On the Minimum Number of Interrupts in Time-Slot Assignments for Time-Division Multiple-Access Systems

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Abstract—**We study the minimum number of interrupts in an optimal time-slot assignment of a time-division multiple-access (TDMA) system. If the number of channels c in the TDMA system equals 2, then there always exists an optimal assignment with at most one interrupt. If $c \geq 3$, then there exist TDMA systems for which each optimal assignment requires at least $(c-1)(c-2)$ interrupts.**

I. Introduction

SUPPOSE a list of requests for communication services of specified duration (measured in terms of number of *time slots*) between certain transmitters and receivers is given. Assume that all communication between transmitters and receivers must be scheduled in a *c-channel* environment in which no transmitter or receiver may use two channels at the same time. Such a system is known as a *time-division multiple-access* (TDMA) system. The problem of how to find the *optimal assignment*, i.e., the assignment which schedule all communication using the minimum number of time slots, can be answered by the theory of doubly stochastic matrices (e.g., see [2]).

Suppose in a time slot assignment, a channel is configured to serve the ith transmitter and the jth receiver at a certain epoch. In the next epoch, the channel has to be reconfigurated to serve the αth transmitter and the βth receiver where $(i,j) \neq (\alpha, \beta)$. These reconfiguration processes could be expensive. If the communication services are demand assigned and centrally controlled, then the bandwidth of the communication resources required to coordinate time slot assignments will increase with the number of channel reconfigurations required. So, besides minimizing the number of time slots required in a schedule of the service, one might also want to minimize the number of reconfigurations. Notice that here we count the number of reconfigurations of individual channels, other authors (e.g., [2]) have counted as one reconfiguration all changes made at the same time.

In many TDMA systems the communication resources required to coordinate time slot assignments are self-hosted. Channel time-slot assignments must be received before they can be activated. Thus in the kth scheduling epoch bandwidth must be assigned for communicating the time slot schedule to be used in the $(k+1)$st epoch. Suboptimal schedules will result if either too much or insufficient bandwidth is reserved. Bounds on the number of reconfigurations required in an optimal assignment can be used to bound the bandwidth required for communication of control information. The purpose of this note is to study the minimum number of reconfigurations in an optimal assignment of a TDMA system.

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II. Mathematical Formulation

Let $x_1, \cdots, x_m$ be the transmitters and $y_1, \cdots, y_n$ be the receivers. Denote by $A = (a_{ij})$ the $m \times n$ matrix such that a_{ij} represents the number of time slots required for transmitting information from transmitter x_i to receiver y_j. We call A the *traffic matrix* of the TDMA system. For a given time slot assignment that requires T time slots, the corresponding *schedule matrix* $S = (s_{ij})$ is the $c \times T$ matrix such that

$$s_{ij} \in \{(p,q) : 1 \leq p \leq m,\ 1 \leq q \leq n\} \bigcup \{(0,0)\}$$

where $s_{ij} = (p,q) \neq (0,0)$ means that at the jth time slot, the ith channel is *active* and is transmitting information from x_p to y_q; and $s_{ij} = (0,0)$ means that the ith channel is *inactive* at the jth time slot. Clearly, the total number of (p,q) pairs in S equals a_{pq}. By the restriction on the system, excluding the $(0,0)$ pairs, the (p,q) pairs in a column of S has different first coordinates, and different second coordinates. In the ith row of S, a nonzero pair (p,q) represents a *configuration* of the ith channel. An active configuration (p,q) is a *reconfiguration* if either it is the first active configuration in a channel or the last previous active configuration of the same channel has a different value. (There is no change of configuration in the channel if some nonzero (p,q) pairs and $(0,0)$ pairs occur alternatively.) Clearly, the number of reconfigurations is at least the number of positive entries in the traffic matrix. If there is a difference between these two numbers, there must be certain service between some transmitters and receivers that are not scheduled continuously. We say that *interrupts* occur at the corresponding entry in the traffic matrix and define the *number of interrupts* of the assignment to be the difference between the number of reconfigurations and the number of positive entries in the traffic matrix. In the following section, we shall study the minimum number of interrupts in an optimal assignment of a TDMA system. Notice that (e.g., see [2]) the problem of finding a time-slot assignment for a TDMA system can be reduced to finding a representation $A = \sum_{i=1}^{t} \alpha_i P_i$ where α_i are positive integers and P_i are $0-1$ matrices of rank not greater than c and have at most one positive entry in each row and in each column. For such a representation, $\sum_{i=1}^{t} \alpha_i$ will be the total number of time slots required for the corresponding assignment. Suppose $A = \sum_{i=1}^{t} \alpha_i P_i$ is a representation corresponding to an optimal assignment. Then a rearrangement of the summands in the representation will change the corresponding schedule matrix and hence will affect the number of interrupts. Moreover, there may be different choices for the set of permutation matrices used to represent an optimal assignment. This will also affect the number of interrupts. As a result, one has to find the optimal set of permutation matrices used for the representation and the best arrangement of them in order to get a minimum number of interrupts in the optimal assignment.

The above question was previously considered in [1]. Here we give a complete proof that, if the number of channels c in the TDMA system equals 2, then there always exists an optimal schedule with at most one interrupt. However, if $c \geq 3$, then there exist TDMA systems for which each optimal assignment requires at least $(c-1)(c-2)$ interrupts. In fact, this is true for almost all TDMA systems in a certain class. This answers a question raised in [1]. A somewhat related problem is studied in [3].

III. Results

The following result is known, e.g., see [2].

Proposition 1: Given a TDMA system with traffic matrix, A, the minimum number of time slots required for an assignment equals

$$\max\{R(A),\, C(A),\, \lceil s(A)/c \rceil\}$$

where $R(A)$ and $C(A)$ denotes the maximum row sum and column sum of A, respectively, $s(A)$ denotes the sum of all entries of A, and $\lceil \cdot \rceil$ denotes the ceiling function.

Recall that a time-slot assignment for a TDMA system is optimal if the time slots used is minimum. In general, there may be more than one optimal assignment. Certainly, one would choose the one with minimum number or interrupts so as to minimize the cost for configurating the system. In the following proposition, we show that one could always find an optimal assignment whose number of interrupts is of the order $O(cn^2)$.

Proposition 2: Let the $m \times n$ matrix A be the traffic matrix of a TDMA system. Suppose $m \leq n$ and A has $p(A)$ positive entries. Then there is an optimal assignment with number of interrupts not more than $cd^2 - p(A)$ where $d = \max\{m, n\}$.

Proof: Using the method in [2], one sees that the problem of finding a time-slot assignment for a TDMA system can be reduced to finding a representation $A = \sum_{i=1}^{t} \alpha_i P_i$ where α_i are positive integers and P_i are $0-1$ matrices of rank not greater than c and have at most one positive entry in each row and in each column. By the discussion after the proof of Theorem 2.2 in [2], we see that there is an optimal assignment requiring t matrices P_i such that t is not greater than d^2. Writing down the corresponding schedule matrix, one sees that reconfigurations can only occur at the first column or at the end of α_1th column, $(\alpha_1 + \alpha_2)$th column, etc. It follows that the maximum number of reconfigurations in the schedule matrix is not greater than cd^2. By the fact that the number of interrupts equals the difference of the number of reconfigurations and $p(A)$, we get the result. □

Actually, the discussion following Theorem 2.2 in [2] shows that the bound in Proposition 2 may be improved in certain circumstances. For example, if $k = m = n$, $cd^2 - p(A)$ may be replaced by $c(n^2 - 2n + 1) - p(A)$, and if, in addition, A is fully indecomposable, then we obtain the bound $n(p(A) - 2n + 2) - p(A)$.

Clearly, if $c = 1$, one can find an optimal assignment with no interrupt. In [1], the authors asserted correctly that, if $c = 2$, one can always find an optimal assignment with at most one interrupt. The proof given there is incomplete. We give a proof for the statement in the sequel.

We first quote some remarks from [1] which are useful in our discussion. Suppose we are dealing with a TDMA system in a c-channel environment. Notice that the scheduling problem is invariant under permutation of the rows and columns of the traffic matrix. Assume that the optimal assignment requires T time slots. If there are two rows (resp. columns) of the traffic matrix A whose entries have sum not greater than T and if A' is obtained from A by deleting the two rows (resp. columns) and adjoining their sum as the last row (resp. column) to the resulting matrix, then the optimal assignment for the TDMA system with traffic matrix A' also requires T time slots. Moreover, if A' has an optimal assignment with k interrupts, then A has an optimal assignment with no more than k interrupts. Notice that if A has more than $2c - 1$ rows or columns, we can always combine rows or columns as described above. By these observations, we can prove the following lemma and proposition.

Lemma: If $m = c = 2$, then there is an optimal time slot assignment with no interrupt.

Proof: Let T be the minimum number of time slots required for the optimal assignment. By the discussion before the lemma, we may assume that $A = (a_{ij})$ is a 2×3 matrix, otherwise, we combine columns of A. The problem is easy if $T = C(A)$, say $T = a_{11} + a_{21}$. In this case, we have $a_{22} + a_{23} \leq a_{11}$ and $a_{12} + a_{13} \leq a_{21}$. Assuming $t = \sum_{i=1}^{3} (a_{1i} - a_{2i}) \geq 0$, we may construct the schedule matrix as

$$S = \begin{bmatrix} a_{11} * (1,1) & & a_{12} * (1,2) & a_{13} * (1,3) \\ a_{22} * (2,2) & a_{23} * (2,3) & t * (0,0) & a_{21} * (2,1) \end{bmatrix}$$

where $r * (p,q)$ denotes r consecutive (p,q) pairs.

Now suppose $T = R(A)$. We may assume that all rows of A have sum T, otherwise increase some of the entries. Suppose a_{11} is the largest entry of A and $a_{13} \leq a_{12}$. Then $a_{11} \leq a_{22} + a_{23}$, otherwise $a_{11} + a_{21} > T$. We consider two cases.

Case 1: $a_{22} + a_{23} \leq a_{11} + a_{12}$. Since $a_{11} \geq a_{22}$ and $a_{22} + a_{23} \geq a_{11}$, we may construct the schedule matrix as

$$S = \begin{bmatrix} a_{11} * (1,1) & a_{12} * (1,2) & a_{13} * (1,3) \\ a_{22} * (2,2) & a_{23} * (2,3) & a_{21} * (2,1) \end{bmatrix}.$$

Case 2: $a_{22} + a_{23} > a_{11} + a_{12}$. Then $a_{21} < a_{13}$ and $a_{23} > a_{12}$. We may construct the schedule matrix as

$$S = \begin{bmatrix} a_{13} * (1,3) & a_{11} * (1,1) & a_{12} * (1,2) \\ a_{21} * (2,1) & a_{22} * (2,2) & a_{23} * (2,3) \end{bmatrix}.$$

In both cases, no interrupt occurs. □

Proposition 3: Given a TDMA system with $c = 2$, there is an optimal assignment with at most one interrupt.

Proof: Let T be the minimum number of time slots required in an optimal assignment for the TDMA system. By combining columns or rows of A as mentioned before, we may assume that $A = (a_{ij})$ is 3×3. Let s_i and $r_i, i = 1,2,3$, be the ith column and ith row sum of A, respectively. We may further assume that

$$\max\{r_1, r_2, r_3, s_1, s_2, s_3\} = r_3 < T, \tag{1}$$

(if $r_3 = T$, we can combine the first two rows and apply the Lemma), and $r_1 + r_2 + r_3 = 2T$. Then $r_1 + r_2 > r_3$, otherwise $r_1 + r_2 + r_3 < 2T$. Let

$$(r_1 + r_2 - r_3)/2 = q, \tag{2}$$

which is an integer as $r_1 + r_2 + r_3 = 2T$ is even. Then

$$\min\{r_1, r_2\} \geq 2q. \tag{3}$$

We now schedule the first q time slots. Consider the first two rows. Assume $a_{11} = \max\{a_{ij} : 1 \leq i \leq 2, 1 \leq j \leq 3\}$. We consider two cases.

Case 1: Suppose $a_{11} \geq q$. If $a_{22} + a_{23} \geq q$, construct the first q columns of S as

$$S_1 = \begin{bmatrix} q * (1,1) & \\ t * (2,2) & (q-t) * (2,3) \end{bmatrix} \tag{4}$$

where $t = \min\{a_{22}, q\}$. Suppose $a_{22} + a_{23} < q$. Then by (3), $a_{21} \geq q$. Moreover, $a_{12} + a_{13} \geq q$, otherwise by (2) we have

$$\begin{aligned} r_3 \geq s_1 \geq a_{11} + a_{21} &= r_1 + r_2 - (a_{12} + a_{13}) - (a_{22} + a_{23}) \\ &> r_1 + r_2 - 2q = r_3, \end{aligned}$$

which is impossible. So we may construct the first q columns of S as

$$S_1 = \begin{bmatrix} t * (1,2) & (q-t) * (1,3) \\ q * (2,1) & \end{bmatrix} \tag{5}$$

where $t = \min\{a_{12}, q\}$.

Case 2: Suppose $a_{11} < q$. Then by (3) and the fact that $a_{11} \geq a_{13}$, we have $a_{11} + a_{12} > q$. Moreover, as $q > a_{11} \geq a_{21}$ and $r_2 \geq 2q$, we have $a_{22} + a_{23} > q$. We may construct the first q columns of S as

$$S_1 = \begin{bmatrix} a_{11} * (1,1) & (q - a_{11}) * (1,2) \\ a_{22} * (2,2) & (q - a_{22}) * (2,3) \end{bmatrix}. \tag{6}$$

After that, we combine the rest of the services required in the first two rows of A to form a new matrix

$$A' = \begin{bmatrix} d_{11} & d_{12} & d_{13} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

whose row sums equal r_3 and columns sums are not greater than s_3. By the Lemma, we get a schedule matrix S_2 of A' with no interrupts. If we construct a schedule matrix by adjoining S_2 to S_1 to get $[S_1 \mid S_2]$ and expanding every entry d_{1j} in the first row of S_2 into two parts corresponding to a_{1j} and a_{2j} according to its original composition, we get an optimal time-slot assignment. However, there may be two interrupts at the qth column of S. To avoid this, we modify the construction as follows. Suppose the last column of S_1 is $[(1,i),(2,j)]^t$ such that the number of $(1,i)$ pairs and $(2,j)$ pairs are less that a_{1i} and a_{2j}, respectively. Let the first and last entries in the first row of S_2 be $(1,\alpha)$ and $(1,\gamma)$, respectively. If $\alpha = i$, then the possible interrupt at a_{1i} can be avoided by constructing S as $[S_1 \mid S_2]$ and splitting $d_{1\alpha}$ with $(1,i)$ pairs preceding $(2,i)$ pairs. If $\alpha = j$, then split $d_{1\alpha}$ with $(2,i)$ pairs preceding $(1,i)$ pairs and interchange the rows of S_2 to get S_2'. The possible interrupt at a_{2j} can then be avoided by constructing S as $[S_1 \mid S_2']$. If $\alpha \neq i,j$, then $\gamma = i$ or $\gamma = j$. Construct S_2' from S_2 by interchanging its tth and $(q-t+1)$th columns for all t. Then we are back to the previous case and one of the possible interrupts in a_{1i} or a_{2j} can be avoided. Consequently, there will be at most one interrupt in the assignment. □

In view of the above results, the question was raised in [1] whether one can always get an optimal assignment with $c-1$ interrupts. Unfortunately, this is not true in general as shown by the following proposition.

Proposition 4: Suppose $c \geq 3$. There exists a TDMA system such that every optimal assignment of it has at least $(c-1)(c-2)$ interrupts.

Proof: Let Ω be the set of all $c \times c$ doubly stochastic matrices. Suppose $k = (c-1)^2$. Then it is well known (e.g., see [4]) that Ω has (affine) dimension k, and every X in Ω can be written as a convex combination of no more than $k+1$ permutation matrices. Note that if $q < k$, then the set of convex combinations of $q+1$ given permutation matrices has (affine) dimension less than k. Since the number of such convex polytopes is finite, their union cannot cover Ω, so there has to be an open set contained in the complement of their union. Thus, there is a rational doubly stochastic matrix which is a convex combination of $k+1$ but not fewer permutation matrices. As a result, we can construct a $c \times c$ traffic matrix A such that all of its row sums and column sums equal r and it is a positive integral combination of $k+1$ but not fewer permutation matrices. Suppose S is a schedule matrix of A corresponding to an optimal time-slot assignment with minimum number of interrupts. Then each column of S corresponds to a $c \times c$ permutation matrix. Assume the ith and the $(i+1)$th column of S are different for $i = j_1, \cdots, j_t$. Then A can be written as the positive integral combination of $t+1$ permutation matrices. Thus, $t \geq k$. Notice that between the ith and $(i+1)$th column of S for $i = j_1, \cdots, j_t$, there are changes in configurations in at least two channels (since one cannot change just one element in a column of S to obtain a new column which still corresponds to a permutation). There are c initial configurations and at least $2k$ additions reconfigurations. Thus, the number of interrupts is at least

$$c + 2k - p(A) \geq c + 2(c-1)^2 - c^2 = (c-1)(c-2). \quad \square$$

We observe that it follows from the proof of Proposition 4, that, when $c \geq 3$, for "almost all" schedule matrices corresponding to a traffic matrix with equal row and column sums the number of interrupts in an optimal assignment must be at least $(c-1)(c-2)$.

Corollary: If $c > 3$, there exists a TDMA system such that every optimal time-slot assignment of it has more than $c-1$ interrupts.

The above Corollary shows that the number of interrupts required in an optimal assignment is greater than $c-1$ in general if $n > 3$. If $n \geq m = c = 3$, then we can always find an optimal assignment with at most two interrupts. We pose the following question: if $c \geq 3$, does every TDMA system have an optimal assignment with exactly $(c-1)(c-2)$ interrupts.

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