times n$ matrices. We start with a few definitions.

DEFINITION 2.1.   A *path* in a digraph is a sequence $\{(t_k, t_{k+1})\}_{k=1}^m$ of m arcs. We denote such a path by $(t_1, \ldots, t_{m+1})$. A path $(t_1, \ldots, t_{m+1})$ is said to be a *cycle* if $t_{m+1} = t_1$. Such a cycle is said to be *simple* if $t_1, \ldots, t_m$ are distinct.

CONVENTION 2.2.   Throughout this paper, δ and δ' are considered to be sequences $(d_1, \ldots, d_p)$ and $(d'_1, \ldots, d'_p)$ of positive integers respectively, and χ is considered to be a sequence $(c_1, \ldots, c_p)$ of numbers. For a positive integer m, $m > p$, we let $d_m = d_{(m-1)(\mathrm{mod}\ p)+1}$ and $d'_m = d'_{(m-1)(\mathrm{mod}\ p)+1}$.

DEFINITION 2.3.

(i) We denote by $\Delta(\delta)$ the arc-weighted digraph with vertex set $\{1, \ldots, p\}$, and with an arc from i to j with weight d_i whenever $j - i \equiv d_i \,(\mathrm{mod}\ p)$. Such an arc will be denoted by $i_{d_i}j$. The weight d_i of the arc $i_{d_i}j$ is also called the length of that arc.

(ii) We denote by $\Delta(\delta, \chi)$ the vertex-weighted and arc-weighted digraph with vertex set $\{1, \ldots, p\}$, with weight c_i on vertex i, $i = 1, \ldots, p$, and with same arcs as in $\Delta(\delta)$.

DEFINITION 2.4.   Let A be an $n \times n$ matrix. The *digraph* $G(A)$ of A is defined to be the digraph with vertex set $\{1, \ldots, n\}$, and where there is an arc from i to j if and only if $a_{ij} \neq 0$.

NOTATION 2.5.   Let d and n be positive integers, $d \leqslant n$, and let c be a number. We denote by $A_{d,c}$ the $n \times n$ matrix defined by

$$
(A_{d,c})_{ij} = \begin{cases} 1, & i = j + 1, \\ c, & (i,j) = (1,d), \\ 0, & \text{otherwise.} \end{cases}
$$

LEMMA 2.6.   *Let B be the matrix product $A_{d_1,c_1} A_{d_2,c_2} \cdots A_{d_p,c_p}$, and let $\beta = (i_0, \ldots, i_t)$, $i_t = i_0$, be a cycle in $G(B)$. Then β corresponds to a cycle $\gamma = (j_1, \ldots, j_l)$ in $\Delta(\delta, \chi)$, where $(\prod_{k=1}^{t} b_{i_{k-1} i_k})^{1/t} = (\prod_{k=1}^{l} c_{j_k})^{p/d}$ for $d = \sum_{k=1}^{l} d_{j_k}$.*

Proof.   Since

$$
b_{ij} = \sum_{m_1, \ldots, m_{p-1} \in \{1, \ldots, n\}} (A_{d_1,c_1})_{i m_1} (A_{d_2,c_2})_{m_1 m_2} \cdots (A_{d_p,c_p})_{m_{p-1} j},
$$

it follows by Notation 2.5 that

$$
b_{ij} = \underbrace{1 \cdot 1 \cdot \cdots \cdot 1}_{i-1} \cdot c_i \cdot \underbrace{1 \cdot 1 \cdot \cdots \cdot 1}_{d_i - 1} \cdot c_{i+d_i} \cdot \underbrace{1 \cdot 1 \cdot \cdots \cdot 1}_{d_{i+d_i} - 1} \cdot \cdots _{\displaystyle p}.
$$

Also, $\prod_{k=1}^{t} b_{i_{k-1} i_k}$ is the (i_0, i_0) diagonal element of the matrix B^t. Therefore, $\prod_{k=1}^{t} b_{i_{k-1} i_k}$ is a product of a total of tp 1's and c's, where if some c occurs in position l then the next c occurs d_l positions further down. Observe that the cycle β in $G(B)$ corresponds to the cycle $\gamma = i_{0 d_{i_0}} (i_0 + d_{i_0})_{d_{i_0 + d_{i0}}} \cdots$ in $\Delta(\delta, \chi)$. Note that the vertices $j_1, \ldots, j_l$ of the cycle γ are given by $j_1 = i_0$ and $j_{k+1} = j_k + d_{j_k}$, $k = 1, \ldots, l-1$. Observe that we have $\prod_{k=1}^{t} b_{i_{k-1} i_k} = \prod_{k=1}^{l} c_{j_k}$. Since the total weight $d = \sum_{k=1}^{l} d_{j_k}$ of arcs in γ is equal to tp, we have $1/t = p/d$, and our claim follows.   ∎

Conversely, we have

LEMMA 2.7.   *Let $B = A_{d_1,c_1} A_{d_2,c_2} \cdots A_{d_p,c_p}$, and let $\gamma = (j_1, \ldots, j_l)$ be a cycle in $\Delta(\delta, \chi)$. Let $d = \sum_{k=1}^{l} d_{j_k}$. Then γ corresponds to a cycle $\beta = (i_0, \ldots, i_t)$, $i_t = i_0$, in $G(B)$, where $(\prod_{k=1}^{t} b_{i_{k-1} i_k})^{1/t} = (\prod_{k=1}^{l} c_{j_k})^{p/d}$.*

Proof.   Let m be the total weight of arcs in γ. Since γ is a cycle, it follows that m is divisible by p. Let α be the sequence

$$
\underbrace{\left(j_1, j_1 - 1, \ldots, 1, d_{j_1}, d_{j_1} - 1, \ldots, 1, d_{j_1 + d_{j_1}}, d_{j_1 + d_{j_1}} - 1, \ldots \right)}_{\displaystyle m}.
$$

Similarly to the discussion in the proof of Lemma 2.6, it follows that $\beta = (\alpha_1, \alpha_{1+p}, \alpha_{1+2p}, \ldots, \alpha_{1+m-p}, \alpha_1)$ is a cycle of m/p arcs in $G(B)$ satisfying our requirements. ∎

DEFINITION 2.8. A *connected component* of a digraph G is a maximal set of vertices such that for every two vertices i and j in the set there exists a sequence $(t_1, \ldots, t_m)$ of vertices such that $t_1 = i$, $t_m = j$, and for every $k \in \{1, \ldots, m - 1\}$ either (t_k, t_{k+1}) or (t_{k+1}, t_k) is an arc in G.

LEMMA 2.9. *Let* $B = A_{d_1, c_1} A_{d_2, c_2} \cdots A_{d_p, c_p}$. *Then every connected component of $G(B)$ has one cycle and maybe some paths terminating at vertices of the cycle.*

Proof. Our claim follows from the fact that in each A_{d_i, c_i} every row contains exactly one nonzero element. Therefore, every row of B contains exactly one nonzero element, that is, every vertex i in $G(B)$ has exactly one arc originating at i. ∎

COROLLARY 2.10. *The spectrum of $B = A_{d_1, c_1} A_{d_2, c_2} \cdots A_{d_p, c_p}$ consists of the multiset* $S = \{t\text{th roots of } \prod_{k=1}^t b_{i_{k-1} i_k} : (i_0, \ldots, i_t), i_t = i_0, \text{ is a cycle in } G(B)\}$ *and* $n - |S|$ *zeros.*

Proof. In view of Lemma 2.9, the irreducible components (diagonal blocks in the Frobenius normal form) of B are the submatrices whose digraphs are the cycles of $G(B)$ as well as maybe some zero 1×1 matrices. The claim follows. ∎

3. SPECTRAL RADII OF PRODUCTS OF MATRICES AND MAXIMAL CYCLE MEANS OF DIGRAPHS

In this section we apply the results of the previous section to evaluate the spectral radius of the product $A_{r_1} A_{r_2} \cdots A_{r_p}$. For the sake of consistency with the previous section we use the sequence $(d_1, \ldots, d_p)$, where $d_i = r_i$, $i = 1, \ldots, p$. Also we use c for the constant ρ of Section 1. From now on in this paper, we assume that $c_1 = \cdots = c_p = c$, where c is a real number, $0 < c < 1$.

NOTATION 3.1

Let γ be a cycle in $\Delta(\delta)$. We denote by $\mu(\gamma)$ the cycle mean of γ, that is, the average of weights of arcs in γ. We denote by $\mu(\Delta(\delta))$ the maximal cycle mean of $\Delta(\delta)$, that is, the maximal $\mu(\gamma)$ where γ is a cycle in $\Delta(\delta)$.

DEFINITION 3.2.   A cycle γ in $\Delta(\delta)$ is said to be *maximal* if $\mu(\gamma) = \mu(\Delta(\delta))$. A cycle γ in $\Delta(\delta)$ is said to be *minimal* if $\mu(\gamma) \leqslant \mu(\gamma')$ for any cycle γ' in $\Delta(\delta)$.

REMARK 3.3.   It is easy to verify that there always exist a *simple* maximal cycle and a *simple* minimal cycle.

We now obtain the following theorem as a corollary of Lemmas 2.6, 2.7 and Corollary 2.10.

THEOREM 3.4.   *The spectrum of $B = A_{d_1, c} A_{d_2, c} \cdots A_{d_p, c}$ consists of a multiset $S = \{t$th roots of $c^{tp/\mu(\gamma)} : \gamma$ is a cycle in $\Delta(\delta)$, t is a positive integer depending on $\gamma\}$ and $n - |S|$ zeros.*

Proof.   By Corollary 2.10, the spectrum of B consists of the multiset $S = \{t$th roots of $\prod_{k=1}^{t} b_{i_{k-1} i_k} : (i_0, \ldots, i_t), i_t = i_0$, is a cycle in $G(B)\}$ and $n - |S|$ zeros. Let γ be a cycle in $\Delta(\delta)$ with l arcs and total weights d of arcs. Then $\mu(\gamma) = d/l$. Our claim now follows by Lemmas 2.6 and 2.7.   ∎

The following corollary of Theorem 3.4 is an important tool in our study.

THEOREM 3.5.   *The spectral radius of $B = A_{d_1, c} A_{d_2, c} \cdots A_{d_p, c}$ is equal to $c^{p/\mu(\Delta(\delta))}$.*

Proof.   By Corollary 2.10, the positive eigenvalues of B are

$$\left\{ c^{p/\mu(\gamma)} : \gamma \text{ is a cycle in } \Delta(\delta) \right\}.$$

Since $0 < c < 1$, it follows that the largest one is $c^{p/\mu(\Delta(\delta))}$.   ∎

NOTATION 3.6.   We denote by $\rho(\delta, c)$ the spectral radius of the product $B = A_{d_1, c} A_{d_2, c} \cdots A_{d_p, c}$.

We conclude this section with a couple of examples that illustrate our results.

EXAMPLE 3.7.   Let $n = 6$, let $c = 0.5$, and let $\delta = (2, 4, 6, 3)$. The cycles in $\Delta(\delta)$ are $1_2 3_6 1$ and $2_4 2$. Both have cycle mean 4, and hence we have $\mu(\Delta(\delta)) = 4$. Indeed, the nonzero eigenvalues of $A_{2, 0.5} A_{4, 0.5} A_{6, 0.5} A_{3, 0.5}$ are the second roots of 0.5^2 and the first root of 0.5, and so we have $\rho(\delta, c) = 0.5 = 0.5^{4/4} = c^{p/\mu(\Delta(\delta))}$. Another example is when we choose $n = 10$,

$c = 0.2$, and $\delta = (7, 4, 10)$. In this case, the only cycle in $\Delta(\delta)$ is $1_7 2_4 3_{10} 1$, and hence we have $\mu(\Delta(\delta)) = 7$. The nonzero eigenvalues of $A_{7, 0.2} A_{4, 0.2} A_{10, 0.2}$ are the seventh roots of 0.2^3, and thus $\rho(\delta, c) = 0.5017 = 0.2^{3/7} = c^{p/\mu(\Delta(\delta))}$. This latter example also shows that the numbers t in the definition of the set S in Theorem 3.4 are not necessarily the numbers of arcs in the cycles γ, as one might think in view of the first example.

4. DOWNWARD OPTIMAL SEQUENCES

In view of Theorem 3.5, in order to study $\rho(\delta, c)$ it is enough to study $\mu(\Delta(\delta))$. Indeed, in the sequel we study maximal cycle means of digraphs $\Delta(\delta)$. Since $0 < c < 1$, it also follows from Theorem 3.5 that the bigger $\mu(\Delta(\delta))$ is, the bigger $\rho(\delta, c)$ is.

NOTATION 4.1. We denote by $\delta \geqslant \delta'$ the case where $d_i \geqslant d'_i$, $i \in \{1, \ldots, p\}$.

DEFINITION 4.2.

(i) The sequence δ is said to be *downward optimal* if $\mu(\Delta(\delta)) \geqslant \mu(\Delta(\delta'))$ whenever $\delta \geqslant \delta'$.
(ii) The sequence δ is said to be *upward optimal* if $\mu(\Delta(\delta)) \leqslant \mu(\Delta(\delta'))$ whenever $\delta \leqslant \delta'$.

The following example is of a sequence which is neither downward optimal nor downward optimal. Examples of optimal sequences will be given in the sequel.

EXAMPLE 4.3. The only cycle in $\Delta(6, 2, 14)$ is $1_6 1$, and hence $\mu(\Delta(6, 2, 14)) = 6$. The cycle $1_5 3_{14} 2_2 1$ is the only cycle in $\Delta(5, 2, 14)$, and so $\mu(\Delta(5, 2, 14)) = 7 > \mu(\Delta(6, 2, 14))$, implying that $(6, 2, 14)$ is now downward optimal. The only cycle in $\Delta(7, 2, 14)$ is $1_7 2_2 1$, and so $\mu(\Delta(7, 2, 14)) = 4.5 < \mu(\Delta(6, 2, 14))$, implying that $(6, 2, 14)$ is not upward optimal.

In this section we shall look for conditions for sequences of positive integers to be downward optimal. For the sake of convenience in stating our results we now define a digraph which is a spread of the digraph $\Delta(\delta)$ over the positive real axis.

DEFINITION 4.4.   We denote by $\tilde{\Delta}(\delta)$ the digraph whose vertices are the positive integers, and with arc from i to j whenever $j - i = d_i$. Such an arc (i, j) is said to be of *length* d_i.

DEFINITION 4.5.   The total length of the arcs in a path $\tilde{\gamma}$ in $\tilde{\Delta}(\delta)$ is said to be the *length* of $\tilde{\gamma}$.

OBSERVATION 4.6.   There is a correspondence between an arc $i_d{}_l j$ in $\Delta(\delta)$ and all arcs (k, l) in $\tilde{\Delta}(\delta)$ such that $i = (k - 1)(\mathrm{mod}\ p) + 1$. Therefore, a path in $\tilde{\Delta}(\delta)$ corresponds to a unique path in $\Delta(\delta)$, but a path in $\Delta(\delta)$ corresponds to infinitely many paths in $\tilde{\Delta}(\delta)$ (with different starting points). Also, a path in $\tilde{\Delta}(\delta)$ whose length is divisible by p corresponds to a unique cycle in $\Delta(\delta)$, and a cycle in $\Delta(\delta)$ corresponds to infinitely many paths in $\tilde{\Delta}(\delta)$, where the length of each is equal to the total weight of the arcs in γ.

DEFINITION 4.7.   Let $\gamma = (i_0, i_1, \ldots, i_q)$, $i_q = i_0$, be a cycle in $\Delta(\delta)$. A *spread* of γ is an infinite sequence $(t_1, t_2, \ldots)$ of positive integers such that

$$t_1 = (i_j - 1)(\mathrm{mod}\ p) + 1 \qquad \text{for some} \quad j \in \{0, \ldots, q\}$$

and

$$t_{k+1} = t_k + d_{t_k}, \qquad k = 1, 2, \ldots .$$

OBSERVATION 4.8.   Every cycle γ in $\Delta(\delta)$ has infinitely many spreads. Also, every truncation (from the beginning) of a spread of γ is a spread of γ.

LEMMA 4.9.   *Let $\delta' \leqslant \delta$, let γ and γ' be cycles in $\Delta(\delta)$ and $\Delta(\delta')$ respectively, and let $(t_1, t_2, \ldots)$ and $(t'_1, t'_2, \ldots)$ be spreads of γ and γ' respectively. If for some i and j we have $t'_i = t_j$, then $t'_{i+1} \leqslant t_{j+1}$.*

Proof.   The claim follows immediately from Definition 4.7 and the fact that $d'_i \leqslant d_i$, $i \in \{1, \ldots, p\}$.   ∎

LEMMA 4.10.   *Let $\delta' \leqslant \delta$, let γ and γ' be cycles in $\Delta(\delta)$ and $\Delta(\delta')$ respectively, and let $(t_1, t_2, \ldots)$ and $(t'_1, t'_2, \ldots)$ be spreads of γ and γ' respectively. Assume that there exists a positive integer k such that no path in $\tilde{\Delta}(\delta)$ corresponding to any k consecutive arcs of γ lies in the interior of the union of any k arcs of $\tilde{\Delta}(\delta)$. Then if for some i and j we have $t'_i < t_j$, then $t'_{i+k} \leqslant t_{j+k}$.*

Proof. Assume to the contrary that $t'_{i+k} > t_{j+k}$. Then the path $\beta = (t_j, t_{j+1}, \ldots, t_{j+k})$ in $\tilde{\Delta}(\delta)$, corresponding to k consecutive arcs of γ, lies in the interior of the path $(t'_i, t'_{i+1}, \ldots, t'_{j+k})$ in $\tilde{\Delta}(\delta')$. Since $\delta' \leqslant \delta$, it follows by Definition 4.4 that every arc in $\tilde{\Delta}(\delta')$ is contained in an arc in $\tilde{\Delta}(\delta)$. Therefore, the path β lies in the interior of the union of k arcs of $\tilde{\Delta}(\delta)$, in contradiction to the conditions of the lemma.   ∎

A repeated application of Lemmas 4.9 and 4.10 yields the following.

COROLLARY 4.11.   *Let $\delta' \leqslant \delta$, let γ and γ' be cycles in $\Delta(\delta)$ and $\Delta(\delta')$ respectively, and let $(t_1, t_2, \ldots)$ and $(t'_1, t'_2, \ldots)$ be spreads of γ and γ' respectively, such that $t'_1 \leqslant t_1$. Assume that there exists a positive integer k such that no path in $\tilde{\Delta}(\delta)$ corresponding to any k consecutive arcs of γ lies in the interior of the union of any k arcs of $\tilde{\Delta}(\delta)$. Then there are infinitely many i's for which $t'_i \leqslant t_i$.*

LEMMA 4.12.   *Let $\delta' \leqslant \delta$, let γ and γ' be cycles in $\Delta(\delta)$ and $\Delta(\delta')$ respectively, and let $(t_1, t_2, \ldots)$ and $(t'_1, t'_2, \ldots)$ be spreads of γ and γ' respectively, such that $t'_1 \leqslant t_1 < t'_2$. If $\mu(\gamma') > \mu(\gamma)$, then there exists a positive integer M such that $t'_i > t_i$ for all $i > M$.*

Proof. Let the total weights of arcs in γ be $m_1 p$, and let the total weights of arcs in γ' be $m_2 p$. Let m be the least common multiple of m_1 and m_2. Then the segment of length mp in $\tilde{\Delta}(\delta)$ starting at t_1 is covered by an integral number of repetitions of γ, and the segment of length mp in $\tilde{\Delta}(\delta')$ starting at t'_1 is covered by an integral number of repetitions of γ'. Let k_1 and k_2 be the total numbers of arcs in these repetitions of γ and γ' respectively. Since $\mu(\Delta(\delta')) > \mu(\Delta(\delta))$, and since $k_1 = mp/\mu(\Delta(\delta))$ and $k_2 = mp/\mu(\Delta(\delta'))$, it follows that $k_1 > k_2$. The segment of length mp in $\tilde{\Delta}(\delta)$ starting at t_1 ends at t_{1+k_1}, while the segment of length mp in $\tilde{\Delta}(\delta')$ starting at t'_1 ends at t'_{1+k_2}. Also, since $t_1 < t'_2$, we have

$$(4.13) \qquad\qquad t_{1+k_1} < t'_{2+k_2}.$$

Let l be a positive integer satisfying $l > k_2/(k_1 - k_2)$ or, equivalently,

$$(4.14) \qquad\qquad lk_1 > (l+1)k_2.$$

We now prove that $M = 1 + (l+1)k_1$ satisfies the required property. Let $m > M$, and let $m - (1 + k_1) = h_1 k_1 + r_1 = h_2 k_2 + r_2$, where h_1 and h_2 are nonnegative integers and $r_1 < k_1$, $r_2 < k_2$. Since $m > M$, we have

$m - (1 + k_1) > lk_1$, and it follows from (4.14) that $h_1 + 1 \leqslant h_2$. Therefore, with (4.13) we obtain

$$t_m = t_{1+k_1+h_1k_1+r_1} < t_{1+k_1+(h_1+1)k_1} = t_{1+k_1} + (h_1 + 1)mp \leqslant t_{1+k_1} + h_2 mp$$

$$< t'_{2+k_2} + h_2 mp = t'_{2+k_2+h_2k_2} \leqslant t'_{2+k_2+m-(1+k_1)} = t'_{1-(k_1-k_2)+m} \leqslant t'_m. \quad \blacksquare$$

Let γ and γ' be cycles in $\Delta(\delta)$ and $\Delta(\delta')$ respectively. We remark that, by Observation 4.8, we can find spreads $(t_1, t_2, \ldots)$ and $(t'_1, t'_2, \ldots)$ of γ and γ', respectively, such that $t'_1 \leqslant t_1 < t'_2$. Therefore, as a corollary of Corollary 4.11 and Lemma 4.12 we now obtain the following theorem.

THEOREM 4.15. *If there exists a cycle γ in $\Delta(\delta)$ and a positive integer k such that for every k consecutive arcs of γ, a path in $\tilde{\Delta}(\delta)$ corresponding to those arcs does not lie in the interior of the union of any k arcs of $\tilde{\Delta}(\delta)$, then γ is a maximal cycle and δ is a downward optimal sequence.*

Proof. Let $\hat{\gamma}$ be any cycle in $\Delta(\delta)$. It follows from Corollary 4.11 and Lemma 4.12 that $\mu(\gamma) \geqslant \mu(\hat{\gamma})$, and therefore γ is a maximal cycle in $\Delta(\delta)$. Also, $\mu(\gamma) \geqslant \mu(\gamma')$ for every cycle γ' in $\Delta(\delta')$ where $\delta' \leqslant \delta$, and hence δ is a downward optimal sequence. $\blacksquare$

COROLLARY 4.16. *If there exists a positive integer k such that for every k consecutive arcs in $\Delta(\delta)$, a path in $\tilde{\Delta}(\delta)$ corresponding to those arcs does not lie in the interior of the union of any k arcs of $\tilde{\Delta}(\delta)$, then every cycle in $\Delta(\delta)$ is a maximal cycle, and δ is a downward optimal sequence.*

If we choose $k = 1$, then Theorem 4.15 and Corollary 4.16 yield the following.

THEOREM 4.17. *If there exists a cycle γ in $\Delta(\delta)$ such for every arc of γ, an arc in $\tilde{\Delta}(\delta)$ corresponding to that arc does not lie in the interior of another arc of $\tilde{\Delta}(\delta)$, then γ is a maximal cycle and δ is a downward optimal sequence.*

COROLLARY 4.18. *If no arc of $\tilde{\Delta}(\delta)$ lies in the interior of another arc of $\tilde{\Delta}(\delta)$, then every cycle in $\Delta(\delta)$ is a maximal cycle, and δ is a downward optimal sequence.*

It is natural to ask whether Theorem 4.15 is equivalent to its weaker form in Theorem 4.17. The answer to this question is negative, as demonstrated by the following example.

EXAMPLE 4.19. The digraph $\Delta = \Delta(11, 14, 1, 1, 13)$ consists of the arcs $1_{11}2$, $2_{14}1$, 3_14, 4_15, and $5_{13}3$. We have $\mu(\Delta) = 12.5$, and the only maximal

cycle is $\gamma = (1, 2, 1)$. Observe that $1_{11}2$ corresponds to $(6, 17)$, which lies in the interior of (5.18), corresponding to $5_{13}3$. Therefore, the condition in Theorem 4.17 is not satisfied. Now, take the consecutive two arcs $1_{11}2$ and $2_{14}1$ of γ. Since their total length is 25, the union of two arcs to contain a corresponding path in $\tilde{\Delta}(11, 14, 1, 1, 13)$ in its interior should be of length at least 27. The only possibilities for such pairs of arcs would therefore be the pairs of arcs corresponding to $2_{14}1, 2_{14}1$ or to $2_{14}1, 5_{13}3$ or to $5_{13}1, 2_{14}1$. It is easy to check that none of these unions contains a path corresponding to the consecutive two arcs $1_{11}2$ and $2_{14}1$ in its interior. Therefore, the condition in Theorem 4.15 is satisfied for $k = 2$.

The converse of Theorem 4.15 and of Corollary 4.18 does not hold in general, as demonstrated by the following example.

EXAMPLE 4.20. The digraph $\Delta(6, 1, 8)$ consists of the loop $1_6 1$ and the arcs $2_1 3$ and $3_8 2$. We have $\mu(\Delta(6, 1, 8)) = 6$, and the only maximal cycle is the loop on 1. The arc $1_6 1$ corresponds to $(4, 10)$, which lies in the interior of $(3, 11)$, corresponding to the arc $3_8 2$. Thus, for every positive integer k, k consecutive arcs $1_6 1$ correspond to the path $(4, 10, \ldots, 4 + 6k)$ in $\Delta(6, 1, 8)$, which lies interior of the union $\bigcup_{i=1}^{k}(6i - 3, 6i + 5)$ of k arcs, each corresponding to $3_8 2$. However, since

$$\mu(\Delta(1, 1, 8)) = \mu(\Delta(2, 1, 8)) = \mu(\Delta(3, 1, 8)) = \mu(\Delta(4, 1, 8))$$

$$= \mu(\Delta(5, 1, 8)) = 4.5,$$

since

$$\mu(\Delta(1, 1, 7)) = 3, \qquad \mu(\Delta(2, 1, 7)) = 4.5, \qquad \mu(\Delta(3, 1, 7)) = 3,$$

$$\mu(\Delta(4, 1, 7)) = 4, \qquad \mu(\Delta(5, 1, 7)) = 6, \qquad \mu(\Delta(6, 1, 7)) = 6,$$

and since clearly $\mu(\Delta(d_1, 1, d_3)) \leqslant 6$ whenever $d_1, d_3 \leqslant 6$, it follows that $6 = \mu(\Delta(6, 1, 8)) \geqslant \mu(\Delta(d_1', 1, d_3'))$ for every positive integers d_1', d_3' such that $6 \geqslant d_1'$, $8 \geqslant d_3'$, and so the sequence $(6, 1, 8)$ is downward optimal.

Yet, in the case $p = 2$ the converse of Theorem 4.15 is true. Observe that $\Delta(d_1, d_2)$ and $\Delta(d_2, d_1)$ are the same, up to relabeling of the vertices, and therefore, without loss of generality, we may assume that $d_1 \geqslant d_2$.

THEOREM 4.21. *Let d_1 and d_2 be positive integers, $d_1 \geqslant d_2$. Then the following are equivalent:*

(i) (d_1, d_2) *is a downward optimal sequence.*

(ii) *Either d_1 is odd and $d_1 - d_2 \leqslant 2$, or d_1 is even.*

(iii) *For every two consecutive arcs of a maximal cycle in $\Delta(d_1, d_2)$, a path in $\tilde{\Delta}(d_1, d_2)$ corresponding to those arcs does not lie in the interior of the union of any two arcs of $\tilde{\Delta}(d_1, d_2)$.*

Proof. (i) $\Rightarrow$ (ii): If d_1 is odd and $d_1 - d_2 > 2$, then we have

$$\mu(\Delta(d_1, d_2)) = \begin{cases} d_2, & d_2 \text{ even,} \\ \dfrac{d_1 + d_2}{2}, & d_2 \text{ odd.} \end{cases}$$

Observe that $\mu(\Delta(d_1 - 1, d_2)) = d_1 - 1 > (d_1 + d_2)/2 \geqslant \mu(\Delta(d_1, d_2))$, and hence (d_1, d_2) is not downward optimal.

(ii) $\Rightarrow$ (iii): Distinguish between three possible case:

(a) d_1 *is even.* The maximal cycle is $1_{d_1}1$, as well as $2_{d_2}2$ if $d_2 = d_1$.

(b) d_1 *and d_2 are odd.* The maximal cycle is $1_{d_1}2_{d_2}1$.

(c) d_1 *is odd and d_2 is even.* The maximal cycle is $2_{d_2}2$.

In any case, it is easy to verify that, since $d_1 \geqslant d_2$ and $d_1 - d_2 \leqslant 2$, (iii) holds.

(iii) $\Rightarrow$ (i) by Theorem (4.15).   ∎

5.   UPWARD OPTIMAL SEQUENCES

The sufficient conditions for downward optimality proven in Theorem 4.15 and in Theorem 4.17 are not sufficient for upward optimality, as demonstrated by the following example.

EXAMPLE 5.1.   The digraph $\Delta(2, 2, 3, 1, 2)$ consists of the arcs $1_2 3$, $2_2 4$, $3_3 1$, $4_1 5$, and $5_2 2$. We have $\mu(\Delta(2, 2, 3, 1, 2)) = 2.5$, and the only maximal cycle is $\gamma = 1_2 3_3 1$. No arc of γ lies in the interior of any arc of $\tilde{\Delta}(2, 2, 3, 1, 2)$. Thus, by Theorem 4.15, $(2, 2, 3, 1, 2)$ is a downward optimal sequence. However, the only cycle in the digraph $\Delta(3, 2, 3, 1, 2)$ is $\gamma' = 2_2 4_1 5_2 2$, and so $\mu(\Delta(3, 2, 3, 1, 2)) = \frac{5}{3} < \mu(\Delta(2, 2, 3, 1, 2))$, implying that $(2, 2, 3, 1, 2)$ is not an upward optimal sequence.

In this section we shall prove, among other things, that the sufficient condition for downward optimality proven in Corollary 4.18 is also sufficient for upward optimality.

DEFINITION 5.2. A set S of arcs in $\tilde{\Delta}(\delta)$ is said to be *nonoverlapping* if for every two arcs (t_1, t_2) and (t_3, t_4) in S we have either $t_2 \leqslant t_3$ or $t_4 \leqslant t_1$.

LEMMA 5.3. *Let $\delta' \geqslant \delta$, let γ and γ' be cycles in $\Delta(\delta)$ and $\Delta(\delta')$ respectively, and let $(t_1, t_2, \ldots)$ and $(t'_1, t'_2, \ldots)$ be spreads of γ and γ' respectively. Assume that there exists a positive integer k such that no union of k nonoverlapping arcs of $\tilde{\Delta}(\delta)$ lies in the interior of a path of k arcs in $\Delta(\delta)$ corresponding to k consecutive arcs of γ. Then if for some i and j we have $t'_i > t_j$, then $t'_{i+k} \geqslant t_{j+k}$.*

Proof. Assume to the contrary that $t'_{i+k} < t_{j+k}$. Then the path $\delta' = (t'_j, t'_{j+1}, \ldots, t'_{j+k})$ in $\tilde{\Delta}(\delta)$ lies in the interior of the path $(t'_i, t_{i+1}, \ldots, t_{i+k})$ in $\Delta(\delta)$. Since $d'_i \geqslant d_i$, $i \in \{1, \ldots, p\}$, it follows that the union of the k nonoverlapping arcs $\{(t'_{j+l}, t'_{j+l} + d_{t'_{j+l}}) : j = 1, \ldots, k\}$ lies in the interior of the path $(t_i, t_{i+1}, \ldots, t_{i+k})$ in $\Delta(\delta)$, in contradiction to the conditions of the lemma. ∎

A repeated application of Lemmas 4.9 and 5.3 yields the following.

COROLLARY 5.4. *Let $\delta' \geqslant \delta$, let γ and γ' be cycles in $\Delta(\delta)$ and $\Delta(\delta')$ respectively, and let $(t_1, t_2, \ldots)$ and $(t'_1, t'_2, \ldots)$ be spreads of γ and γ' respectively, such that $t'_1 \geqslant t_1$. Assume that there exists a positive integer k such that no union of k nonoverlapping arcs of $\tilde{\Delta}(\delta)$ lies in the interior of a path of k arcs in $\tilde{\Delta}(\delta)$, corresponding to k consecutive arcs of γ. Then there are infinitely many i's for which $t'_i \geqslant t_i$.*

Let γ and γ' be cycles in $\Delta(\delta)$ and $\Delta(\delta')$ respectively. We remark that, by Observation 4.8, we can find spreads $(t_1, t_2, \ldots)$ and $(t'_1, t'_2, \ldots)$ of γ and γ', respectively, such that $t_1 \leqslant t'_1 < t_2$. Therefore, as a corollary of Corollary 5.8 and Lemma 4.12 we now obtain the following results.

COROLLARY 5.5. *If there exists a cycle γ in $\Delta(\delta)$ and a positive integer k such that no union of k nonoverlapping arcs of $\tilde{\Delta}(\delta)$ lies in the interior of a path of k arcs in $\tilde{\Delta}(\delta)$ corresponding to k consecutive arcs of γ, then γ is a minimal cycle.*

Proof. It follows from Corollary 5.4 and Lemma 4.12 that $\mu(\gamma) \leqslant \mu(\hat{\gamma})$ for any cycle $\hat{\gamma}$ in $\Delta(\delta)$. ∎

THEOREM 5.6.   *If there exists a maximal cycle γ in $\Delta(\delta)$ and a positive integer k such that no union of k nonoverlapping arcs of $\tilde{\Delta}(\delta)$ lies in the interior of a path of k arcs in $\tilde{\Delta}(\delta)$ corresponding to k consecutive arcs of γ, then δ is an upward optimal sequence.*

Proof.   It follows from Corollary 5.4 and Lemma 4.12 that $\mu(\gamma) \leqslant \mu(\gamma')$ for every cycle γ' in $\Delta(\delta')$ where $\delta' \geqslant \delta$, and hence δ is an upward optimal sequence.   ∎

One cannot relax the requirement that γ is maximal from the condition in Theorem 5.6, as is demonstrated by the following example.

EXAMPLE 5.7.   The digraph $\Delta(6, 1, 2)$ consists of the loop $1_6 1$ and the arcs $2_1 3$ and $3_2 2$. We have $\mu(\Delta(6, 1, 2)) = 6$, and the only maximal cycle is the loop on 1. Since the cycle $\gamma = 2_1 3_2 2$ consists of an arc of length 1 and an arc of length 2, it follows that no arc of $\tilde{\Delta}(\delta)$ lies in the interior an arc of γ. Nevertheless, $(6, 1, 2)$ is not an upward optimal sequence, as $\mu(\Delta(7, 1, 2)) = 1.5 < 6 = \mu(\Delta(6, 1, 2))$. We remark that by Theorem 5.6, the sequence $(7, 1, 2)$ is upward optimal.

If we choose $k = 1$, then Theorem 5.6 yields the following.

THEOREM 5.8.   *If there exists a maximal cycle γ in $\Delta(\delta)$ such that no arc of $\tilde{\Delta}(\delta)$ lies in the interior of an arc in $\tilde{\Delta}(\delta)$ corresponding to an arc of γ, then δ is an upward optimal sequence.*

Theorem 5.6 is not equivalent to its weaker form in Theorem 5.8. The following example demonstrates a case where the condition in Theorem 5.8 is not satisfied, while the condition in Theorem 5.6 is satisfied for $k = 2$.

EXAMPLE 5.9.   The digraph $\Delta(5, 2, 1)$ consists of the arcs $1_5 3$, $2_2 1$, and $3_1 1$. The only cycle γ in $\Delta(5, 2, 1)$ is $1_5 3_1 1$. Observe that the arc $2_2 1$ and $3_1 1$ correspond to $(2, 4)$ and $(3, 4)$ respectively, which lie in the interior of $(1, 6)$, corresponding to the arc $1_5 3$ of γ. Thus, the condition in Theorem 5.8 is not satisfied. However, it is easy to verify that no union of two nonoverlapping arcs of $\tilde{\Delta}(5, 2, 1)$ lies in the interior of a path of two arcs in $\tilde{\Delta}(5, 2, 1)$ corresponding to the two consecutive arcs of γ, and therefore, by Theorem 5.6, $(5, 2, 1)$ is an upward optimal sequence.

The converse of Theorem 5.6 does not hold in general, as demonstrated by the following example.

EXAMPLE 5.10.   The digraph $\Delta(3, 1, 7)$ consists of the loop $1_3 1$ and the arcs $2_1 3$ and $3_7 1$. We have $\mu(\Delta(3, 1, 7)) = 3$, and the only cycle γ is the loop on 1. The arc $2_1 3$ corresponds to $(2, 3)$, which lies in the interior of $(1, 4)$, corresponding to the arc $1_3 1$ of γ. Nevertheless, it is easy to verify that whenever $d'_1 \geqslant 3$, $d'_2 \geqslant 1$, and $d'_3 \geqslant 7$, the maximal cycle in the digraph $\Delta(d'_1, d'_2, d'_3)$ is either a loop of length at least 3, or a two arc cycle of length at least 6, or a three arc cycle of length at least 12. In any case, we have $\mu(\Delta(d'_1, d'_2, d'_3)) \geqslant 3 = \mu(\Delta(3, 1, 7))$, and so $(3, 1, 7)$ is an upward optimal sequence.

Yet, in the case $p = 2$ the converse of Theorem 5.6 is true.

THEOREM 5.11.   *Let d_1 and d_2 be positive integers, $d_1 \geqslant d_2$. Then the following are equivalent:*

(i) *(d_1, d_2) is an upward optimal sequence.*
(ii) *Either d_1 is odd and d_2 is even, or d_2 is odd and $d_1 - d_2 \leqslant 2$, or d_1 is even and $d_2 = d_1$.*
(iii) *No union of two nonoverlapping arcs of $\tilde{\Delta}(d_1, d_2)$ lies in the interior of a path of two arcs in $\tilde{\Delta}(d_1, d_2)$ corresponding to two consecutive arcs of a maximal cycle in $\Delta(d_1, d_2)$.*

Proof.   (i) $\Rightarrow$ (ii): Distinguish between three possible cases:

(a) *d_1 and d_2 are even and $d_1 > d_2$.* We have $\mu(\Delta(d_1 + 1, d_2)) = d_2 < d_1 = \mu(\Delta(d_1, d_2))$.
(b) *d_1 is even, d_2 is odd, $d_1 - d_2 > 2$.* Here $\mu(\Delta(d_1 + 1, d_2 + 1)) = d_2 + 1 < d_1 = \mu(\Delta(d_1, d_2))$.
(c) *d_1 and d_2 are odd, $d_1 - d_2 > 2$.* Here $\mu(\Delta(d_1, d_2 + 1)) = d_2 + 1 < (d_1 + d_2)/2 = \mu(\Delta(d_1, d_2))$.

In any case, it follows that (d_1, d_2) is not downward optimal.
(ii) $\Rightarrow$ (iii): Distinguish between four possible case:

(a) *d_1 is odd and d_2 is even.* Here the maximal cycle is $2_{d_2} 2$.
(b) *d_1 is even and $d_2 = d_1$.* Here the maximal cycles are $1_{d_1} 1$ and $2_{d_2} 2$.
(c) *d_1 is even and $d_2 = d_1 - 1$.* Here the maximal cycle is $1_{d_1} 1$.
(d) *d_1 and d_2 are odd, and $d_1 - d_2 \leqslant 2$.* Here the maximal cycle is $1_{d_1} 2_{d_2} 1$.

In any case, it is easy to verify that (iii) holds.
(iii) $\Rightarrow$ (i) by Theorem (5.6).   ■

Finally for this section, we use our results to prove a necessary and sufficient condition for a sequence to be both downward optimal and upward optimal. Our result yields an assertion of [2].

The following is an immediate corollary of Theorem 5.6.

COROLLARY 5.12.   *If there exists a positive integer k such that no union of k nonoverlapping arcs of $\tilde{\Delta}(\delta)$ lies in the interior of a path of k arcs in $\tilde{\Delta}(\delta)$, then every cycle in $\Delta(\delta)$ is a minimal cycle, and δ is an upward optimal sequence.*

Corollary 5.12, together with Corollary 4.18, yields

COROLLARY 5.13.   *If no arc of $\tilde{\Delta}(\delta)$ lies in the interior of another arc of $\tilde{\Delta}(\delta)$, then every cycle in $\Delta(\delta)$ is both a maximal cycle and a minimal cycle, and δ is both a downward optimal sequence and an upward optimal sequence.*

An immediate consequence of Corollary 5.13 is the following theorem, proven in [2].

THEOREM 5.14.   *If $d_i \leqslant r + d_{i+r}$ for all $i, r \in \{1, \ldots, p\}$, then δ is both a downward optimal sequence and an upward optimal sequence.*

Proof.   Let i and r be any positive integers, and let $\bar{i}, \bar{r} \in \{1, \ldots, p\}$ be such that $i \pmod p = \bar{i} \pmod p$ and $r \pmod p = \bar{r} \pmod p$. Then

$$(5.15) \quad d_i = d_{\bar{i}} \leqslant \bar{r} + d_{\bar{i}+\bar{r}} = \bar{r} + d_{i+r} \leqslant r + d_{i+r} \quad \text{for all } i \text{ and } r.$$

Now, let $(i, i + d_i)$ and $(j, j + d_j)$ be two arcs in $\tilde{\Delta}(\delta)$, and assume that $j < i$. By (5.15) we obtain that

$$j + d_j \leqslant j + (i - j) + d_{j+(i-j)} = i + d_i,$$

and so $(i, i + d_i)$ does not lie in the interior of $(j, j + d_j)$. Therefore, no arc in $\tilde{\Delta}(\delta)$ lies in the interior of another arc in $\tilde{\Delta}(\delta)$, and our claim follows from Corollary 5.13.   ∎

6.   ORDER INVARIANCE

DEFINITION 6.1.   The sequence δ is said to be *order invariant* [*for the graph* $\Delta(\delta)$] if $\mu(\Delta(\delta))$ is order invariant, that is, if $\mu(\Delta(d_{\bar{1}}, \ldots, d_{\bar{p}})) = \mu(\Delta(\delta))$ for every permutation $d_{\bar{1}}, \ldots, d_{\bar{p}}$ of $d_1, \ldots, d_p$.

In this section we study conditions for order invariance of δ.

DEFINITION 6.2. Let $\alpha = (\alpha_1, \ldots, \alpha_p)$ be a sequence of positive integers. The *maximal mean* of α, denoted by $m(\alpha)$, is defined to be the maximal average of any consecutive elements of the *cycle* $(\alpha_1, \ldots, \alpha_p)$ (that is, α_1 is considered subsequent to α_p) whose sum is an integer multiple of p.

PROPOSITION 6.3. *There exists a permutation* $(\hat{1}, \ldots, \hat{p})$ *of* $(1, \ldots, p)$ *such that* $\mu(\Delta(d_{\hat{1}}, \ldots, d_{\hat{p}})) = m(d_1, \ldots, d_p)$.

Proof. Without loss of generality we may assume that the consecutive elements of the $(d_1, \ldots, d_p)$ whose sum is an integer multiple of p and whose average is maximal are the first m elements of $(d_1, \ldots, d_p)$. We now use the following algorithm for finding the permutation $(\hat{1}, \ldots, \hat{p})$.

Step 1: Let $\hat{1} = 1$ and let $h_1 = 1$.
Step $k + 1$, $k = 1, \ldots, p - 1$: Let $h = (h_k + d_k - 1)(\mathrm{mod}\ p) + 1$.
We choose

$$h_{k+1} = \begin{cases} h, & \text{if } \hat{h} \text{ has not yet been determined} \\ \text{the smallest } i \text{ such that } \hat{i} \text{ has not yet been determined,} & \text{otherwise,} \end{cases}$$

and we let $\hat{h}_{k+1} = k + 1$.

It is easy to verify that $\Delta(d_{\hat{1}}, \ldots, d_{\hat{p}})$ has a cycle with m arcs weighted $d_1, \ldots, d_m$, and that the weights of any other cycle in $\Delta(d_{\hat{1}}, \ldots, d_{\hat{p}})$ are consecutive elements in $(d_1, \ldots, d_p)$. Therefore, by Notation 3.1 and Definition 2.2 it follows that $\mu(\Delta(d_{\hat{1}}, \ldots, d_{\hat{p}})) = m(d_1, \ldots, d_p)$. ∎

The following example illustrates the algorithm defined in the proof of Proposition 6.3.

EXAMPLE 6.4. Let $p = 5$ and $(d_1, \ldots, d_5) = (3, 7, 7, 1, 8)$. We have $m(d_1, \ldots, d_p) = 5$, obtained as the average of d_1 and d_2. We now find the permutation $(\hat{1}, \ldots, \hat{5})$.

Step 1: $\hat{1} = 1$.
Step 2: $h = (1 + d_1 - 1)(\mathrm{mod}\ 5) + 1 = 4$. $\hat{4}$ has not yet been determined, and so $\hat{4} = 2$.
Step 3: $h = (4 + d_2 - 1)(\mathrm{mod}\ 5) + 1 = 1$. $\hat{1}$ has already been determined, and the smallest i such that $\hat{i}$ has not yet been determined is 2. Hence $\hat{2} = 3$.
Step 4: $h = (2 + d_3 - 1)(\mathrm{mod}\ 5) + 1 = 4$. $\hat{4}$ has already been determined, and the smallest i such that $\hat{i}$ has not yet been determined is 3. Hence $\hat{3} = 4$.

Step 5: $h = (3 + d_4 - 1)(\bmod 5) + 1 = 4$. $\hat{4}$ has already been determined, and the smallest i such that $\hat{i}$ has not yet been determined is 5. Hence $\hat{5} = 5$.

The required permutation of $(1, 2, 3, 4, 5)$ is thus $(1, 3, 4, 2, 5)$. Indeed, the only cycle is $\Delta(d_{\tilde{1}}, \ldots, d_{\tilde{p}}) = \Delta(3, 7, 1, 7, 8)$ is $1_3 4_7 1$, and we have $\mu(\Delta(d_{\tilde{1}}, \ldots, d_{\tilde{p}})) = 5$.

An immediate corollary of Theorem 3.5 and Proposition 6.3 is the following necessary condition for order invariance of δ.

THEOREM 6.5. *If δ is order invariant, then all permutations of* $(d_1, \ldots, d_p)$ *have the same maximal mean.*

The converse of Theorem 6.5 is not true in general, as demonstrated by the following example.

EXAMPLE 6.6. All permutations of $(2, 4, 5)$ have the same maximal mean 4.5. Nevertheless, $\Delta(2, 4, 5)$ consists of the arcs $1_2 3$, $2_4 3$, and $3_5 2$, and so $\mu(\Delta(2, 4, 5)) = 4.5$, while $\Delta(2, 5, 4)$ consists of the arcs $1_2 3$, $2_5 1$, and $3_4 1$, and so $\mu(\Delta(2, 4, 5)) = 3$.

In the converse direction we still have the following.

PROPOSITION 6.7. *There exists a permutation* $(\tilde{1}, \ldots, \tilde{p})$ *of* $(1, \ldots, p)$ *such that* $\mu(\Delta(\delta)) \leqslant m(d_{\tilde{1}}, \ldots, d_{\tilde{p}})$.

Proof. Let $d_{i_1}, \ldots, d_{i_t}$ be the weights of the arcs in a maximal cycle in $\Delta(\delta)$. Clearly, for every permutation $(\tilde{1}, \ldots, \tilde{p})$ of $(1, \ldots, p)$ in which $i_1, \ldots, i_t$ are consecutive elements, we have $\mu(\Delta(\delta)) = \text{average}(d_{i_1}, \ldots, d_{i_t}) \leqslant m(d_{\tilde{1}}, \ldots, d_{\tilde{p}})$. ∎

Example 6.6 above shows that the " $\leqslant$ " in the statement of Proposition 6.7 cannot be replaced by " $=$ ".

We thus do not have a necessary condition for order invariance of δ which is also a sufficient condition. We conclude the paper with five different sufficient conditions for order invariance of δ.

THEOREM 6.8. *If all the d_i's but one are the same, then δ is order invariant.*

Proof. Observe that $\Delta(\delta)$ is isomorphic to $\Delta(d_k, \ldots, d_p, d_1, \ldots, d_{k-1})$ for all $k \in \{1, \ldots, p\}$. Since in our case every permutation of $(d_1, \ldots, d_p)$ is equal to a cyclic shift $(d_k, \ldots, d_p, d_1, \ldots, d_{k-1})$, our claim follows. ∎

THEOREM 6.9.   *If there exists a positive integer d, relatively prime to p, such that $d_i \equiv d \pmod{p}$, $i \in \{1, \ldots, p\}$, then δ is order invariant.*

Proof.   Since $d_i \equiv d \pmod{p}$, $i \in \{1, \ldots, p\}$, and since d is relatively prime to p, it follows that the only possibility of partial sum of elements of $(d_1, \ldots, d_p)$ which is divisible by p is the sum of all elements. Therefore, $\Delta(\delta)$ consists of one cycle involving all p vertices, and it follows that δ is order invariant.   ■

THEOREM 6.10.   *If the largest d_i that is divisible by p is greater than or equal to the average of the two largest d_i's that are not divisible by p, then δ is order invariant.*

Proof.   Let d_i be the largest element in $(d_1, \ldots, d_p)$ that is divisible by p. It is immediate to check that under the conditions of the theorem, the average of elements of $(d_1, \ldots, d_p)$ whose sum is divisible by p does not exceed d_i. Therefore, the loop $i_{d_i} i$ (up to relabeling) is a maximal cycle in $\Delta(d_{\bar{1}}, \ldots, d_{\bar{p}})$ for every permutation $(\bar{1}, \ldots, \bar{p})$ of $(1, \ldots, p)$.   ■

THEOREM 6.11.   *If the largest d_i is divisible by p, then δ is order invariant.*

Proof.   The claim follows from Theorem 6.10.   ■

THEOREM 6.12.   *If no partial sum of the set of d_i's that are not divisible by p is divisible by p, then δ is order invariant.*

Proof.   It follows from the conditions of the theorem that, for every permutation $(\bar{1}, \ldots, \bar{p})$ of $(1, \ldots, p)$, the only cycles in $\Delta(d_{\bar{1}}, \ldots, d_{\bar{p}})$ are loops. Therefore, $\mu(\Delta(d_{\bar{1}}, \ldots, d_{\bar{p}}))$ is equal to the largest d_i is divisible by p, which is order invariant.   ■

REFERENCES

1   R. Bru, L. Elsner, and M. Neumann, Models of parallel chaotic iteration methods, *Linear Algebra Appl.* 103:175–192 (1988).
2   L. Elsner and M. Neumann, Monotonic sequences and rates of convergence of asynchronized iterative methods, *Linear Algebra Appl.*, 188:17–33 (1993).
3   L. Elsner, M. Neumann, and B. Vemmer (-Bohnhorst), The effect of the number of processors on the convergence rate of the parallel block Jacobi method, *Linear Algebra Appl.* 154/156:154–156 (1991).