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Integral Bases and p-Twisted Digraphs

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A well-known theorem in network flow theory states that for a strongly connected digraph $D = (V, A)$ there exists a set of directed cycles the incidence vectors of which form a basis for the circulation space of D and integrally span the set of integral circulations; that is, every integral circulation can be written as an integral combination of these vectors. In this paper, we extend this result to general digraphs. Following a definition of Hershkowitz and Schneider, we call a digraph p-twisted if each pair of vertices is contained in a closed (undirected) walk with the property that as the walk is traversed there are no more than p changes in the orientations of the arcs. We show that for every p-twisted digraph there exists a set of p-twisted cycles the incidence vectors of which form a basis for the circulation space and integrally span the set of integral circulations. We show that such a set can be computed in $O(|V|\,|A|)$ time.

1. INTRODUCTION

The circulation space of a directed graph $D = (V, A)$ with $\omega(D)$ (weak) components is a subspace of dimension $|A| - |V| + \omega(D)$ of the vector space of all real-valued vectors indexed on the arcs A. A basis for the circulation space can be generated as the incidence vectors of the fundamental cycles associated with any spanning forest of D; that is, the set of cycles created by adding to a spanning forest the arcs not contained in the forest. This basis has the additional property that every integral circulation can be written as an integral combination of the basis vectors. We call such a set of cycles an *integral cycle-basis* for D.

The fundamental cycles associated with a spanning forest are not, in general, directed even if D is strongly connected. An *integral directed cycle-basis,* that is an integral cycle-basis consisting of directed cycles, can be constructed for strongly connected D as follows (see [1, p. 29]). Find any directed cycle C for D and contract the arcs of C. While there are uncontracted arcs, repeat this operation on the resulting digraph and *lift* the cycle to a cycle of D by adding previously contracted arcs. The set of cycles generated by this algorithm is an integral directed cycle-basis for D.

The purpose of this paper is to extend the notion of an integral directed cycle-basis to arbitrary digraphs. We use the notion of a p-twisted digraph—introduced by Hershkowitz and Schneider [3]—to describe a measure of the degree to which D fails to be strongly-connected. Specifically, a digraph is p-*twisted* if each pair of vertices is contained in a closed walk with the property that as the walk is traversed once there are no more than p changes in orientation of the arcs; that is, *twists.* This generalizes the definition of a strongly connected digraph, since a digraph is strongly connected iff it is 0-twisted.

In Theorem 6, the main result of this paper, we show that every p-twisted digraph has an integral p-twisted cycle-basis. In our approach (for convenience, we assume that D is connected) we first generate an integral directed cycle-basis for each strong component of D, using for example Berge's algorithm. For the *condensed* diagraph of D (the digraph formed by contracting each strong component to a point), we then generate an integral cycle-basis associated with a spanning tree rooted at a source s that has the property that for each vertex v the unique (s, v) path in the tree can be decomposed into fewer than $p/2$ directed paths. We use these two sets of cycles to form an integral p-twisted cycle-basis for D.

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Partial results for this problem are described in Hershkowitz and Schneider [3], where it is shown that the set of p-twisted cycles is an integral spanning set for the set of integral circulations. Moreover, the authors pose as an open question the problem of determining whether a p-twisted digraph has an integral p-twisted cycle-basis. We show that such a basis exists and can be computed in $O(|V|\,|A|)$ time.

Next, we describe our paper in more detail. In Section 2 we describe the notation and definitions needed in our paper, and in Section 3 we define a p-twisted cycle-basis. In Section 4 we present some structural results for p-twisted digraphs, and in Section 5 we describe the main theorem on the existence of an integral p-twisted cycle-basis. Finally, in Section 6 we show that such a basis can be computed in $O(|V|\,|A|)$ time and consider related complexity issues.

2. Preliminaries

Let $D = (V, A)$ be a directed graph with *vertex set* V and *arc set* A. The digraphs that we consider may contain loops and parallel arcs. We use the notation $a = (u, v)$ to denote the arc a with *tail* u and *head* v. Note that there is a slight abuse of notation here, since there may be many arcs with tail u and head v.

For vertices $u, v \in V$, a (u, v)-*walk* for D is an alternating sequence of vertices and arcs $W = (v_0, a_1, v_1, \ldots, a_k, v_k)$ such that $v_0 = u$, $v_k = v$, and for $i = 1, 2, \ldots, k$ either $a_i = (v_{i-1}, v_i)$ or $a_i = (v_i, v_{i-1})$; arcs $a_i = (v_{i-1}, v_i)$ are called *forward* arcs, and the remaining arcs are called *reverse* arcs. The *length* of W, written $|W|$, is k the number of arcs of the sequence. A *path* is a walk containing no repeated vertices. A walk that contains only forward or only reverse arcs is called *directed*, and we will refer to directed walks and directed paths as *diwalks* and *dipaths*, respectively. If $v_0 = v_k$, then W is called a *closed walk*. A closed walk containing no repeated vertices other than v_0 and v_k is called a *cycle*. For vertices $u, v \in V$, we use walk(u, v) to denote the set of all (u, v)-walks of D, and cwalk(u, v) to denote the set of all closed walks containing u and v. For a (u, v) walk W and a (v, w) walk W', we use $W + W'$ to denote the (u, w) walk formed by *concatenating* W and W' and, conversely, we say that the walk $W'' = W + W'$ can be *decomposed* into walks W and W'. A *section* of W is a walk W' that is a subsequence of consecutive terms of W.

The *diameter* of the digraph D, written diameter (D), is defined by

$$\text{diameter}(D) = \max_{u,v \in V} \min_{W \in \text{walk}(u,v)} |W|.$$

The *condensation* of D, written condense(D), is the digraph formed by contracting each strong component of D to a point (see Figure 1). Thus, there are one-to-one

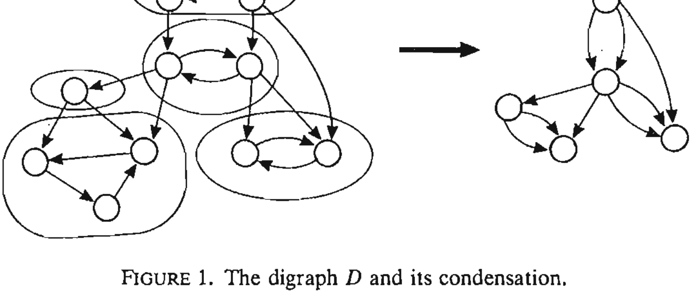

FIGURE 1. The digraph D and its condensation.

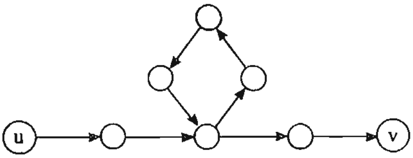

FIGURE 2. Walk W with $\text{seg}(W) = 1$ or $\text{seg}(W) = 3$.

correspondences between the vertices of condense(D) and the strong components of D, and between the arcs of condense(D) and the arcs of D not contained in some strong component. The digraph D is called *acyclic* if it contains no directed cycles. It is easy to see that for any digraph D, condense(D) is acyclic. A vertex v of an acyclic digraph D is called a *source*, respectively *sink*, if D contains no arc directed into, respectively out of, v.

For a walk W, we define seg(W) to be the minimum number of directed walks into which W can be decomposed. Note that seg(W) cannot in general be determined from the underlying graph of W. For example, a (u, v)-walk with underlying graph shown in Figure 2 and with no repeated arcs has seg(W) of either 1 or 3 depending on whether the cycle occurs in W in the forward or the reverse direction. For vertices $u, v \in V$ of the digraph D, we define tdist(u, v) by

$$\text{tdist}(u, v) = \min_{W \in \text{walk}(u, v)} \text{seg}(W).$$

For a closed walk C, we define the *twist number* of C, written twist(C), as follows. Starting from any arc $a \in C$ we traverse C once and count the number of changes in the orientations of the arcs. In particular, $\text{twist}(C) = 0$ iff C is directed. Further, it is easy to see that twist(C) is an even non-negative integer. Cycles with twist numbers of 2 and 6, respectively, are shown in Figure 3. Note that the twist number cannot be determined from the underlying graph since, for example, a closed walk containing no repeated arcs, the underlying graph of which is shown in Figure 4, has twist number either 0 or 2. We define twist(u, v) by

$$\text{twist}(u, v) = \min\{\text{twist}(C) \mid C \in \text{cwalk}(u, v)\}.$$

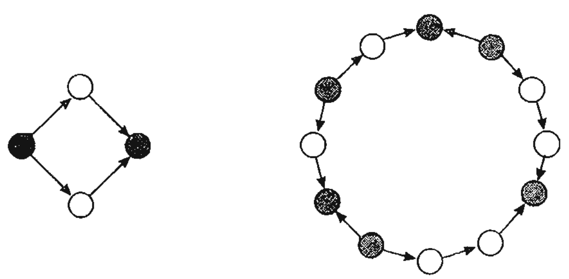

FIGURE 3. Cycles with twist numbers 2 and 6, respectively.

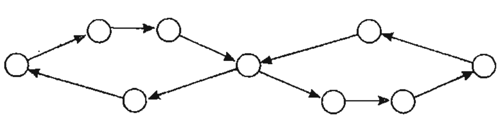

FIGURE 4. Closed walk C with $\text{twist}(C) = 0$ or 2.

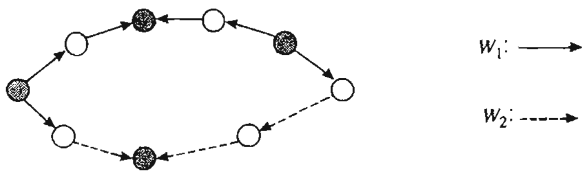

FIGURE 5. $k = \text{seg}(W_1) + \text{seg}(W_2)$ odd; $\text{twist}(C) = k - 1$.

We say that a closed walk C is *p-twisted* if $\text{twist}(C) \leq p$. For the digraph D, we define the *twist number of D*, written $\text{twist}(D)$, by

$$\text{twist}(D) = \max_{u,v \in V} \text{twist}(u, v).$$

We summarize some elementary results for $\text{tdist}(u, v)$ and $\text{twist}(u, v)$.

LEMMA 1. *Let $D = (V, A)$ be a connected p-twisted digraph. Then*:
(i) *If C is a closed walk of D and $C = W_1 + W_2$,*

$$seg(W_1) + \text{seg}(W_2) - 2 \leq \text{twist}(C) \leq \text{seg}(W_1) + \text{seg}(W_2).$$

(ii) *If s is a source of D and $q = \max_{v \in V} \text{tdist}(s, v)$,*

$$2q \leq p \leq 4q.$$

PROOF. (i) There are three cases for $\text{twist}(C)$, and we illustrate these graphically. First, if $k = \text{seg}(W_1) + \text{seg}(W_2)$ is odd then, as shown in Figure 5, a change in orientation occurs at exactly one of the endpoints of W_1 and W_2, and therefore $\text{twist}(C) = k - 1$. If k is even, then as shown in Figure 6 $\text{twist}(C) = k$ or $k - 2$ depending on whether a change in orientation occurs at both or neither endpoints.

(ii) Let $v \in V$ satisfy $\text{tdist}(s, v) = q$ and let $C \in \text{cwalk}(s, v)$. Let $C = W_1 + W_2$ be a decomposition of C into an (s, v)- and a (v, s)-walk, respectively, as shown in Figure 7.

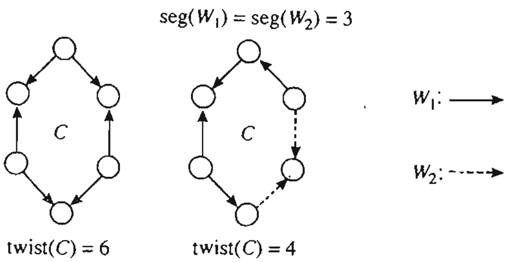

FIGURE 6. $k = \text{seg}(W_1) + \text{seg}(W_2)$ even; $\text{twist}(C) = k$ or $k - 2$.

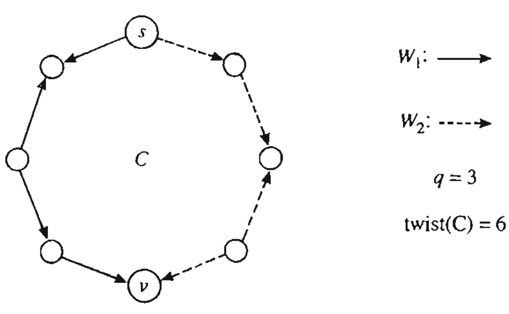

FIGURE 7. $C = W_1 + W_2 \in \text{cwalk}(s, v)$; $\text{twist}(C) \geq 2q$.

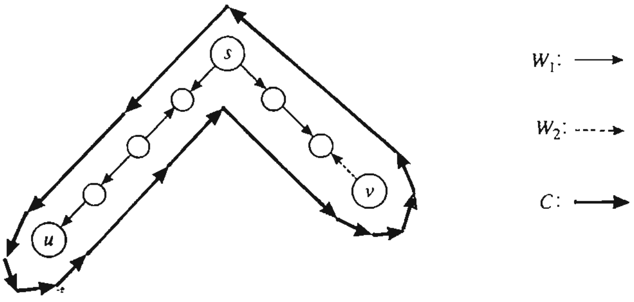

FIGURE 8. $p = \text{twist}(u, v) \leqslant \text{twist}(C) \leqslant 4q$.

Then $\text{seg}(W_1)$, $\text{seg}(W_2) \geqslant q$ and since s is a source, a change in orientation occurs at s and therefore $\text{twist}(C) \geqslant 2q - 1$. Since $\text{twist}(C)$ is even, we must have $\text{twist}(C) \geqslant 2q$, and the first inequality follows.

To see that $p \leqslant 4q$ choose $u, v \in V$ such that $\text{twist}(u, v) = p$. Let W_1 and W_2 be, respectively, (s, u)- and (s, v)-walks, as shown in Figure 8, with $\text{seg}(W_1)$, $\text{seg}(W_2) \leqslant q$, and let C be the closed (u, v)-walk determined by concatenating W_1 and W_2 in the forward and reverse directions. Thus,

$$p = \text{twist}(u, v) \leqslant 2(\text{seg}(W_1) + \text{seg}(W_2)) \leqslant 4q,$$

and this completes the proof. $\square$

The examples shown in Figure 9 show that the inequalities in part (ii) are the best possible.

3. CYCLE BASES

We define the *circulation space* of the digraph $D = (V, A)$, written circulation(D), to be the set of all real-valued vectors x indexed on the arcs A such that

$$\sum_{(u,v) \in A} x_{(u,v)} = \sum_{(v,u) \in A} x_{(v,u)} \qquad \text{for all } v \in V.$$

The *integral circulation space* of D, written icirculation(D), is the set of vectors of circulation(D) with integral co-ordinates.

For a cycle C of D, the *incidence vector* of C is the vector x indexed on A defined by

$$x_a = \begin{cases} +1 & \text{if } a \text{ is a forward arc of } C, \\ -1 & \text{if } a \text{ is a reverse arc of } C, \\ 0 & \text{if } a \notin C. \end{cases}$$

We will use χ_C to denote the incidence vector of C. A set of cycles $\mathscr{C}$ of D is called a *cycle-basis* for D if the set $\{\chi_C \mid C \in \mathscr{C}\}$ is a basis (in the usual linear algebraic sense)

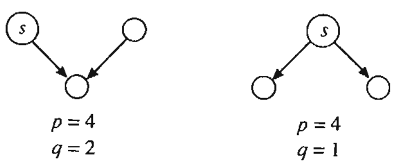

FIGURE 9. Digraphs with $p = 2q$, and $p = 4q$.

for circulation(D). A cycle-basis is called *integral* if in addition every integral vector $x \in$ circulation(D) can be written as

$$x = \sum_{C \in \mathscr{C}} \alpha_C \chi_C$$

for some integral constants α_C, $C \in \mathscr{C}$.

It is straightforward to see that any digraph $D = (V, A)$ has an integral cycle-basis. Specifically, let T be any spanning forest for D and let $\mathscr{C} = \{C(T, a) \mid a \in A, a \notin T\}$, where $C(T, a)$ for $a \notin T$ is the unique cycle formed by adding a to T. Then it is straightforward to show that $\mathscr{C}$ is a cycle-basis for D. We will refer to $\mathscr{C}$ as the set of *fundamental cycles* associted with the forest T. As a consequence it follows that the dimension of circulation(D) is $|A| - |V| - \omega(D)$, where $\omega(D)$ is the number of (weak) components of D. Using the construction in Berge [1, p. 29], it follows that a strongly connected digraph D has integral directed cycle-basis. An integral cycle-basis $\mathscr{C}$ is called an *integral p-twisted cycle-basis* if every cycle $C \in \mathscr{C}$ is p-twisted.

4. The Structure of p-Twisted Digraphs

Clearly, the twist number of a digraph D equals the maximum of the twist numbers of the connected components of D. Furthermore, we observe that the twist number of D is the same as the twist number of condense(D). Thus, for simplicity of exposition in this section we will assume that D is connected and acyclic, although the results extend to general digraphs D.

For the connected, acyclic digraph D, let S and T be, respectively, the set of sources and sinks of D. We define the *bipartite digraph of D*, written $B(D)$, to be the bipartite digraph with bipartition (S, T) containing arc (s, t) for $s \in S$ and $t \in T$ whenever D contains an (s, t)-dipath.

Theorem 2. *For a connected acyclic digraph D,*

$$\text{twist}(D) = \text{twist}(B(D)) = 2 * \text{diameter}(B(D)). \tag{1}$$

Proof. To see the second equality in (1), note that for vertices u and v, $\text{twist}_{B(D)}(u, v)$ is obtained by traversing a shortest (u, v)-path P in the forward and reverse directions. The number of twists of this closed walk is twice the length of P, and the lemma follows from the definitions of twist number and diameter.

To see the first equality, let $p = \text{twist}(D)$. We show that $\text{twist}(B(D)) \leq p$. Assume that $V(D) = \{1, 2, \ldots, n\}$ and that every arc (u, v) satisfies $u < v$. Let u and v be distinct vertices of $B(D)$. Since u and v are also vertices of D, they are contained in a closed walk C of D with q twists, where $q \leq p$. By extending C down to a sink or up to a source at each change in orientation, as shown in Figure 10, we can construct a

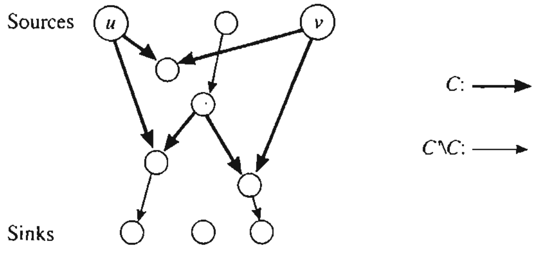

Figure 10. Extending the closed walk C to C'.

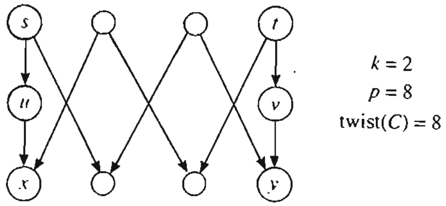

FIGURE 11. A p-twisted cycle containing u and v; $p = 4k$.

q-twisted closed walk C' of D containing u and v such that every change of orientation occurs at a source or a sink of D. Now C' induces a closed walk in $B(D)$ with q twists, and therefore

$$\text{twist}_{B(D)}(u, v) \leqslant \text{twist}_D(u, v).$$

It follows that

$$\begin{aligned}\text{twist}(B(D)) &= \max_{u,v \in V(B(D))} \text{twist}_{B(D)}(u, v) \\ &\leqslant \max_{u,v \in V(B(D))} \text{twist}_D(u, v) \\ &\leqslant \max_{u,v \in V(D)} \text{twist}_D(u, v) \\ &= \text{twist}(D).\end{aligned}$$

Conversely, suppose that $p = \text{twist}(B(D))$, and let u and v be distinct vertices of D. Let $\{s, t\}$ and $\{x, y\}$ be, respectively, sources and sinks of D, as shown in Figure 11, such that D contains directed (s, u)-, (u, x)-, (t, v)- and (v, y)-paths. First, suppose that $p = 4k$ for some positive integer k. It follows from the second equality in (1) that $B(D)$ contains (t, x) and (s, y) walks of length at most $2k$, and that since these walks must be odd, they must have length at most $2k - 1$. Note that each arc (w, z) of $B(D)$ can be *lifted* to a (w, z) dipath in D. Concatenating the directed walks associated with each arc of the (t, x) and (s, y) walks of $B(D)$ together with the (s, u)-, (u, x)-, (t, v)- and (v, y)-dipaths, as shown in Figure 11, produces a closed walk $C \in \text{cwalk}(u, v)$ of D with at most p twists. Since u and v are arbitrary vertices of D, it follows that $\text{twist}(D) \leqslant \text{twist}(B(D))$.

Next, suppose that $p = 4k - 2$; then $B(D)$ contains (s, t)- and (x, y)-paths of length at most $2k - 1$, which must have length at most $2k - 2$ since these paths are even. By lifting the arcs of these paths and concatenating as in the previous case (see Figure 12), we again have a cycle $C \in \text{cwalk}(u, v)$ with at most p twists. $\square$

The following corollary is a direct consequence of Theorem 2.

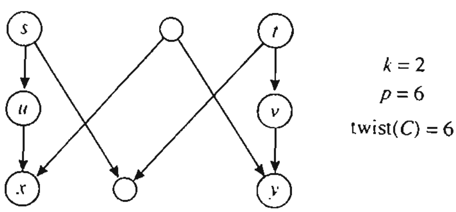

FIGURE 12. A p-twisted cycle containing u and v; $p = 4k - 2$.

Corollary 3. *Let D be an connected acyclic graph. Then*:
(i) $\text{twist}(D)=2$ *iff $B(D)$ is a line (i.e. two vertices and one arc) or, equivalently, iff D has exactly one source and one sink which are distinct.*
(ii) $\text{twist}(D)=4$ *iff $B(D)$ is a complete bipartite graph with at least three vertices or, equivalently, iff D contains dipaths from each source to each sink and D has at least three vertices.*

Note that part (ii) follows since the diameter of a bipartite graph with at least three vertices is 2 iff D is complete.

5. The Main Theorem

In this section, we prove that every p-twisted digraph has an integral p-twisted cycle-basis. First, we show that it suffices to consider acyclic digraphs.

Lemma 4. *For a digraph $D=(V,A)$, if* condense(D) *has an integral p-twisted cycle-basis, then D has such a basis.*

Proof. Without loss of generality, we can assume that D is connected. Let $D_1, D_2, \ldots, D_{\omega'}$ be the strong components of D. For each $i=1,2,\ldots,\omega'$, let $\mathscr{C}_i$ be a integral directed cycle-basis for D_i. Suppose that condense(D) has a integral p-twisted cycle-basis $\mathscr{C}^*$. Note that a cycle C of condense(D) can be expanded to a cycle C' of D such that $\text{twist}(C)=\text{twist}(C')$ by replacing each vertex of C with a directed path in the corresponding strong component, as shown in Figure 13. Let $\mathscr{C}^{**}$ be a set of cycles for D formed by expanding each element of $\mathscr{C}^*$, and define

$$\mathscr{C}=\mathscr{C}^{**}\cup\mathscr{C}_1\cup\cdots\cup\mathscr{C}_{\omega'}. \tag{2}$$

We argue that $\mathscr{C}$ is a integral p-twisted cycle-basis for D. First, the independence of the set $\{\chi_C \mid C\in\mathscr{C}\}$ follows directly from the independence of the incidence vectors for each of the sets on the right-hand side of (2). Second, a straightforward counting argument shows that the cardinality of $\mathscr{C}$ is $|A|-|V|+1$, the dimension of circulation(D), and therefore $\mathscr{C}$ is a basis. Third, the cycles of each $\mathscr{C}_i$ are directed and the expansion of a cycle $C\in\mathscr{C}^*$ to a cycle $C'\in\mathscr{C}^{**}$ does not change the twist number. Thus, $\mathscr{C}$ is a p-twisted cycle-basis for D.

Finally, note that if x is a circulation for D, then the restriction of x to the arcs of condense(D), which we will denote by x^*, is a circulation for condense(D). (Note that

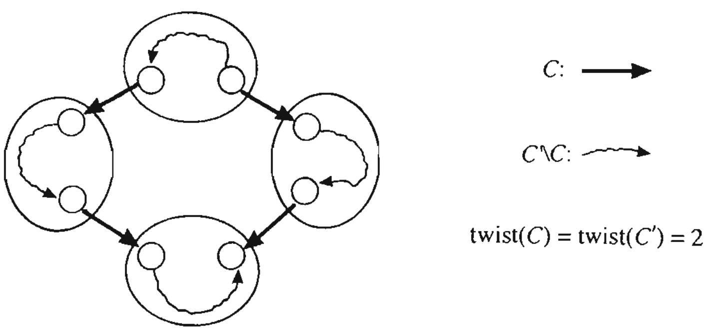

Figure 13. Expanding a cycle C of condense(D) to a cycle C' of D with $\text{twist}(C)=\text{twist}(C')$.

we are using the fact that, in the definition of condense(D), parallel arcs formed by contracting a strong component of D are *not* identified.) If $x \in$ icirculation(D), then

$$x^* = \sum_{C \in \mathscr{C}^*} \alpha_C \chi_C$$

for some integers α_C, $C \in \mathscr{C}^*$. Let

$$x' = x - \sum_{C \in \mathscr{C}^*} \chi_C \chi_{C'}.$$

Then x' is integer, and $x'_a = 0$ for each arc $a \in A(D)$ not contained in a strong component of D. Thus, for each $i = 1, 2, \ldots, \omega'$, the restriction of x' to the arcs of D_i is an integral circulation for D_i and therefore can be expressed as an integral combination of the incidence vectors of $\mathscr{C}_i$. □

Next, we show that every p-twisted digraph has a integral p-twisted cycle-basis. Clearly, if D is disconnected, then the union of integral p-twisted cycle-bases for each component forms an integral p-twisted cycle-basis for D. Thus, by Lemma 4, it suffices to consider the case of acyclic connected digraphs only. Our proof is based on showing that the following algorithm constructs a spanning tree the fundamental cycles of which form the desired basis.

Low-Twist Spanning Tree Algorithm

Input: a connected acyclic p-twisted digraph $D = (V, A)$ and source s.
Output: a spanning tree T for D and a function ϕ mapping V onto $\{0, 1, \ldots, q\}$ for some $0 \leq q \leq p/2$ such that for each vertex $v \in V$ the (unique) (s, v)-path in T can be decomposed into $\phi(v)$ directed segments.
(0) Set $\phi(s) = 0$, $k = 0$, $T = \varnothing$ and $U = \{s\}$.
(1) If $U = V$, return T and ϕ, and STOP; otherwise, set $k = k + 1$.
(2) If k is odd, then while there exists $(u, v) \in A$ with $u \in U$ and $v \in V \backslash U$, do

$$T = \cup \{(u, v)\}$$
$$U = U \cup \{v\}$$
$$\phi(v) = k.$$

If k is even, then while there exists $(v, u) \in A$ with $u \in U$ and $v \in V \backslash U$, do

$$T = \cup \{(v, u)\}$$
$$U = U \cup \{v\}$$
$$\phi(v) = k.$$

(3) Go to (1).

Theorem 5. *The Low-twist Spanning Tree Algorithm terminates with the desired spanning tree T and function $\phi(v) = \mathrm{tdist}(s, v)$.*

Proof. A simple induction shows that the algorithm terminates in step (1) with a spanning tree T in which each vertex v can be reached from s by a path composed of

$\phi(v)$ directed walks. This implies that tdist $(s, v) \leqslant \phi(v)$ for all $v \in V$ by definition. On the other hand, if there is an (s, v)-walk which can be decomposed into k directed walks, then another simple induction shows that $\phi(v) \leqslant k$. Therefore $\phi(v) \leqslant \text{tdist}(s, v)$ for all $v \in V$. Finally, it follows from Lemma 1 that $q = \max_{v \in V} \text{tdist}(s, v) \leqslant p/2$. □

THEOREM 6. *Every p-twisted digraph D has a integral p-twisted cycle-basis.*

PROOF. Since the twist number of D is the maximum of the twist numbers of the components of D, it suffices to prove the result under the assumption that D is connected. Further, it follows from Lemma 4 that we may assume that D is acyclic.

Let T be the spanning tree computed by the Low-twist Spanning Tree Algorithm, and let $\mathscr{C}$ be the set of fundamental cycles associated with T. We claim that $\mathscr{C}$ is the desired basis. It is straightforward to show that $\mathscr{C}$ is an integral basis for D. Suppose that C is the unique cycle formed by adding $(u, v) \notin T$ to T. Since $\text{tdist}(s, u)$, $\text{tdist}(s, v) \leqslant p/2$, it follows from Lemma 1 that

$$\text{twist}(C) \leqslant \text{tdist}(s, u) + \text{tdist}(s, v) + 1 \leqslant p + 1.$$

Since $\text{twist}(C)$ must be even and p is even, it follows that $\text{twist}(C) \leqslant p$. This completes the proof. □

6. ALGORITHMIC CONSIDERATIONS

In this section, we describe efficient algorithms for computing $\text{twist}(D)$ and for computing an integral p-twisted cycle-basis for an arbitrary digraph $D = (V, A)$. Since the (weakly) connected components of D can be determined in $O(|A|)$ time, throughout this section we will assume for convenience that D is connected.

Although the bipartite digraph $B(D)$ of $\text{condense}(D)$ can be determined efficiently, since it is a subdigraph of the transitive closure of $\text{condense}(D)$, it does not make sense to compute the diameter of $B(D)$ directly to determine $\text{twist}(D)$, as Lemma 1 would suggest. Instead, our algorithm computes the diameter of $B(D)$ implicitly from $\text{condense}(D)$, which can be constructed in $O(|A|)$ time using depth-first search. Let S and T be the set of sources and sinks in $\text{condense}(D)$, respectively, so that $S \cup T$ is the vertex set of $B(D)$. Since the Low-twist Spanning Tree Algorithm applied to $\text{condense}(D)$ computes the quantities $\text{tdist}(s, v)$ for each vertex v in $\text{condense}(D)$, it computes the shortest distance in $B(D)$ from s to every vertex in $S \cup T$. Thus we can find the shortest distance in $B(D)$ from each $s \in S$ to every vertex in $S \cup T$ in $O(|S|\,|A|)$ time, as each arc of $\text{condense}(D)$ is only examined once in the Low-twist Spanning Tree algorithm. Let k_S be the maximum length of any shortest path in $B(D)$ from a vertex $s \in S$ to a vertex in $S \cup T$. If k_S is even, then $\text{twist}(D) = 2k_S$. On the other hand, if k_S is odd, then analogously we can find the shortest distance in $B(D)$ from each $t \in T$ to every vertex in $S \cup T$ in $O(|T|\,|A|)$ time. Then $\text{twist}(D) = 2k_T$, where k_T is the maximum length of any shortest path in $B(D)$ from a vertex $t \in T$ to a vertex in $S \cup T$. This gives an $O(|S \cup T|\,|A|)$ algorithm for computing $\text{twist}(D) = \max\{k_S, k_T\}$. To see that it is correct, suppose that k_S is even. If $k_T > k_S$, then $B(D)$ would contain a shortest path of length $k_S + 1$ from some $t \in T$ to some $s \in S$, since $k_S + 1$ is odd. Since this implies that $k_S \geqslant k_T$, we have a contradiction. Likewise, if k_S is odd, then $B(D)$ contains a shortest path of length k_S from some $s \in S$ to some $t \in T$, which implies that $k_T \geqslant k_S$ (in fact, either $k_T = k_S$ or $k_T = k_S + 1$).

As Theorem 4 suggests, our algorithm for finding an integral p-twisted cycle-basis requires that we first find directed integral cycle-bases for the strongly connected components of D. Since our algorithm for finding an integral directed cycle-basis depends heavily on Berge's result on integral directed cycle-bases, we include a proof for the sake of completeness.

THEOREM 7. *Every strongly connected digraph has an integral directed cycle-basis*

PROOF. This statement is clearly true for digraphs which have a single vertex, so suppose that it holds for all strongly connected graphs with fewer vertices than $D=(V, A)$ and that $|V| \geqslant 2$.

Let C^* be a directed cycle for D with length $k \geqslant 2$ (such a cycle exists since D is strongly connected). Contract the arcs of C^* to form a new digraph where the vertices $v_1, \ldots, v_k$ of C^* are replaced by a new vertex v^* and any arc incident to one (or two) vertices of C^* is replaced by an arc (or loop) incident to v^*. The new digraph $D^*=(V^*, A^*)$ is strongly connected, and has $|V^*|=|V|-k+1<|V|$ vertices and $|A^*|=|A|-k$ arcs. By virtue of the induction hypothesis, D^* has an integral directed cycle-basis $\mathscr{C}^*$ consisting of

$$|A^*|-|V^*|+1=(|A|-k)-(|V|-k+1)+1=|A|-|V|$$

directed cycles. Note that a directed cycle C of D^* can be lifted to a directed cycle C' of D by replacing v^* by the unique directed path in C^* connecting the two vertices of C^* which correspond to v^* in C. Let $\mathscr{C}^{**}$ be a set of directed cycles for D formed by lifting each element of $\mathscr{C}^*$, and define $\mathscr{C}=\mathscr{C}^{**} \cup\{C^*\}$.

We argue that $\mathscr{C}$ is a integral directed cycle-basis for D. First, the independence of the set $\{\chi_C \mid C \in \mathscr{C}\}$ follows directly from independence of the set $\{\chi_C \mid C \in \mathscr{C}^*\}$. Second, the cardinality of $\mathscr{C}$ is $|A|-|V|+1$, the dimension of circulation(D), and therefore $\mathscr{C}$ is a basis. Third, the lifting of a directed cycle $C \in \mathscr{C}^*$ results in a directed cycle $C' \in \mathscr{C}^{**}$. Thus, $\mathscr{C}$ is a directed cycle-basis for D.

Finally, if x is a circulation for D, then under the contraction of C^* in D, x becomes a circulation x^* for D^*. If $x \in$ icirculation(D) then, by the induction hypothesis,

$$x^*=\sum_{C \in \mathscr{C}^*} \alpha_C \chi_C$$

for some integers α_C, $C \in \mathscr{C}^*$. Let

$$x'=x-\sum_{C \in \mathscr{C}^*} \alpha_C \chi_{C'}.$$

Then x' is integer, and $x'_a=0$ for each arc $a \notin C^*$. Since x' is a circulation for D, it must be the case that $x'=\alpha_{C^*} \chi_{C^*}$ for some integer α_{C^*}. □

Next, we describe an $O(|V|\,|A|)$ algorithm for finding an integral directed cycle-basis. The algorithm does $O(|V|^2)$ preprocessing, and then constructs an integral directed cycle basis in time proportional to the total length of all $|A|-|V|+1$ directed cycles in the cycle-basis.

Berge's construction for the proof above directly yields an algorithm which finds an integral directed cycle-basis in at most $|V|-1$ iterations. Initially we delete any loops (which will become elements of the cycle-basis) and all but one of any number of parallel arcs (which will eventually become loops, and be lifted to elements of the cycle-basis). Then, in each iteration, we find a directed cycle C^* of length at least two and contract it to form the digraph D^*, deleting any loops and all but one of any number of parallel arcs which are created (along with the directed cycle C^*, these will be lifted to elements of the cycle-basis). The only difficulty lies in lifting the $|A|-|V|+1$ loops and directed cycles. If we neglect the lifting, the algorithm only requires $O(|V|^2)$ time. To see this, note that we may delete loops and all but one of any number of parallel arcs initially in $O(|A|)$ time. Then, since each digraph D^* will be loopless and strongly connected, we can find a directed cycle C^* of length at least two in $O(|V|)$ time. Contracting the directed cycle C^* and forming the new digraph D^* can be done in time proportional to the number of arcs incident to vertices in C^*, which will be at most $2|V|\,|C^*|$, since the old digraph will not contain parallel arcs. Thus the algorithm requires $O(|A|+|V|\,|C_1^*|+\cdots+|V|\,|C_n^*|)$ time, where C_i^* is the directed cycle contracted in the ith of n iterations, and here a simple induction shows that $\sum_{i=1}^n |C_i^*| \leq 2|V|$.

Next, we show that it is unnecessary to lift the directed cycles and loops in each iteration. Consider the partition

$$A = A_1 \cup A_2 \cup \cdots \cup A_n \cup A_{n+1}$$

induced by Berge's algorithm, where A_i is the set of original arcs corresponding to the arcs of C_i^* contracted in the ith of n iterations, and A_{n+1} consists of the original arcs which correspond to loops (including parallel arcs) deleted during the course of the algorithm. We can clearly set the values $x_a = i$ for $a \in A_i$ during the course of the algorithm. Now any arc $a \in A$ can be lifted to a directed cycle for D by connecting its head and tail by a directed path which is contained in the smallest possible subgraph of the form $A_1 \cup A_2 \cup \cdots \cup A_j$ for some $j \leq n$. This can be accomplished by computing a family of $|V|$ bottleneck path trees rooted at each vertex of the digraph with vertex set V and arc set $A_1 \cup A_2 \cup \cdots \cup A_n$. Here, a (u, v) bottleneck path is a directed (u, v)-path P for which $\max_{a \in P} x_a$ is minimized, and a bottleneck path tree rooted at u contains (u, v) bottleneck paths for each $v \in V$. Since $\sum_{i=1}^n |A_i| \leq 2|V|$, the algorithm given in Hartmann and Schneider [2] finds the $|V|$ bottleneck path trees in $O(|V|^2)$ time. To compute the integral cycle-basis using the bottleneck path trees, an original arc corresponding to each directed cycle and loop is lifted to a directed cycle by following the pointers in the bottleneck path tree rooted at the head of the arc, starting from the tail of the arc to the root.

The algorithm just described for finding an integral directed cycle-basis of a strongly connected digraph yields an $O(|V|\,|A|)$ algorithm for finding an integral p-twisted cycle-basis for an arbitrary p-twisted digraph. As was the case for the previous algorithm, this algorithm does $O(|V|^2)$ preprocessing, and then constructs an integral p-twisted cycle-basis in time proportional to the total length of all $|A|-|V|+1$ cycles in the cycle-basis. First, we can find the strongly connected components $D_1, D_2, \ldots, D_{\omega'}$ of D and form the condensation of D in $O(|A|)$ time using depth-first search. Then, since $\sum_{i=1}^{\omega'} |V_i|^2 \leq |V|^2$, we can run Berge's algorithm and compute a family of bottleneck path trees for each strongly connected component $D_i = (V_i, A_i)$ in $O(|V|^2)$ time. Further, we can find a source s and apply the Low-twist Spanning Tree Algorithm to condense(D) in $O(|A|)$ time. Given this preprocessing, we can find an integral p-twisted cycle-basis for condense(D) in time proportional to the total length of all cycles in its cycle-basis, and extend this to a p-twisted cycle-basis for D, as in the proof of Lemma 4, by following pointers in the bottleneck path trees.

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