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ALGORITHMS FOR COMPUTING BASES FOR THE PERRON EIGENSPACE WITH PRESCRIBED NONNEGATIVITY AND COMBINATORIAL PROPERTIES*

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Abstract. Let P be an $n \times n$ nonnegative matrix. In this paper the authors introduce a method called the SCANBAS algorithm for computing a union of (Jordan) chains C corresponding to the Perron eigenvalue of P, such that C consists of nonnegative vectors only and such that at each height, C contains the maximal number of nonnegative vectors of that height possible in a height basis for the Perron eigenspace of P. It is further shown that C can be extended to a height basis for the Perron eigenspace of P. The chains are extracted from transform components of P that are, in turn, polynomials in P. When the Perron eigenspace has a Jordan basis consisting of nonnegative vectors only, this algorithm computes such a basis. The paper concludes with various examples computed by the algorithm using MATLAB. The work here continues and deepens work on computing nonnegative bases for the Perron eigenspace from polynomials in the matrix already begun by Hartwig, Neumann, and Rose and by Neumann and Schneider.

Key words. nonnegative matrices, M-matrices, Perron eigenspace, computations

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1. Introduction. In this paper we continue an investigation begun by Hartwig, Neumann, and Rose [6] and Neumann and Schneider [11] on nonnegative and combinatorial properties of bases for the Perron eigenspace of a nonnegative matrix, which can be extracted from certain polynomials in the matrix.

More specifically, let P be an $n \times n$ nonnegative matrix and $\rho(P)$ its spectral radius which is well known to be an eigenvalue of P, called its *Perron root.* A more comprehensive explanation and detailed background to some of the concepts used in this introduction and appropriate references are given in the next sections. Let Z be the eigenprojection of P at $\rho(P)$. In [6] it was shown that for sufficiently small $\epsilon > 0$, the matrix

$$J(\epsilon) = [(\epsilon + \rho(P))I - P]^{-1}Z \tag{1.1}$$

is nonnegative and its columns contain a basis of nonnegative vectors for the (generalized) eigenspace of P corresponding to $\rho(P)$ known as the *Perron eigenspace of P.* Furthermore, an algorithm for computing ϵ and hence a method for computing such a basis was suggested in [6, Thm. 2.2].

In [11] it was observed that since $J(\epsilon)$ is an analytic function in P, it is a polynomial in P, so that if P is put, say, in block lower triangular Frobenius normal form, then $J(\epsilon)$ would be a block lower triangular matrix conforming to the block partitioning in the Frobenius normal form of P. Thus combinatorial properties possessed by certain nonnegative bases for the Perron eigenspace of P that were present in the first proofs of the existence of such basis for the Perron eigenspace of P obtained by Rothblum [13] and Richman and Schneider [12] could be extracted from the columns

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of $J(\epsilon)$, and this idea led to the investigation in [11]. The combinatorial properties that the authors of [11] had in mind to recapture from the columns of $J(\epsilon)$ were access relations in the directed graph of P or, more precisely, in the block directed graph that can be associated with the Frobenius normal form of P known as the *reduced graph of* P. Indeed, in [11], it was shown that for sufficiently small $\epsilon > 0$, nonnegative bases from the columns of $J_T(\epsilon)$ could be chosen that are *strongly combinatorial.* (See §2 and [11] for precise definitions of these terms.)

The purpose of this paper is to go one step deeper in search of combinatorial and algebraic properties of bases that can be extracted from the columns of $J(\epsilon)$ and certain other nonnegative matrices that are polynomials in the matrix P and to compute such bases. Actually, we think of $J^{(0)}(\epsilon) := \epsilon J(\epsilon)$ as a zeroth *transform component of* P. Other nonnegative matrices that are polynomials in P, which we will work with later, are the higher-order transform components

$$J^{(k)}(\epsilon) = \epsilon^{k+1}(P-\rho(P)I)^k J(\epsilon), \quad k = 1, \ldots, \nu - 1.$$

Here ν is the index of the Perron root as an eigenvalue of P.

Hershkowitz [7] defines the *peak characteristic tuple* $(\xi_1, \ldots, \xi_\nu)$ of the Perron root as an eigenvalue of P, and he shows that for each $h = 1, \ldots, \nu$, ξ_h is the maximal number of nonnegative vectors of height h in a height basis for the Perron eigenspace and that a height basis for the eigenspace exists with that many nonnegative vectors at each height $h = 1, \ldots, \nu$.

The principal question that we shall answer in this paper is this: Can we extract from the columns of the transformed components $J^{(0)}(\epsilon), \ldots, J^{(\nu-1)}(\epsilon)$, a union $\mathcal{C}$ of nonnegative Jordan chains that contains exactly ξ_h vectors of height h, $h = 1, \ldots, \nu$, where the tuple $(\xi_1, \ldots, \xi_\nu)$ is the peak characteristics mentioned above? Moreover, can this union be extended to a height basis for the eigenspace? We achieve this via a *scanning process of the transform components* that begins by stacking the transform components on each other from the lowest to the highest. We call this process the SCANBAS algorithm. The algorithm is set out in §5 after some preparations and preliminary results from §§2–4. We go on to show in Corollary 4 that if $(\eta_1, \ldots, \eta_\nu)$ is the *height characteristic of the* Perron root as an eigenvalue of P with $\eta_k = \xi_k$ for $k = t, \ldots, \nu$, so that by Hershkowitz and Schneider [8, Thm. (6.6)] there is a Jordan basis for the Perron eigenspace of P corresponding to its Perron root such that all Jordan chains of length t and higher consist of nonnegative vectors only, then our SCANBAS algorithm produces such chains. In particular, if the Perron eigenspace of P has a Jordan basis consisting entirely of nonnegative chains, our SCANBAS algorithm computes such a basis. In §6 we conclude the paper by presenting various examples of bases that were produced by our SCANBAS algorithm implemented by using MATLAB. These examples show that generally $\mathcal{C}$ cannot be extended to a Jordan basis for the eigenspace. We end the section with a brief description of the MATLAB programs that were actually used in the computation of the examples.

Finally, we find it convenient to work and state the results of this paper in terms of the minus M–matrix $A = P - \rho(P)I$ that can be associated with our nonnegative matrix P.

2. Notations and preliminaries. For a positive integer n, we denote by $\langle n \rangle$ the set $\{1, \ldots, n\}$.

In all our considerations we assume that A is an $n \times n$ real matrix given in a block lower triangular form with p square diagonal blocks as follows:

$$A = \begin{pmatrix} A_{1,1} & 0 & \dots & 0 \\ A_{2,1} & A_{2,2} & & 0 \\ \vdots & & \ddots & \vdots \\ A_{p,1} & \dots & \dots & A_{p,p} \end{pmatrix}, \tag{2.1}$$

where each diagonal block is either an irreducible matrix or the 1×1 null matrix. The above form is called the *Frobenius normal form* of A. It is well known that any square matrix is symmetrically permutable to such a form. The *reduced graph of* A, $\mathcal{R}(A)$, is defined to be the graph with vertices $\{1, \dots, p\}$, where (i, j) is an arc from i to j if $A_{i,j} \neq 0$. A vertex i in $\mathcal{R}(A)$ is said to be singular if $A_{i,i}$ is singular. Otherwise the vertex is called *nonsingular.* The set of all singular vertices in $\mathcal{R}(A)$ will be denoted by $\mathcal{S}(A)$. A sequence of vertices $(i_1, \dots, i_k)$ in $\mathcal{R}(A)$ is said to be a *path* from i_1 to i_k if there is an arc in $\mathcal{R}(A)$ from i_j to i_{j+1} *for all* $j \in \langle k-1 \rangle$. The path is said to be *simple* if $i_1, \dots, i_k$ are distinct. The empty path will be considered a simple path linking every vertex $i \in \mathcal{R}(A)$ to itself. If there is a path (in $\mathcal{R}(A)$) from i to j, we write that $i \succeq j$. If $i \neq j$ and there is a path from i to j, we write that $i \succ j$.

Let $x \in R^n$. We partition $x = ((x_1)^T, \dots, (x_p)^T)^T$ in conformity with (2.1). Let $i \in \langle p \rangle$. We say that that the *level of* i in $\mathcal{R}(A)$ is k ($\text{lev}(i) = k$) if the maximal number of singular vertices on a path ending at i is k. We say that the *level of* $x \in R^n$ is k ($\text{lev}(x) = k$) if

$$k = \max\{\text{lev}(i) \mid x_i \neq 0\}.$$

For an $n \times n$ matrix A we denote by:

$N(A)$, the nullspace of A;

$E(A)$, the generalized nullspace of A, viz., $N(A^n)$;

$\nu(A)$, the index of 0 as an eigenvalue of A, viz., the size of the largest Jordan block associated with 0. Where no confusion is likely to arise, we write ν for $\nu(A)$.

We let $Z^{(0)}(A)$ be the eigenprojection of A corresponding to the eigenvalue 0 and we put $Z^{(k)}(A) = A^k Z^{(0)}(A)$, $k = 0, \dots, \nu - 1$. Where no confusion is likely to arise, we write $Z^{(k)}$ for $Z^{(k)}(A)$, $k = 1, \dots, \nu - 1$. The matrices $Z^{(k)}$, $k = 0, \dots, \nu - 1$, are called *the principal components of* A (*corresponding to the eigenvalue* 0). For background material on the principal components, see Lancaster and Tismenetski [10, p. 314] and, in the case of nonnegative matrices, see Neumann and Schneider [11].

Let $\alpha \subseteq \langle n \rangle$. By $A[\alpha]$ we denote the principal submatrix of A whose rows and columns are determined by α. Similarly, for an n-vector x, we denote by $x[\alpha]$ the subvector of x whose entries are indexed by α. For an array C, we use $C \geq 0$ to denote when all its entries are nonnegative numbers. $C > 0$ denotes the fact that $C \geq 0$, but $C \neq 0$. $C \gg 0$ denotes the fact that all of the entries of C are positive numbers.

Let P be an $n \times n$ nonnegative matrix. The Perron Frobenius theory (cf. Berman and Plemmons [2]) tells us that the spectral radius of P, given by the quantity

$$\rho(P) = \max\{|\lambda| : \det(P - \lambda I) = 0\},$$

is an eigenvalue of P that corresponds to a nonnegative eigenvector. In particular, if P is irreducible, then $\rho(P)$ is simple and the corresponding eigenvector is, up to a multiple by a scalar positive. The matrix $A = P - \rho(P)I$, which has all its off-diagonal entries nonnegative, is the $n \times n$ minus M–matrix that we associate with P and, in several sections of our paper, it will be convenient to work with A rather than with P. (We call A a *minus* M-*matrix* if $-A$ is an M-matrix. For the many equivalent conditions for a real matrix with nonpositive off-diagonal entries to be an M-matrix, see Berman and Plemmons [2, Chap. 6].) Suppose now that $m = \dim(E(A))$. It is known that m is equal to the number of singular vertices in $\mathcal{R}(A)$. Rothblum [13] has shown that $\nu(A)$ is equal to the maximum over all lengths of the simple paths in $\mathcal{R}(A)$, a result we shall refer to as the *Rothblum index theorem.* Let $\mathcal{S}(A) = \{\alpha_1, \ldots, \alpha_m\}$. Rothblum [13] and, independently, Richman and Schneider [12] (See also [14]) have shown that $E(A)$ possesses a basis of nonnegative vectors that is strongly combinatorial in the sense defined in Definition 1.

DEFINITION 1. *Let A be the $n \times n$ minus* M-*matrix given in form* (2.1) *and consider $\mathcal{R}(A)$.*

(i) *A nonnegative basis $u^{(1)}, \ldots, u^{(m)}$ is a* (nonnegatively) proper combinatorial basis *for $E(A)$ if*

$$u^{(j)}[i] > 0 \Rightarrow i \succeq \alpha_j$$

and

$$u^{(j)}[\alpha_j] \gg 0$$

for all $i \in \langle p \rangle$ and $j \in \langle m \rangle$.

(ii) *A nonnegative basis $u^{(1)}, \ldots, u^{(m)}$ is called a* (nonnegatively) strongly combinatorial basis *for $E(A)$ if*

$$u^{(j)}[i] = \begin{cases} \gg 0 & \textit{iff } i \succeq \alpha_j, \\ 0 & \textit{otherwise.} \end{cases}$$

Let $x \in E(A)$. We say that the *height of* x is k ($\mathrm{ht}(x) = k$) if k is the smallest nonnegative integer such that $A^k x = 0$. The *fundament of* x is, according to Hershkowitz and Schneider [9], the vector $A^{k-1}x$. For a set of vectors $\mathcal{S} = \{x, y, \ldots\}$ in $E(A)$, the *fundament of* $\mathcal{S}$ is the set of vectors formed from the fundaments of the elements of $\mathcal{S}$.

Let A be a minus M-matrix. Then it is known that $\mathrm{ht}(x) \leq \mathrm{lev}(x)$ for all $x \in E(A)$, cf. [8, Cor. (4.17)]. A vector $x \in E(A)$ for which $\mathrm{ht}(x) = \mathrm{lev}(x)$ is called a *peak vector.* If x is a peak vector, then $\mathrm{lev}(Ax) = \mathrm{lev}(x) - 1$ by [9, Prop. 6.5]. Also every nonnegative vector in $E(A)$ is a peak vector.

3. Jordan bases with nonnegative chains. Let $A \in R^{nn}$ be given as in (2.1). We shall use the following notation subsequently:

R^n_+ – the set of nonnegative vectors in $\mathcal{R}^n$.
$F = E(A) \cap R^n_+$.
$F_k = N(A^k) \cap R^n_+$, $k = 0, \ldots, \nu$.
$E_k = \mathrm{span}(F_k)$, $k = 0, \ldots, \nu$.
$S_k = A^{k-1}E_k$, $k = 1, \ldots, \nu$.

Henceforth, we let $A \in R^{nn}$ be a (singular) minus M-matrix of index ν. Since every nonnegative vector in $E(A)$ is a peak vector, we have the following graph theoretic classification of F_k and E_k.

$$F_k = \{x \in F \ : \ \mathrm{lev}(x) \leq k \}, \quad k = 0, \ldots, \nu. \tag{3.1}$$

$$E_k = \{x \in E(A) \ : \ \text{lev}(x) \leq k\}, \quad k = 0, \dots, \nu. \tag{3.2}$$

We note that

$$\{0\} = E_0 \subseteq E_1 \subseteq \cdots \subseteq E_\nu = E(A). \tag{3.3}$$

Now by (3.1) and (3.2) and because lev(Ax) =lev$(x) - 1$ for $x \in F_k$, it follows that

$$AE_k \subseteq E_{k-1}, \quad k = 1, \dots, \nu. \tag{3.4}$$

Hence we have

$$\{0\} \subseteq S_\nu \subseteq \cdots \subseteq S_1 = E_1 \subseteq N(A). \tag{3.5}$$

DEFINITION 2. (i) (Hershkowitz and Schneider [8, Def. (2.6)]) *The* height characteristic *of A is defined to be the ν-tuple*

$$\eta(A) = (\eta_1(A), \dots, \eta_\nu(A)),$$

where $\eta_k(A) = \dim(N(A^k)) - \dim N(A^{k-1}), \ k = 1, \dots, \nu$.

(ii) *The* peak characteristic *of A is defined to be the ν-tuple*

$$\xi(A) = (\xi_1(A), \dots, \xi_\nu(A)),$$

where $\xi_k(A) = \dim(S_k), \ k = 1, \dots, \nu$.

Where no confusion is likely to arise, we denote the height characteristic of A by $\eta = (\eta_1, \dots, \eta_\nu)$ and the peak characteristic of A by $\xi = (\xi_1, \dots, \xi_\nu)$. In [7, Def. (4.1)], Hershkowitz defines the peak characteristic of A by letting $\xi_k = \dim E_k - \dim(N(A^{k-1}) \cap E_k)$. Since S_k is isomorphic to $E_k/(N((A)^{k-1}) \cap E_k), \ k = 1, \dots, \nu$, it follows that his definition of the peak charactersitic of A coincides with the definition given above.

For the sake of completeness, we prove the following proposition that forms part of [7, Thm. (6.5)]. Recall, cf. [8, Def. (3.1)], that a basis $\mathcal{B}$ for E(A) is called a *height basis* if the number of basis elements of height k is equal to $\eta_k, \ k = 1, \dots, \nu$.

PROPOSITION 1. *Let A be a minus* M-*matrix and let $\mathcal{B}$ be a height basis for $E(A)$. Let β_k be the number of peak vectors in $\mathcal{B}$ of height k, $k = 1, \dots, \nu$. Then $\beta_k \leq \xi_k, \ k = 1, \dots, \nu$.*

Proof. Let $1 \leq k \leq \nu$ and let $x^{(1)}, \dots, x^{(s)}$ be peak vectors of height k in the height basis $\mathcal{B}$ for E. Then $x^{(1)}, \dots, x^{(s)}$ are linearly independent mod $N(A^{k-1})$ by [8, Prop. (3.14)]. Hence $A^{k-1}x^{(1)}, \dots, A^{k-1}x^{(s)}$ are linearly independent vectors. □

Hershkowitz [7, Thm. (6.5)] also proves that for every minus M-matrix, there exists a height basis that has ξ_k nonnegative vectors of height k, for $k = 1, \dots, \nu$. In §5 we give an algorithm that (in exact arithmetic) computes such a basis.

4. The transform components. We begin by introducing the (ϵ) transform components of A.

DEFINITION 3. *For $\epsilon > 0$ and for $k = 0, \dots, \nu - 1$, we define the* kth transform component *of A by*

$$(4.1) \qquad J^{(k)}(\epsilon) \;=\; Z^{(k)} + \frac{Z^{(k+1)}}{\epsilon} + \cdots + \frac{Z^{(\nu-1)}}{\epsilon^{\nu-k-1}}.$$

In [6] it was shown that provided $\epsilon > 0$ is sufficiently small, a basis of nonnegative vectors for $E(A)$ can be chosen from the columns of $J^{(0)}(\epsilon)$. Moreover, a method was given for determining such ϵ's, cf. [6, Thm. 2.2]. The results of [6] were improved in [11], where it was shown that provided $\epsilon > 0$ is sufficiently small, the nonnegative basis for $E(A)$ chosen from the columns of $J^{(0)}(\epsilon)$ can be chosen to be strongly combinatorial. We now strengthen the results of both papers by showing that the method used in [6] can be adapted to compute ϵ's that ensure that the columns of $J^{(0)}(\epsilon)$ contain a strongly combinatorial basis and that all transform components are nonnegative and have interesting combinatorial properties. To this end, for $1 \le i,j \le p$ and for $0 \le k \le \nu - 1$, let $\mu_{i,j}^{(k)}$ be the least element in $Z_{i,j}^{(k)}$ and let

$$\gamma_{i,j}^{(k)} = |\min\{\mu_{i,j}^{(k)}, 0\}|.$$

We note that by [11, Thm. 1], $\mu_{i,j}^{(d-1)} > 0$, when $d = d(i,j) \ge 1$. Let

$$(4.2) \qquad \mu = \min \frac{\mu_{i,j}^{(d-1)}}{\gamma_{i,j}^{(k)} + \cdots + \gamma_{i,j}^{(d-2)}},$$

where the minimum is taken over all i, j, k such that $1 \le i,j \le p$, $0 \le k \le \nu - 1$, and $d = d(i,j) > k$. We comment that we here take a ratio $p/0$, where $p > 0$, to be $+\infty$.

LEMMA 1. *Let A be a minus* M*-matrix and suppose $1 \le i,j \le p$ and $0 \le k \le \nu-1$. Let $0 < \epsilon \in \min\{1, \mu\}$, where μ is given by* (4.2).

(i) *If $d(i,j) \le k$, then $J_{i,j}^{(k)}(\epsilon) = 0$.*

(ii) *If $d(i,j) > k$, then $J_{i,j}^{(k)}(\epsilon) \gg 0$.*

Proof. (i) By [11, Lemma 2], if $d(i,j) \le k$, then $Z_{i,j}^{(q)} = 0$ for all q such that $k \le q \le \nu - 1$ and the result follows.

(ii) Let $d = d(i,j) > k$. Then

$$J_{i,j}^{(k)}(\epsilon) = Z_{i,j}^{(k)} + \frac{Z_{i,j}^{(k+1)}}{\epsilon} + \cdots + \frac{Z_{i,j}^{(d-1)}}{\epsilon^{d-k+1}}.$$

By [11, Thm. 1] $Z_{i,j}^{(d-1)} \gg 0$ and so $\mu_{i,j}^{(d-1)} > 0$. Let $0 < \epsilon < 1$ and let α be the least element in $J_{i,j}^{(k)}(\epsilon)$. Then

$$\begin{aligned} \alpha &\ge -\gamma_{i,j}^{(k)} - \cdots - \frac{\gamma^{(d-2)}}{\epsilon^{d-k-2}} + \frac{\mu_{i,j}^{(k-1)}}{\epsilon^{d-k-1}} \\ &\ge -\frac{1}{\epsilon^{d-k-2}}(\gamma_{i,j}^{(k)} + \cdots + \gamma_{i,j}^{(d-2)}) + \frac{\mu_{i,j}^{(k-1)}}{\epsilon^{d-k-1}}. \end{aligned}$$

Hence $\alpha > 0$ if

$$\epsilon < \frac{\mu_{i,j}^{(k-1)}}{\gamma_{i,j}^{(k)} + \cdots + \gamma_{i,j}^{(d-2)}}$$

and the result follows. □

We now make more precise a result mentioned in [11].

COROLLARY 1. *Let $\alpha_1, \ldots, \alpha_m$ be the singular vertices of $\mathcal{R}(A)$. Let $v^{(j)}$ be a column of $J^{(0)}(\epsilon)$ chosen from the columns of the α_jth block column of $J^{(0)}(\epsilon)$, $j = 1, \ldots, m$. Then $v^{(1)}, \ldots, v^{(m)}$ is a strongly combinatorial basis for $E(A)$ and, what is more, they satisfy:*

$$\begin{aligned} (A^k v^{(j)})_i &\gg 0 \quad \text{if } d(i, \alpha_j) > k, \\ (A^k v^{(j)})_i &= 0 \quad \text{if } d(i, \alpha_j) \le k. \end{aligned}$$

Proof. We observe that $A^k v^{(j)}$ is a column of $J^{(k)}(\epsilon)$ belonging to the α_jth block column $J^{(k)}(\epsilon)$, $k = 0, \ldots, \nu - 1$ and $j = 1, \ldots, m$. □

We note that this basis satisfies the properties of Rothblum, [13, Thm. 3.1]. Additionally we have the following corollary.

COROLLARY 2. *Let $v^{(1)}, \ldots, v^{(m)}$ be a basis of $E(A)$ which satisfies the conclusion of Corollary 1. The subset consisting of those vectors whose level does not exceed k forms a basis for E_k, $k = 0, \ldots, \nu - 1$.*

Proof. It holds that $v^{(1)}, \ldots, v^{(m)}$ is a strongly combinatorial basis for $E(A)$. □

5. The SCANBAS algorithm.

From now on we shall assume that ϵ has been chosen so that the transform components $J^{(k)}(\epsilon)$, $k = 0, \ldots, \nu - 1$ satisfy the conclusions of Lemma 1.

Observe that in the algorithm below, the index h is decreased in each iteration. Thus when we determine the sets $\mathcal{F}_h$ and the chains $\mathcal{C}_{i,h}$, the sets $\mathcal{F}_k$ and $\mathcal{C}_{i,k}$ are already determined for $k = h + 1, \ldots, \nu$.

THE SCANBAS ALGORITHM

Set $h = \nu$.

Step 1. Scan $J^{(h-1)}(\epsilon)$ to extract a set

$$\mathcal{F}_h = \{u^{(h,1,h)}, \ldots, u^{(h,s_h,h)}\}$$

of null vectors of A, which is maximal with respect to the property that the union $\mathcal{G}_h$ of $\mathcal{F}_h$ and the sets $\mathcal{F}_k$, $k = h + 1, \ldots, \nu$ is linearly independent.

Step 2. Then for each $u^{(h,i,h)}$, $i = 1, \ldots, s_h$, select the chain

$$\mathcal{C}_{i,h} = \{u^{(j,i,h)} \mid j = 1, \ldots, h\},$$

which consists of the columns in $J^{(0)}(\epsilon), \ldots, J^{(h-1)}(\epsilon)$ corresponding to $u^{(h,i,h)}$, i.e., if $u^{(h,i,h)}$ is the rth column of $J^{(h-1)}(\epsilon)$, then $u^{(j,i,h)}$ is the rth column of $J^{(j-1)}(\epsilon)$, $j = 1, \ldots, h$.

If $h > 1$, reduce h by 1, and repeat.
If $h = 1$, then stop.

Remark. Note that $u^{(j,i,h)}$ is the vector of height $h - j + 1$ in the ith chain of length h.

THEOREM 1. *Let $\mathcal{C}$ be the union of the chains*

$$\mathcal{C}_{i,h} = \{u^{(j,i,h)} \mid j = 1, \ldots, h\}, \quad i = 1, \ldots, s_h, \quad h = 1, \ldots, \nu.$$

Then

(i) *$\mathcal{C}$ consists of nonnegative vectors.*
(ii) *$\mathcal{C}$ is a linearly independent set of vectors.*
(iii) *Let $1 \le h \le \nu$. Then $\mathcal{G}_h = \cup_{k=h}^{\nu} \mathcal{F}_k$ is a basis for S_h.*
(iv) *$\mathcal{C}$ contains exactly ξ_h vectors of height h, $h = 1, \ldots, \nu$.*
(v) *$\mathcal{C}$ can be extended to a height basis for $E(A)$.*

Proof. (i) Each vector in $\mathcal{C}$ appears in a column in some $J^{(h)}$, $h = 0, \ldots, \nu - 1$, and these matrices are nonnegative.

(ii) The set $\mathcal{G}_1$ defined above is the fundament of $\mathcal{C}$ and, by construction, $\mathcal{G}_1$ is linearly independent. Hence $\mathcal{C}$ is linearly independent, e.g., Bru and Neumann [3].

(iii) Let $1 \le h \le \nu$. First, let $x \in \mathcal{G}_h$. Then $x \in \mathcal{F}_k$ for some k, $h \le k \le \nu$. Hence x is a column of $J^{(k-1)}(\epsilon)$ and therefore $x = A^{k-1}y$, where y is a column of $J^{(0)}(\epsilon)$. Since $x \in N(A)$, it follows that y must be in F_k. Hence $x \in A^{k-1}E_k = S_k \subseteq S_h$.

Conversely, let $x \in S_h$. Then $x = A^{h-1}y$, where $y \in E_h$ and so, by Corollary 2, y is a linear combination of columns of $J^{(0)}(\epsilon)$ that lie in F_h. Hence x is a linear combination of columns of $J^{(h-1)}(\epsilon)$ that lie in $N(A)$. Since by the first part of the proof of (iii), $\mathcal{F}_k \subseteq S_h$, $k = h+1, \ldots, \nu$, it now follows that the set $\mathcal{G}_h$ obtained in Step 2 of the SCANBAS algorithm is a basis for S_h.

(iv) Let $\mathcal{C}_h$ be the set of all vectors in $\mathcal{C}$ of height h. Then the map $x \to A^{h-1}x$ is a bijection of $\mathcal{C}_h$ onto $\mathcal{F}_h$, and hence (iv) follows from (iii).

(v) Since, by (iv), $A^{h-1}\mathcal{C}_h = \mathcal{F}_h$ and $\mathcal{F}_h$ is linearly independent, it follows that $\mathcal{C}_h$ is linearly independent mod E_{h-1}. Hence we can extend $\mathcal{C}_h$ to a set $\mathcal{B}_h$, which is a basis E_h mod E_{h-1}. It follows that $\mathcal{B} = \cup_{h=1}^{\nu} \mathcal{B}_h$ is a height basis for $E(A)$, cf. [8, Prop. 3.14]. □

COROLLARY 3. *It holds that*

$$\xi_h = s_h + \cdots + s_\nu, \quad h = 1, \ldots, \nu.$$

Let $x \in E(A)$ be a vector of height k. Then the *the chain derived from* x is defined to be the set $\{x, Ax, \ldots, A^{k-1}x\}$. The *chains derived from a subset* of $E(A)$ is the union of all chains derived from the vectors in this set. The technique used to prove the following important corollary is related to the proof of Hershkowitz and Schneider [8, Prop. (6.1)].

COROLLARY 4. *Let $1 \le t \le \nu$ and let $\eta_k = \xi_k$, $k = t, \ldots, \nu$. Then the chains $\mathcal{C}_{i,h}$, where $1 \le i \le s_h$ and $t - 1 \le h \le \nu$, can be embedded (extended) to a Jordan basis for $E(A)$.*

Proof. For $k = t, \ldots, \nu$, let $\mathcal{H}_k$ consist of all vectors $u^{(1,i,k)} \in \mathcal{C}$, $i = 1, \ldots, s_k$. Since $\mathrm{ht}(u^{(1,i,k)}) = k$ and $\xi_k = \eta_k$, it follows that $\cup_{r=k}^{\nu} A^{r-k}\mathcal{H}_r$ is a basis for $N(A^k)$ mod $(N(A^{k-1}))$. Now let $k = t - 1$. Then the set of vectors $u^{(1,i,k)}$, $i = 1, \ldots, s_k$, can be completed to a set $\mathcal{H}_k$ such that $\cup_{r=k}^{\nu} A^{r-k}\mathcal{H}_r$ is a basis for $N(A^k)$ mod $(N(A^{k-1}))$. Furthermore, for $k = 1, \ldots, t-2$, there exist sets $\mathcal{H}_k$ such that $\cup_{r=k}^{\nu} A^{r-k}\mathcal{H}_r$ is a basis for $N(A^k)$ mod $N(A^{k-1})$. The chains derived from $\cup_{k=1}^{\nu}\mathcal{H}_k$ now form a Jordan basis for $E(A)$ with the required properties. □

Remark. In [7, Thm. (6.6)] it is shown that there exists a Jordan basis for $E(A)$ such that all chains of length greater than or equal to t are nonnegative if and only if $\xi_k = \eta_k,\ k = t, \dots, \nu$. Thus, if a Jordan basis exists such that all chains of length t or greater are nonnegative, then the SCANBAS algorithm will produce such chains. In particular, if a nonnegative Jordan basis for $E(A)$ exists (viz., $\xi_k = \eta_k,\ k = 2, \dots, \nu$ or see [9, Thm. 6.6] for many other equivalent conditions), then the SCANBAS algorithm produces a nonnegative Jordan basis for $E(A)$. Finally, since we always have $\xi_\nu = \eta_\nu$, cf. [8, Prop. (4.2)], the SCANBAS algorithm always produces a set of nonnegative chains of length ν which can be extended to a Jordan basis $E(A)$ by adding chains of length at most $\nu - 1$. The result that there is a Jordan basis for $E(A)$ such that all chains of length ν are nonnegative is known; see [8, Cor. (6.12)] for the existence of such chains.

6. Examples and concluding remarks. We call a set $\mathcal{C}$ of vectors a maximal nonnegative union of chains (MNUC) provided $\mathcal{C}$ is a union of nonnegative chains, $\mathcal{C}$ is linearly independent, and $\mathcal{C}$ contains ξ_h vectors of height h. By Theorem 1, the SCANBAS alogorithm produces an MNUC. In this section we give several examples of MNUCs for various matrices and the relation of these MNUCs to Jordan bases.

We call a diagram of pluses with ξ_h pluses in row h (counting from the bottom) the *Peak diagram* of the matrix. Similarly we call a diagram of stars with η_h pluses in row h (counting from the bottom) the *Jordan diagram* of the matrix. (As is very well known, the number of stars in each column, read from the left, yields the Jordan (Segre) characteristic of the matrix.)

Example 1. We begin with an example where the MNUC consists of complete Jordan chains and may be completed to a Jordan basis by adjoining an eigenvector.

Let

$$a = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & -1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 & 0 \\ 2 & 1 & 2 & 1 & 1 & 0 & 0 \end{pmatrix}.$$

We put $sca = \text{scanbas}(a)$. Then

$$sca = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 \\ 2 & 3 & 3 & 1 & 1 \\ 4 & 6 & 6 & 3 & 2 \end{pmatrix}.$$

Then

$$jna = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 1 & 1 & 0 & 1 & -1 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \\ 2 & 3 & 3 & 1 & 1 & 0 \\ 4 & 6 & 6 & 3 & 2 & 0 \end{pmatrix}$$

is a Jordan basis since

$$a \times jna = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 2 & 3 & 0 & 1 & 0 \\ 0 & 4 & 6 & 0 & 3 & 0 \end{pmatrix}.$$

We observe that the Jordan and Peak diagrams can be combined as

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+ +
+ + * .
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Example 2. We now give an example of a minus M-matrix whose Perron eigenspace has a nonnegative Jordan basis and the basis with such specifications produced by our SCANBAS algorithm. Let

$$b = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 & 0 & 0 \\ 2 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 2 & 1 & 0 & 0 \end{pmatrix}.$$

Here the SCANBAS algorithm yields the MNUC $scb = \text{scanbas}(b)$ given by

$$scb = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 & 1 \\ 0 & 2 & 2 & 0 & 1 & 1 \\ 3 & 5 & 5 & 2 & 3 & 3 \\ 4 & 5 & 5 & 3 & 4 & 4 \end{pmatrix}.$$

This is easily seen to be a nonnegative Jordan basis consisting of two chains each of length 3.

Example 3. We give an example of a matrix that possesses an MNUC that can be embedded in a Jordan basis, but where no MNUC can consist of complete Jordan chains.

Let

$$c = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 2 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 \end{pmatrix}.$$

Then the index of c is 3 and, if we choose $\epsilon = 1$, we obtain the transform components

$$j0c = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 \\ 4 & 3 & 1 & 1 & 1 & 0 \\ 3 & 3 & 1 & 1 & 0 & 1 \end{pmatrix},$$

$$j1c = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 4 & 3 & 1 & 1 & 0 & 0 \\ 3 & 3 & 1 & 1 & 0 & 0 \end{pmatrix},$$

and

$$j2c = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & 2 & 0 & 0 & 0 & 0 \\ 2 & 2 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

As we can see by inspection of the transform components, our SCANBAS algorithm yields $scc = \text{scanbas}(c)$ given by

$$scc = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 2 & 4 & 4 & 1 \\ 2 & 3 & 3 & 0 \end{pmatrix}.$$

This set may be extended to a Jordan basis

$$jnc = \begin{pmatrix} 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & -1 \\ 2 & 4 & 4 & 1 & 0 & 0 \\ 2 & 3 & 3 & 0 & 0 & 0 \end{pmatrix},$$

where

$$c \times jnc = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 2 & 4 & 0 & 1 & 0 \\ 0 & 2 & 3 & 0 & 0 & 0 \end{pmatrix}.$$

Thus the combined Peak and Jordan diagrams here are

$$\begin{array}{lll} + & & \\ + & * & \\ + & + & * . \end{array}$$

We label the columns of jnc (from left to right) by v^{11}, v^{12}, v^{13}, v^{21}, v^{22}, v^{31}. Then we have the Jordan chains (v^{13}, v^{12}, v^{11}), (v^{22}, v^{21}) and (v^{31}). If some MNUC can be extended to a Jordan basis, then we would also get a combined Peak and Jordan diagram for c of the form

$$\begin{array}{lll} + & & \\ + & * & \\ + & * & + . \end{array}$$

We shall show that this is impossible; for let (w^{13}, w^{12}, w^{13}), (w^{22}, w^{21}), and (w^{31}) be another Jordan basis. Note that w^{31} is a linear combination of v^{11}, v^{21}, and v^{31} with nonzero coefficients for v^{31}, since w^{31} does not belong to range(c), see Bru, Rodman, and Schneider [4] for arguments of this type. But then, by inspection of the vectors, w^{31} cannot be nonnegative.

Example 4. We give an example of a matrix for which it is impossible to embed the chains of any MNUC into a Jordan basis.

Let

$$d = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \end{pmatrix}.$$

Then the SCANBAS algorithm yields the MNUC $scd = \text{scanbas}(d)$

$$scd = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 2 & 4 & 4 & 4 & 1 & 1 \\ 2 & 3 & 3 & 3 & 0 & 0 \end{pmatrix}.$$

A Jordan basis for d is given by

$$jnd = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & -1 \\ 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 2 & 4 & 4 & 4 & 1 & 0 & 0 & 0 & 0 \\ 2 & 3 & 3 & 3 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Thus the peak diagram for d is

$$\begin{array}{cc} + & \\ + & \\ + & + \\ + & + \end{array}$$

and the Jordan diagram is

$$\begin{array}{cccc} * & & & \\ * & * & & \\ * & * & & \\ * & * & * & * \end{array}.$$

We claim that no Jordan basis for d is an extension of an MNUC. We label the columns of jnd (from left to right) as x^{11}, x^{12}, x^{13}, x^{14}, x^{21}, x^{22}, x^{23}, x^{31}, x^{41}.

Suppose that there is a Jordan basis whose elements of height 3 are w^{13} and w^{23}, where w^{13} is of form $d(w^{14})$. Then, w^{13} is a multiple of x^{13}, while w^{23} is a linear combination of x^{13} and x^{23}, where x^{23} must have a nonzero coefficient. Hence w^{23} is not nonnegative. But if the Jordan basis is an extension of an MNUC, w^{23} must be nonnegative. Our claim follows.

Finally, we outline how our SCANBAS algorithm is implemented using MATLAB. The entire process is controlled by a function called *scanbas.m* whose input is the minus M-matrix A and whose output is an MNUC. This function first calls another MATLAB function *nnb.m* that returns an $\epsilon > 0$ and $J^{(0)} \geq 0$. The value value of $\epsilon > 0$, which is returned, is also sufficient to ensure that all higher-order transform components of A are nonnegative. To achieve its purpose, *nnb.m* initially determines the eigenprojection $Z^{(0)}$ by calling on a function *drazin.m*. The original version of *drazin.m* was written by Professor Robert E. Hartwig of North Carolina State University. This function computes the eigenprojection via the evaluation of the Drazin inverse A^D, viz., $Z^{(0)} = I - AA^D$, which is carried out using an algorithm due to Hartwig [5]. (For other methods of computing the Drazin inverse of a matrix, see the *shuffle* algorithm due to Anstreicher and Rothblum [1].) We mention that in *drazin.m*, the reduction steps used to implement Hartwig's algorithm are executed using the [q,r]=qr($\cdot$) command of MATLAB, not only for accuracy, but for the convenience of having the reducing matrices that this method needs from step to step [5]. The function *drazin.m* also returns ν, the index of A at 0. With $Z^{(0)}$ and ν at hand, *nnb.m* calls the function *macse.m*, which computes an $\epsilon > 0$ such that all transform components are nonnegative. This is done by generating iteratively all the principal components of A. With all this data at hand, *nnb.m* finally computes $J^{(0)}$ and returns the control to *scanbas.m* which now proceeds to compute an MNUC according to Steps 1 and 2 of the SCANBAS algorithm given in §5. This segment of *scanbas.m* starts by setting up an array W that contains, juxtaposed, all, say up to a multiple, transform components generated iteratively from $J^{(0)}$. Steps 1 and 2 are now carried out using a nested for/if loops.

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