SIAM J. MATRIX ANAL. APPL.
Vol. 11, No. 4, pp. 565–574, October 1990

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ON THE INERTIA OF INTERVALS OF MATRICES*

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Abstract. The inertia of intervals and lines of matrices is investigated. For complex $n \times n$ matrices A and B it is shown that, under mild nonsingularity conditions, $A + tB$ changes inertia at no more than n^2 real values of t. Conditions are given for the constancy of the inertia of $A + tB$, where t lies in a real interval. These conditions generalize and organize some known results.

Key words. inertia, constant inertia, inertia change point, interval of matrices, matrix stability, Lyapunov operators, Z-matrices

AMS(MOS) subject classification. 15

1. Introduction. Bialas [1], Johnson and Rodman [4], Väliaho [7], and Fu and Barmish [2], [3] have recently studied the inertia of intervals and lines of matrices. We extend these investigations under nonsingularity conditions. While some of our results are not difficult and are related to known results, taken together they show interrelations between various types of conditions, and as such they organize knowledge in this area of inertia theory.

Let A and B be square complex matrices and suppose there is a real t such that the Lyapunov matrix $L(A + tB)$ associated with $A + tB$ is nonsingular. We show that $A + tB$ changes inertia at no more than n^2 values of t. Let T be an interval, i.e., a connected subset of the real numbers. Under the assumption that $L(A)$ is nonsingular, we state our principal condition,

(CI) $\quad A + tB$ has constant inertia of type $(\pi, \nu, 0)$ for every t in T,

and we compare several other conditions (some obviously equivalent) to (CI). Some of these conditions involve the real eigenvalues of $A^{-1}B$ and of $L(A)^{-1}L(B)$. Each of the conditions either implies or is implied by (CI), but not all are equivalent in general. By adding additional requirements on a single matrix or on the interval, such as stability, the reality of all eigenvalues, or a condition we call Property X (which Z-matrices satisfy), some implications in one direction become equivalences.

Section 2 of our paper contains notation, definitions, and some well-known results stated for easy reference. Section 3 contains preliminary results on eigenvalues and results on changes of inertia. Our main results on intervals with constant inertia, summarized above, may be found in § 4. In § 5 we give some applications to the convex hull of two matrices. We derive results from [1]–[4] and [7].

* Received by the editors December 19, 1988; accepted for publication (in revised form) September 8, 1989. This research was supported by grant 85-00153 from the United States–Israel Binational Science Foundation (BSF), Jerusalem, Israel.

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Our principal theorems are proved for the case of general complex matrices, and we then apply the results to Hermitian matrices and Z-matrices.

Properties of the Lyapunov operator $A \to L(A)$ that are crucial to our results are the following:

(1.1) If λ is an eigenvalue of A, then $2\,\mathrm{Re}(\lambda)$ is an eigenvalue of $L(A)$.

(1.2) If t is the maximal (minimal) eigenvalue of $L(A)$, then there is an eigenvalue λ of A with $2\,\mathrm{Re}(\lambda) = t$.

Similar results may be proved for any real linear operator, from the space of complex $n \times n$ matrices into a space of matrices, which satisfies (1.1) and (1.2). For spaces of real matrices, another operator that satisfies these conditions is found in [1] and [3]. The results in [1] are proved for that operator, whereas in [3] results are proved for all operators satisfying (1.1) and (1.2). The results in [2] are proved for the Lyapunov operator, as in the present paper. Only real matrices are considered in [1]–[3]. In referring to the results of these papers in the sequel, we do not distinguish between the various operators involved. We also observe that the results in [7] deal with real symmetric matrices (where there is no need to employ the Lyapunov operator), but some results in [7] hold under weaker nonsingularity assumptions.

2. Notation and preliminaries. As usual, $\mathbb{R}$ and $\mathbb{C}$ denote the real and complex fields, respectively, and $\mathbb{C}^{nn}$ denotes the complex space of all complex matrices. By $\mathscr{H}_n$ we denote the *real* space of all $n \times n$ Hermitian matrices. In this paper, A and B will always be $n \times n$ complex matrices that may be considered fixed throughout. The convex hull of A and B is denoted by conv (A, B). The spectrum of a matrix A is denoted by spec (A). The spectrum is considered to be a multiset, that is, every eigenvalue is counted as many times as its multiplicity.

Notation 2.1. We denote the following:

$\pi(A)$—the number of eigenvalues of A in the open right halfplane,

$\nu(A)$—the number of eigenvalues of A in the open left halfplane,

$\delta(A)$—the number of eigenvalues of A on the imaginary axis.

DEFINITION 2.2. The inertia In (A) of A is defined to be the triple $(\pi(A), \nu(A), \delta(A))$.

DEFINITION 2.3. (i) The matrix A is said to be *positive* [*negative*] *stable* if all its eigenvalues are in the open right [left] halfplane.

(ii) The matrix A is said to be *positive* [*negative*] *semistable* if all its eigenvalues are in the closed right [left] halfplane.

(iii) The matrix A is said to be *positive* [*negative*] *near-stable* if A is positive [negative] semistable but not positive [negative] stable.

In this paper "stable," "semistable," and "near-stable" may be interpreted consistently to mean either "positive stable," "positive semistable," and "positive near-stable" or "negative stable," "negative semistable," and "negative near-stable."

DEFINITION 2.4. The *Lyapunov operator* (*or Lyapunov matrix*) $L(A)$ of A is defined to be the linear operator of $\mathscr{H}_n$ into itself given by

$$L(A)H = AH + HA^*.$$

For reference, we collect some properties of the operator $L(A)$. We follow the notation of [5] for the Kronecker (or tensor) product of matrices.

PROPOSITION 2.5. *We have*

(i) $L(A) = I \otimes A + \bar{A} \otimes I$.

(ii) *The spectrum of* $L(A)$ *is the multiset* $\{\lambda + \bar{\mu} : \lambda, \mu \in \operatorname{spec}(A)\}$.

(iii) $L(A)$ *is nonsingular if and only if* $\lambda + \bar{\mu} \neq 0$ *for* $\lambda, \mu \in \operatorname{spec}(A)$.

(iv) A *is stable* [*semistable*] (*near-stable*) *if and only if* $L(A)$ *is*.

(v) *The mapping* $A \to L(A)$ *is real linear, i.e.,* $L(sA + tB) = sL(A) + tL(B)$, *for all real numbers* s *and* t.

Proof. Parts (i) and (ii) are standard (e.g., see [5, Chap. 12]). Parts (iii) and (iv) follow immediately from (ii). Part (v) follows from the definition of $L(A)$. □

In our proofs (as in the proof of almost any inertia theorem) we use properties often called "continuity of eigenvalues." The basic result is stated as Lemma 3 in [6]. Here we state consequences of this lemma in the forms needed for our applications.

LEMMA 2.6. (i) *Let* $A(t)$ *be a continuous matrix function of the real variable* t. *Let* λ *be an eigenvalue of* $A(0)$. *Let* S *be a disc in the complex plane with center at* λ *such that* S *does not contain any other eigenvalue of* $A(0)$. *If there exists a positive* δ *such that for all* t, $0 < t < \delta$, $A(t)$ *has an even number of eigenvalues in* S, *then the multiplicity of* λ *as an eigenvalue of* $A(0)$ *is even*.

(ii) *If* A *has no imaginary eigenvalues, then for all sufficiently small* ε, *we have* $\operatorname{In}(A + \varepsilon B) = \operatorname{In}(A)$.

(iii) *If* $\operatorname{In}(A) \neq \operatorname{In}(B)$, *then there is a matrix* $C \in \operatorname{conv}(A, B)$ *that has an imaginary eigenvalue*.

(iv) *If* A *is stable but* B *is not stable, then there is a matrix* $C \in \operatorname{conv}(A, B)$ *that is near-stable*.

Proof. Parts (i) and (ii) follow from Lemma 3 of [6]. Parts (iii) and (iv) follow from (ii) using the completeness of the real numbers and the connectedness of the interval $[0, 1]$. □

Convention 2.7. By the term "*interval*" we mean a connected subset of the real line. That is, open intervals, closed intervals, half-open intervals, halflines and the whole real line are intervals.

DEFINITION 2.8. Let T be an interval. The *matrix interval* $S(A, B; T)$ of matrices is defined to be the set $\{A + tB : t \in T\}$.

DEFINITION 2.9. Let $t_0 \in \mathbb{R}$. We say that t_0 is an *inertia change point* for $S(A, B; \mathbb{R})$ if for every $\varepsilon > 0$ there exists $t \in \mathbb{R}$ such that $|t - t_0| < \varepsilon$ and Inertia $(A + tB) \neq$ Inertia $(A + t_0 B)$.

DEFINITION 2.10. Let T be an interval.

(i) The interval T is called an *interval of constant inertia* (π, ν, δ) for $S(A, B; \mathbb{R})$ if every matrix in $S(A, B; T)$ has inertia (π, ν, δ).

(ii) The interval T is called an *interval of semiconstant inertia* $(\pi, \nu, 0)$ for $S(A, B; \mathbb{R})$ if every matrix $C \in S(A, B; T)$ such that $\delta(C) = 0$ has $\operatorname{In}(C) = (\pi, \nu, 0)$.

If T is an interval of constant inertia for $S(A, B; \mathbb{R})$ we may also say that $S(A, B; T)$ has constant inertia or, when every matrix in $S(A, B; T)$ is stable [semistable], that $S(A, B; T)$ is stable [semistable].

DEFINITION 2.11. We call (A, B) a *regular pair of matrices* if there exists a $t \in \mathbb{R}$ such that $L(A + tB)$ is nonsingular.

PROPOSITION 2.12. *If* (A, B) *is a regular pair of matrices, then the number of complex numbers* t *for which* $L(A + tB)$ *is singular is at most* n^2.

Proof. Since $L(A + tB)$ is singular if and only if $\det(L(A + tB)) = 0$, and since $p(t) = \det(L(A + tB))$ is a polynomial of degree at most n^2, it follows that either (A, B) is a regular pair, in which case $p(t)$ has at most n^2 roots, or (A, B) is not a regular pair, in which case $p(t) \equiv 0$. □

COROLLARY 2.13. *If (A, B) is a regular pair of matrices, then the number of complex numbers t for which $A + tB$ has an imaginary eigenvalue is at most n^2.*

Proof. The claim follows from Proposition 2.5(ii) and Proposition 2.12. □

Another corollary of Proposition 2.12 is the following.

COROLLARY 2.14. *(A, B) is a regular pair of matrices if and only if (B, A) is a regular pair of matrices.*

Proof. If (A, B) is a regular pair of matrices then, by Proposition 2.12, there exists a nonzero number t such that $L(A + tB)$ is nonsingular. Therefore, $L(A/t + B)$ is nonsingular, and so (B, A) is a regular pair of matrices. □

Since for all complex $n \times n$ matrices A, (I, A) is a regular pair, it follows from Proposition 2.12 and Corollary 2.14 that $L(A + tI)$ is nonsingular for all but at most n^2 complex numbers t.

3. Observations on eigenvalues and inertia. We start with an immediate observation.

OBSERVATION 3.1. Let A be nonsingular, and let t be a nonzero real number. Then the following are equivalent:

(a_1) $-1/t$ is an eigenvalue of $A^{-1}B$.

(a_2) $I + tA^{-1}B$ is singular.

(a_3) $A + tB$ is singular.

Accordingly, we label the three equivalent conditions in Observation 3.1 (under the assumption that A is nonsingular) as condition (a).

If $L(A)$ is nonsingular then, applying Observation 3.1 to $L(A)$ and $L(B)$, we obtain the following equivalent conditions:

(la_1) $L(A) + tL(B)$ is singular.

(la_2) $I + tL(A)^{-1}L(B)$ is singular.

(la_3) $-1/t$ is an eigenvalue of $L(A)^{-1}L(B)$.

By Proposition 2.5(v), condition (la_1) is equivalent to

(la_4) $L(A + tB)$ is singular.

We now label the four equivalent conditions (la_1)–(la_4) (under the assumption that $L(A)$ is nonsingular) as condition (la).

A third condition we will discuss is

(ie) $A + tB$ has an imaginary eigenvalue.

THEOREM 3.2. *Let A and B be $n \times n$ complex matrices, let t be a nonzero real number, and assume that $L(A)$ is nonsingular. Then we have*

$$(a) \to (ie) \to (la).$$

Proof. First observe that if $L(A)$ is nonsingular then A is nonsingular, so both conditions (a) and (la) are well defined. The implication (a) → (ie) follows from the trivial implication (a_3) → (ie). The implication (ie) → (la) follows from (ie) → (la_4), which follows from Proposition 2.5(ii). □

Clearly, the converses of the implications (a) → (ie) and (ie) → (la) do not hold under the stated hypotheses.

We now add three more conditions that relate to the previous eight.

(ic) t is an inertia change point for $S(A,B;\mathbb{R})$.

(ns) $A+tB$ is near stable.

(us) $A+tB$ is not stable.

THEOREM 3.3. *Let A and B be $n\times n$ matrices, let t be a nonzero real number, and assume that $L(A)$ is nonsingular. Then we have*

$$\begin{array}{l}(\mathrm{a})\rightarrow(\mathrm{ie})\Leftrightarrow(\mathrm{ic})\rightarrow(\mathrm{la})\rightarrow(\mathrm{us}).\\ (\mathrm{ns})\nearrow\end{array}$$

Proof. The implication (ns) → (ie) is clear by Definition 2.3(iii). The implication (ic) → (ie) follows from Lemma 2.6(ii). The implication (ie) → (ic) follows from Corollary 2.13. The implication (la) → (us) follows from (la_4) → (us), which follows from Proposition 2.5(iv). □

The converses of the implications (ns) → (ie) and (la) → (us) do not hold. Also, neither (a) → (ns) nor (ns) → (a) holds.

Theorem 3.3 yields the following corollary.

COROLLARY 3.4. *Suppose that (A, B) is a regular pair of matrices. Then the number of inertia change points for $S(A, B; \mathbb{R})$ is at most n^2.*

Proof. If (A, B) is a regular pair of matrices, then (ic) → (ie) holds even if $L(A)$ is singular. To see that, let $A' = A + t'B$, where $t' \in \mathbb{R}$ is chosen so that $L(A')$ is nonsingular. Let t be an inertia change point for $S(A, B; \mathbb{R})$. Obviously, $t - t'$ is an inertia change point for $S(A', B; \mathbb{R})$. By Theorem 3.3 (applied to A' and B), $A' + (t - t')B = A + tB$ has an imaginary eigenvalue. Our claim now follows from Corollary 2.13. □

THEOREM 3.5. *Let (A, B) be a regular pair of matrices and let $t_1, \cdots, t_m$, where $t_1 < \cdots < t_m$, be the inertia change points of $S(A, B; \mathbb{R})$. Let $T_0 = (-\infty, t_1)$, $T_i = (t_i, t_{i-1})$, $i = 1, \cdots, m$, and $T_m = (t_m, \infty)$. Then the intervals T_i, $i = 0, \cdots, m$ are maximal intervals of constant inertia for $S(A, B; \mathbb{R})$ and the inertia of each matrix in $S(A, B; T_i)$ is of the form $(\pi_i, \nu_i, 0)$, $i = 0, \cdots, m$.*

Proof. By standard results in analysis, T is an interval of constant inertia for $S(A, B; \mathbb{R})$ if and only if T contains no inertia change point for $S(A, B; \mathbb{R})$ and so the first part of the theorem follows. The second part of the theorem follows from the equivalence of (ie) and (ic) in Theorem 3.3. □

4. Inertia of intervals. In this section we apply the observations made in the previous section in order to study the relation between global conditions. The global conditions correspond to the negations of the local conditions in the previous section. In these global conditions as well as in the rest of the paper T denotes an interval.

The equivalent conditions

(A_1) $A^{-1}B$ has no eigenvalue with negative reciprocal in T.

(A_2) $I + tA^{-1}B$ is nonsingular for every t in T.

(A_3) $A + tB$ is nonsingular for every t in T.

will be labeled condition (A).

The equivalence of the following four conditions follows from the equivalence of (la_1)–(la_4):

(LA_1) $L(A)+tL(B)$ is nonsingular for every t in T.

(LA_2) $I+tL(A)^{-1}L(B)$ is nonsingular for every t in T.

(LA_3) $L(A)^{-1}L(B)$ has no eigenvalue with negative reciprocal in T.

(LA_4) $L(A+tB)$ is nonsingular for every t in T.

These conditions will be labeled condition (LA).

We also consider the conditions

(IE) $A+tB$ has no imaginary eigenvalue for any t in T.

(CI) T is an interval of constant inertia $(\pi,\nu,0)$ for $S(A,B;\mathbb{R})$.

THEOREM 4.1. *Let A and B be $n\times n$ complex matrices, let T be an interval, and assume that $L(A)$ is nonsingular. Then we have*

$$(\mathrm{LA})\rightarrow(\mathrm{IE})\Leftrightarrow(\mathrm{CI})\rightarrow(\mathrm{A}).$$

Proof. In view of Theorem 3.3 it is enough to prove the equivalence $(\mathrm{IE})\Leftrightarrow(\mathrm{CI})$. From Theorem 3.3 and the proof of Theorem 3.5 it follows that (IE) implies that $S(A,B;T)$ has constant inertia. By (IE) it follows that the inertia is of type $(\pi,\nu,0)$. The implication $(\mathrm{CI})\rightarrow(\mathrm{IE})$ is trivial. □

By adding additional requirements, some of the implications in Theorem 4.1 become equivalences, as we will show presently.

THEOREM 4.2. *Let A and B be $n\times n$ complex matrices, let T be an interval, assume that $L(A)$ is nonsingular, and assume that $A+tB$ is stable for some t in T. Then we have*

$$(\mathrm{LA})\Leftrightarrow(\mathrm{IE})\Leftrightarrow(\mathrm{CI})\rightarrow(\mathrm{A}).$$

Proof. In view of Theorem 4.1 it is enough to prove the implication $(\mathrm{CI})\rightarrow(\mathrm{LA})$. Observe that under our additional assumption, (CI) implies that $A+tB$ is stable for every t in T. By the implication $(\mathrm{la})\rightarrow(\mathrm{us})$ in Theorem 3.3 we now obtain (LA). □

The following theorem is found in [2] and [3] for real matrices.

THEOREM 4.3. *Let A and B be $n\times n$ complex matrices, and assume that A is positive stable.*

(i) *If $L(A)^{-1}L(B)$ has no real eigenvalue, then $A+tB$ is stable for every real number t.*

(ii) *If $L(A)^{-1}L(B)$ has real eigenvalues, then let r_1 and r_2 be the greatest and the least real eigenvalues of $L(A)^{-1}L(B)$. Define*

$$t_1=\begin{cases}-\dfrac{1}{r_1}, & r_1>0,\\ -\infty, & r_1\leqq 0,\end{cases}$$

$$t_2=\begin{cases}-\dfrac{1}{r_2}, & r_2<0,\\ \infty, & r_2\geqq 0.\end{cases}$$

Then the interval $T=(t_1,t_2)$ is the maximal interval of constant inertia $(n,0,0)$ that contains the point $t=0$.

Proof. Part (i) follows immediately from the equivalence (LA) $\Leftrightarrow$ (CI) in Theorem 4.2.

(ii) Observe that $L(A)^{-1}L(B)$ has no real eigenvalue in $T_1 = (-\infty, -1/t_2)$, nor in $T_2 = (-1/t_1, \infty)$. Therefore, $L(A)^{-1}L(B)$ has no real eigenvalue with negative reciprocal in $(0, t_2)$ or in $(t_1, 0)$. By Theorem 4.2 it follows that $(t_1, 0)$ and $(0, t_2)$ are intervals of constant inertia $(\pi, \nu, 0)$ for $S(A, B; \mathbb{R})$. Since A is stable, it follows from Theorem 3.3 that zero is not an inertia change point for $S(A, B; \mathbb{R})$. Hence, it follows that $T = (t_1, t_2)$ is an interval of constant inertia $(n, 0, 0)$ that contains the point $t = 0$. If $t_1 \neq -\infty$, then it follows that $-1/t_1$ is an eigenvalue of $L(A)^{-1}L(B)$, and by Theorem 4.2 $[t_1, t_2)$ is not an interval of constant inertia $(n, 0, 0)$. Similarly, if $t_2 \neq \infty$, then it follows that $-1/t_2$ is an eigenvalue of $L(A)^{-1}L(B)$, and by Theorem 4.2, $(t_1, t_2]$ is not an interval of constant inertia $(n, 0, 0)$. The maximality of T follows. □

DEFINITION 4.4. A square matrix A is said to have *Property X* if the minimal real part of an eigenvalue of A is an eigenvalue of A.

For example, Hermitian matrices and Z-matrices have Property X.

THEOREM 4.5. *Let A and B be $n \times n$ complex matrices, let T be an interval, assume that $L(A)$ is nonsingular, assume that $A + tB$ is positive stable for some t in T, and assume that $A + tB$ has Property X for every t in T. Then we have*

$$(\mathrm{CI}) \Leftrightarrow (\mathrm{LA}) \Leftrightarrow (\mathrm{IE}) \Leftrightarrow (\mathrm{A}).$$

Proof. Since $A + tB$ is positive stable for some t in T, and since $A + tB$ has Property X for every t in T, it follows, using continuity arguments (see Lemma 2.6(iv)) that $(\mathrm{A}_3) \rightarrow (\mathrm{IE})$. So $(\mathrm{A}) \rightarrow (\mathrm{IE})$, and our claim follows from Theorem 4.2. □

THEOREM 4.6. *Let A and B be $n \times n$ complex matrices, let T be an interval, assume that $L(A)$ is nonsingular, and assume that all eigenvalues of $A + tB$ are real for every t in T. Then we have*

$$(\mathrm{LA}) \rightarrow (\mathrm{IE}) \Leftrightarrow (\mathrm{CI}) \Leftrightarrow (\mathrm{A}).$$

Proof. The implication $(\mathrm{A}_3) \rightarrow (\mathrm{CI})$ follows immediately by continuity (see Lemma 2.6(iii)). So $(\mathrm{A}) \rightarrow (\mathrm{CI})$, and the claim follows from Theorem 4.1. □

We now consider matrix intervals with the same inertia except for a finite number of points.

First, we restate the implications (ic) $\rightarrow$ (la) and (a) $\rightarrow$ (ic) of Theorem 3.3 in a somewhat different form together with a partial converse.

PROPOSITION 4.7. *Let A and B be $n \times n$ complex matrices and assume that $L(A)$ is nonsingular. Let G be the set of inertia change points for $S(A, B; \mathbb{R})$, and let t be a nonzero number.*

(i) *If $t \in G$, then $-1/t$ is an eigenvalue of $L(A)^{-1}L(B)$.*

(ii) *If $-1/t$ is an eigenvalue of $A^{-1}B$, then $t \in G$.*

The converses of Proposition 4.7(i) and (ii) do not hold in general. We give a counterexample to the converse of Proposition 4.7(i).

Example 4.8. Consider the matrices

$$A = \begin{pmatrix} 1 & 0 \\ 0 & -2 \end{pmatrix}, \qquad B = \begin{pmatrix} 2 & 0 \\ 0 & -1 \end{pmatrix}.$$

Observe that $L(A)$ is nonsingular, and that $A + tB$ has the inertia $(1, 1, 0)$ for all t in $[-0.5, \infty)$ except $t = -0.5$. However, it is easy to verify that $L(A + B)$ is singular and hence, by the equivalence of conditions (la_3) and (la_4), $-1/t$ is an eigenvalue of $L(A)^{-1}L(B)$ also for $t = 1$.

Proposition 4.7 does not give necessary and sufficient conditions for a point t to belong to the exceptional set G of inertia change points for $S(A, B; \mathbb{R})$. However it does lead to a finite algorithm for finding these points.

ALGORITHM 4.9. For the sake of simplicity, we assume that $L(A)$ is nonsingular.
Step 1. Find the real nonzero eigenvalues of $L(A)^{-1}L(B)$.
Step 2. Take the negative reciprocals $t_1, \cdots, t_m$ of the numbers found in Step 1. The inertia change points for $S(A, B; \mathbb{R})$ are those t_i, $i \in \{1, \cdots, m\}$, for which $A + t_iB$ has an imaginary eigenvalue.

Necessary and sufficient conditions for a point t to be an inertia change points for $S(A, B; \mathbb{R})$ may be obtained under additional assumptions, as will be demonstrated in the sequel.

First we consider intervals of semistability.

THEOREM 4.10. *Let A and B be $n \times n$ complex matrices, and assume that $L(A)$ is nonsingular. Let T be an interval of semistability for $S(A, B; \mathbb{R})$. Then*

(i) *For $t \in T$, $A + tB$ is near-stable if and only if $t \neq 0$ and $-1/t$ is an eigenvalue of $L(A)^{-1}L(B)$.*

(ii) *All the eigenvalues of $L(A)^{-1}L(B)$ whose negative reciprocals lie in the interior of T have even multiplicity.*

(iii) *If A and B are real, then all the eigenvalues of $A^{-1}B$ whose negative reciprocals lie in the interior of T have even multiplicity.*

Proof. (i) If T consists of one point t_0 then, since $A + t_0B$ is semistable, it follows by Proposition 2.5(ii) that $A + t_0B$ is near stable if and only if $L(A + t_0B)$ is singular (so $t_0 \neq 0$ since $L(A)$ is nonsingular), which is true if and only if $-1/t_0$ is an eigenvalue of $L(A)^{-1}L(B)$. If T consists of more than one point, then it consists of infinitely many points. By Theorem 3.3, every t for which $A + tB$ is near stable is an inertia change point for $S(A, B; \mathbb{R})$. In view of Corollary 3.4, $A + tB$ is stable for all $t \in T$ except for a finite number of t's. Part (i) now follows immediately from the equivalence (LA) $\Leftrightarrow$ (CI) in Theorem 4.2.

(ii) Let λ be an eigenvalue of $L(A)^{-1}L(B)$ whose negative reciprocal lies in the interior of T, and let m be its multiplicity. Let Γ be a disc with center at λ that contains no other eigenvalue of $L(A)^{-1}L(B)$, and such that the negative reciprocals of real numbers in Γ lie in T. Without loss of generality assume that $A + tB$ is positive semistable for every t in T. Since $L(A)$ is nonsingular, it follows that for all sufficiently small positive δ, $L(A + \delta I)$ is nonsingular. For such δ, $(A + \delta I) + tB$ is positive stable for all t in T. By Theorem 4.2, the operator $F(\delta) = L(A + \delta I)^{-1}L(B)$ has no eigenvalue with negative reciprocal in T. Since $F(\delta)$ is an operator on the *real* space $\mathscr{H}_n$, its complex eigenvalues appear in conjugate pairs. Consequently, $F(\delta)$ has an even number of eigenvalues in Γ. By Lemma 2.6(i) it now follows that the multiplicity of λ as an eigenvalue of $L(A)^{-1}L(B)$ is even.

(iii) Let δ be a positive number. If A and B are real then $C(\delta) = (A + \delta I)^{-1}B$ is real, and hence the complex eigenvalues of $C(\delta)$ appear in conjugate pairs. By Theorem 3.2, if $-1/t$ is an eigenvalue of $(A + \delta I)^{-1}B$, then $-1/t$ is an eigenvalue of $L(A + \delta I)^{-1}L(B)$. As in the proof of part (ii), for δ sufficiently small, $L(A + \delta I)^{-1}L(B)$ has no eigenvalue with negative reciprocal in T. Therefore, $(A + \delta I)^{-1}B$ has no eigenvalue with negative reciprocal in T. Since the complex eigenvalues of $C(\delta)$ appear in conjugate pairs, it follows that $C(\delta)$ has an even number of eigenvalues in Γ. By Lemma 2.6(i) it now follows that the multiplicity of λ as an eigenvalue of $A^{-1}B$ is even. $\square$

Remark 4.11. In general, if $A + tB$ is near stable then the multiplicity of $-1/t$ as an eigenvalue of $L(A)^{-1}L(B)$ is not necessarily even. For example, take A to be an identity matrix of odd order, and let $B = A$. Then $L(A)^{-1}L(B)$ is an identity matrix of odd order and hence its only eigenvalue, 1, has odd multiplicity. Yet, $A - B$ is near stable.

Next we assume that all eigenvalues of $A + tB$ are real for t in an interval T. The following result is essentially due to Väliaho [7], where it is stated for Hermitian matrices. It is stated here for the sake of completeness.

THEOREM 4.12. *Let A and B be complex $n \times n$ matrices and assume that A is nonsingular. Assume that all eigenvalues of $A + tB$ are real for all $t \in \mathbb{R}$. Let $-1/t_i$, $i = 1, \cdots, m$, where $t_1 < \cdots < t_m$, be the distinct nonzero eigenvalues of $A^{-1}B$. Let $T_0 = (-\infty, t_1)$, $T_i = (t_i, t_{i-1})$, $i = 1, \cdots, m$, and $T_m = (t_m, \infty)$. Then the intervals T_i, $i = 0, \cdots, m$ are maximal intervals of constant inertia for $S(A, B; \mathbb{R})$ and the inertia of each matrix in $S(A, B; T_i)$ is of the form $(\pi_i, \nu_i, 0)$, $i = 0, \cdots, m$.*

Proof. As in Theorem 3.5, by standard results in analysis, T is an interval of constant inertia for $S(A, B; \mathbb{R})$ if and only if T contains no inertia change point for $S(A, B; \mathbb{R})$. By Theorem 3.3, if $-1/t$ is an eigenvalue of $A^{-1}B$, then t is an inertia change point for $S(A, B; \mathbb{R})$. Our claim now follows from Theorem 4.6. □

5. Stable convex hull of matrices.

The results of the previous section can be applied in several directions. We conclude the paper by demonstrating a sample of such applications.

The following result was proved for real matrices in [1] and [2].

THEOREM 5.1. *Let A and B be $n \times n$ complex matrices. Then the convex hull* conv (A, B) *is stable if and only if A is stable and $L(A)^{-1}L(B)$ has no nonpositive real eigenvalue.*

Proof. If A is stable, it follows from the equivalence (LA) ⇔ (CI) in Theorem 4.2, applied to the matrices A and $B - A$ and the interval $T = [0, 1]$, that conv (A, B) is stable if and only if $L(A)^{-1}L(B - A)$ has no real eigenvalue less than -1, which is equivalent to saying that $L(A)^{-1}L(B)$ has no nonpositive real eigenvalue. Since the stability of conv (A, B) of course implies that A is stable, the result now follows. □

THEOREM 5.2. *Let A and B be $n \times n$ complex matrices, and assume that all the matrices in* conv (A, B) *have Property X. Then the following are equivalent.*

(i) *The convex hull* conv (A, B) *is stable.*

(ii) *A is stable and $L(A)^{-1}L(B)$ has no nonpositive real eigenvalue.*

(iii) *A is stable and $A^{-1}B$ has no nonpositive real eigenvalue.*

Proof. Our claim follows from the equivalences (A) ⇔ (LA) ⇔ (CI) in Theorem 4.5, applied to the matrices A and $B - A$ and the interval $[0, 1]$. □

The following theorem is found in [4], where it is stated for Hermitian matrices.

THEOREM 5.3. *Let A and B be $n \times n$ complex matrices, and assume that all the matrices in* conv (A, B) *have all eigenvalues real. Then the following are equivalent.*

(i) *A is nonsingular and $A^{-1}B$ has no nonpositive real eigenvalue.*

(ii) *All matrices in* conv (A, B) *are nonsingular.*

(iii) conv (A, B) *has constant inertia of type* $(\pi, \nu, 0)$.

Proof. (i) ⇔ (ii) follows from the equivalence of conditions (A_1) and (A_2) applied to the matrices A and $B - A$ and the interval $T = [0, 1]$.

(ii) ⇒ (iii) by Lemma 2.6 (iii), since all matrices in conv (A, B) have all eigenvalues real.

(iii) ⇒ (ii) is trivial. □

We end with an example that illustrates Theorem 5.2 and the analogue for the convex hulls of Theorem 4.10(iii).

Example 5.4. Let

$$A=\begin{pmatrix} 2 & -3 \\ -1 & 2 \end{pmatrix}=B^T.$$

Then it is easy to show that all matrices in conv (A, B) are M-matrices and hence are semistable. Furthermore, each matrix in conv (A, B) is stable, except for $(A+B)/2$. Note that -1 is an eigenvalue of $A^{-1}B$ of multiplicity two. For every positive ε, conv $(A+\varepsilon I, B)$ is stable, and hence, by Theorem 5.2, $(A+\varepsilon I)^{-1}B$ has no nonpositive real eigenvalue. Indeed, the eigenvalues of $(A+.1I)^{-1}B$ are approximately $-.8 \pm .8775i$. Note also that every matrix in conv $(A+.1I, B)$ has Property X, but $(A+.1I)^{-1}B$ does not.

Acknowledgment. The authors are grateful to Professor David Carlson for his helpful comments, which helped to improve the presentation of the paper.

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