Linear and Multilinear Algebra, 1991, Vol. 29, pp. 21–42
Reprints available directly from the publisher
Photocopying permitted by license only

Printed in the United States of America

Combinatorial Bases, Derived Jordan Sets and the Equality of the Height and Level Characteristics of an M-Matrix

DANIEL HERSHKOWITZ*
Mathematics Department, Technion–Israel Institute of Technology, Haifa 32000, Israel

HANS SCHNEIDER†
Mathematics Department, University of Wisconsin–Madison, Madison, Wisconsin 53706, U.S.A.

(Received February 26, 1990; in final form July 5, 1990)

We continue our series of papers on the graph theoretic spectral theory of matrices. Let A be an M-matrix. We introduce the concepts of combinatorial vectors and proper combinatorial vectors in the generalized nullspace $E(A)$ of A. We explore the properties of combinatorial bases for $E(A)$ and Jordan bases for $E(A)$ derived from proper combinatorial sets of vectors. We use properties of these bases to prove additional new conditions for the equality of the (spectral) height (or Weyr) characteristic and the (graph theoretic) level characteristic of A. We also explore the role of the Hall Marriage Condition, well structured graphs and their anchored chain decompositions in the study of the equality of the two characteristics.

1. INTRODUCTION

With this paper we continue the series of papers [6], [4], [7], [5], and [8] on the graph theoretic spectral theory of matrices. In these papers we put emphasis on the relation between the combinatorial structure and the spectral structures of the generalized nullspace of the eigenvalue 0 of an M-matrix (or, equivalently, of the generalized eigenspace of the spectral radius of a nonnegative matrix). In this topic, conditions for the equality of the height (Weyr) and level characteristics for the eigenvalue 0 of an M-matrix are of particular interest. This question was raised in Schneider [13], where conditions for the equality of the two characteristics are proved under some special hypotheses on the singular graph of the matrix. In the general case, necessary and sufficient conditions are found in Richman–Schneider [11], and in our recent paper [8], see [8] and the survey [14] for further information and references. Related results appear in Richman [10], Bru-Neumann [1], and Huang [9].

*The research was completed while the author was a visiting professor at the University of Wisconsin–Madison.
†The research of this author was supported in part by NSF grants DMS-8521521, ECSE 8718971 and DMS-8901445.
The research of both authors was supported also by their joint grant No. 85-00153 from the United States–Israel Binational Science Foundation (BSF), Jerusalem, Israel.

Let A be an M-matrix and let $E(A)$ be the generalized nullspace of A. Let $\eta(A)$ and $\lambda(A)$ be the height and level characteristics of A respectively. In [11] it was proved that $\eta(A) = \lambda(A)$ if and only if $E(A)$ has nonnegative Jordan basis for $-A$, and several other equivalent conditions were proved. In [8] we introduced the concepts of height and level bases for $E(A)$ and, using these concepts, we proved a number of other conditions equivalent to $\eta(A) = \lambda(A)$. In this paper we give further equivalent conditions bringing the total to 36. Some of the new conditions are stated in terms of the concept of combinatorial bases for $E(A)$ which we introduce here. Combinatorial bases are more general than the Rothblum bases found in [12] and, for example, in [1], as well as the preferred bases, found in [11], [6] and many other references. Later in this introduction we describe further main results of our paper. These involve the Hall marriage condition (first used in this particular context by [10]), well structured graphs and anchored chain decompositions of graphs (both of which were introduced in [1], the latter under the name of covering strategies).

We now describe our paper in more detail. Section 2 is devoted to notation and definitions. Here we give the definitions of the height of a vector and of the level of a vector in $E(A)$. We define peak vectors and peak bases, height bases and level bases for $E(A)$. We also define the height characteristic $\eta(A)$ and the level characteristic $\lambda(A)$.

In Section 3 we introduce (proper) combinatorial vectors, and (proper) combinatorial bases. We show that every combinatorial basis is a proper combinatorial basis and also a peak level basis, see Corollaries (3.15) and (3.17).

In the graph theoretic Section 4 we explore the role of the Hall Marriage Condition. In [10] it was shown that this condition (or equivalently the existence of systems of distinct representatives) for certain sets of predecessors in a graph S is equivalent to a condition on the combinatorial dual of the level characteristic $\lambda(A)$. Here we show these conditions hold if and only if S is well structured, see Theorem (4.13).

In Section 5 we apply the concepts and results developed in Sections 3 and 4 and we link graph theoretic and spectral properties. We show that every Jordan basis for $E(A)$ derived from a proper combinatorial set of vectors corresponds to an anchored chain decomposition such that the proper combinatorial set corresponds to the final elements of the chains in the decomposition, see Theorem (5.9). We show by an example that the converse is false.

In Section 6 we prove the conditions equivalent to $\eta(A) = \lambda(A)$ that have been mentioned earlier, see Theorem (6.6). One new equivalent condition, for example, is the existence of a Jordan basis for $E(A)$ derived from a proper combinatorial set of vectors. Another is that some combinatorial basis for $E(A)$ is a height basis. Other new equivalent conditions involve level bases, or height bases, and the linear independence of the fundaments of Jordan sets derived from certain subsets of the bases. We also show that some conditions that appear to have a similar flavor are in fact not equivalent to $\eta(A) = \lambda(A)$. Our Theorem (6.6) proves a somewhat stronger form of a theorem in [1] together with a converse that was conjectured there, see Remark (6.13). At the conclusion of our paper another conjecture on well structured graphs in [1] is proved as a corollary to a result which extends a theorem in [10], see our Theorem (6.14) and Corollary (6.15).

This paper and our paper [8] discuss conditions for the equality of the level characteristic and the height characteristic of an M-matrix. A more general question concerning the relation between the two characteristics for M-matrices is raised in [14]. Some results for a similar question for general matrices over an arbitrary field are found in [5] and [3]. We hope that the concepts of height bases, level bases, peak vectors and bases, and combinatorial vectors and bases, defined in [8] and in this paper, will prove useful in further study of these questions.

2. NOTATION AND DEFINITIONS

In this paper we always assume that A is an $n \times n$ matrix. Most of our results are on M-matrices (see Definition (2.27)). However some of them and almost all the definitions and notation in this section hold for general matrices over an arbitrary field. Some definitions and notation are given in the rest of the paper. Almost all the definitions and notation given in this section are given in [8], where some further explanations may be found.

(2.1) *Notation* For a positive integer n we denote by $\langle n \rangle$ the set $\{1, \ldots, n\}$.

(2.2) *Notation* For a set α we denote by $|\alpha|$ the cardinality of α.

(2.3) *Notation* For the matrix A we denote:

$N(A)$—the nullspace of A.

$n(A)$—the nullity of A (the dimension of $N(A)$.

$E(A)$—the generalized nullspace of A, viz. $N(A^n)$.

$m(A)$—the algebraic multiplicity of 0 as an eigenvalue of A (the dimension of $E(A)$).

index(A)—the index of 0 as an eigenvalue of A, viz., the size of the largest Jordan block associated with 0.

(2.4) *Definition* An $m \times n$ matrix is said to have *full column rank* if its rank equals n.

(2.5) *Definition* Let B and C be $m \times n$ matrices. We say that B and C *have the same zero pattern* if $b_{ij} = 0$ if and only if $c_{ij} = 0$ for all $i \in \langle m \rangle$, $j \in \langle n \rangle$.

(2.6) *Definition* For a vector x in $E(A)$ we define the *height* of x, denoted by height(x), to be the minimal nonnegative integer k such that $A^k x = 0$.

(2.7) *Definition* Let $p = \text{index}(A)$. For $i \in \langle p \rangle$ let $\eta_i(A) = n(A^i) - n(A^{i-1})$. The sequence $(\eta_1(A), \ldots, \eta_p(A))$ is called the *height characteristic* of A, and is denoted by $\eta(A)$. Normally we write η_i for $\eta_i(A)$ where no confusion should result.

We remark that in many references the height characteristic of a matrix A is called the *Weyr characteristic* of A, e.g. [13].

(2.8) *Definition* Let A be a square matrix and let $\text{index}(A) = p$.

(i) Let S be a set of vectors in $E(A)$, and let $\eta_k(S)$ be the number of vectors in S of height k. We define the ***height signature*** $\eta(S)$ of S as the p-tuple $(\eta_1(S), \eta_2(S), \ldots, \eta_q(S))$.

(ii) A basis B for $E(A)$ is said to be a ***height basis*** for $E(A)$ if $\eta(B) = \eta(A)$.

(2.9) *Definition* Let A be a singular matrix.

(i) A sequence $(x^1, \ldots, x^t)$ of vectors in $E(A)$ is said to be a *Jordan chain* for A if $Ax^i = x^{i-1}$, $i \in \{2, \ldots, t\}$, and $Ax^1 = 0$. We call x^t the *top* of the chain $(x^1, \ldots, x^t)$.

(ii) A basis for $E(A)$ that consists of disjoint Jordan chains for A is said to be a *Jordan basis* for $E(A)$.

As is well known, $E(A)$ always has a Jordan basis.

(2.10) *Remark* Observe that every Jordan basis for A is a height basis, but clearly a height basis need not be a Jordan basis.

We continue with some graph theoretic definitions. All the graphs we deal with are simple directed graphs.

(2.11) *Definition* A graph G is said to be a *subgraph* of a graph H $(G \subseteq H)$ if G and H have the same vertex set, and if every arc of G is an arc of H.

(2.12) *Definition* Let G be a graph.

(i) Let i be a vertex of G. A vertex j is said to be a *predecessor* of i if $j = i$ or if there is a chain from j to i in G. The set of all predecessors of i is denoted by $\Delta_G(i)$.

(ii) Let T be a set of vertices in G. We denote by $\Delta_G(T)$ the set $\bigcup_{i \in T} \Delta_G(i)$.

(iii) Normally we write $\Delta(i)$ and $\Delta(T)$ for $\Delta_G(i)$ and $\Delta_G(T)$ respectively where no confusion should result.

(2.13) *Notation* Let G be a graph, and let T be a set of vertices in G. We denote by $\text{top}(T)$ the set $\{i \in T : i \notin \Delta(T \setminus \{i\})\}$.

(2.14) *Definition* Let G be an acyclic graph, i.e., a graph that contains no simple cycle other than loops. Let i be a vertex of G. We define the *level* of i, $\text{level}(i)$, as the maximal length (number of vertices) of a simple chain in G that terminates at i. We call the set of all vertices of level j the *jth level* of G. Let G have q levels, and let λ_j be the cardinality of the jth level of G. The sequence $(\lambda_1, \ldots, \lambda_q)$ is called the *level characteristic* of G.

Let A be a square matrix over some field. As is well known, after performing an identical permutation on the rows and the columns of A we may assume that A is in *Frobenius normal form*, namely a (lower) triangular block form, where the diagonal blocks are square irreducible matrices.

(2.15) *Convention* We shall always assume that A is an $n \times n$ matrix in Frobenius normal form $(A_{ij})_1^r$. Also, every n-vector b will be assumed to be partitioned into r vector components b_i conformably with A.

(2.16) *Notation* For an n-vector b we denote by supp(b) the set $\{i \in \langle r \rangle : b_i \neq 0\}$.

(2.17) *Definition* The *reduced graph* $R(A)$ of A is defined to be the graph with vertices $1, \ldots, r$ and where (i, j) is an arc if and only if $A_{ij} \neq 0$. Note that $R(A)$ is acyclic.

(2.18) *Definition* A vertex i of $R(A)$ is said to be *singular* if A_{ii} is singular. The set of all singular vertices of $R(A)$ is denoted by S.

(2.19) *Definition* The *singular graph* $S(A)$ of A is defined to be the graph with the vertex set S, and where (i, j) is an arc if and only if $i \in \Delta_{R(A)}(j)$. Note that $S(A)$ is a transitive acyclic graph.

(2.20) *Definition* Let b be an n-vector. The *level* of b, denoted by level(b), is defined to be the maximal level in $S(A)$ of a singular vertex i such that $b_i \neq 0$.

(2.21) *Definition* A vector $x \in E(A)$ is said to be a *peak vector* if height(x) = level(x). A subset of $E(A)$ that consists of peak vectors is called a *peak set* of vectors. A basis for $E(A)$ that consists of peak vectors is called a *peak basis* for $E(A)$.

(2.22) *Definition* The cardinality of the jth level of $S(A)$ is denoted by $\lambda_j(A)$. Let $S(A)$ have q levels. The level characteristic $(\lambda_1(A), \ldots, \lambda_q(A))$ of $S(A)$ is called the *level characteristic* of A, and is denoted by $\lambda(A)$. Normally we write λ_i for $\lambda_i(A)$ where no confusion should result.

(2.23) *Convention* We shall always assume that the levels of $S(A)$ are $L_1, \ldots, L_q$. The level characteristic of A will be assumed to be $(\lambda_1, \ldots, \lambda_q)$. The height characteristic of A will be assumed to be $(\eta_1, \ldots, \eta_p)$.

(2.24) *Definition*

- (i) Let S be a set of vectors in $E(A)$, and let $\lambda_k(S)$ be the number of vectors in S of level k. We define the *level signature* $\lambda(S)$ of S as the q-tuple $(\lambda_1(S), \lambda_2(S), \ldots, \lambda_q(S))$.
- (ii) A basis B for $E(A)$ is said to be a *level basis* for $E(A)$ if $\lambda(B) = \lambda(A)$.

(2.25) *Remark* Usually we order a level basis such that the levels of the vectors are non-increasing.

(2.26) *Definition* A basis B for $E(A)$ is said to be a *height-level basis* for $E(A)$ if B is both a height basis and a level basis for $E(A)$.

(2.27) *Definition* A *Z-matrix* is a square matrix of the form $A = \alpha I - P$, where α is a real number and P is a (entrywise) nonnegative matrix. Such a Z-matrix is an *M-matrix* if α is greater than or equal to the spectral radius of P.

(2.28) *Remark* It is well known that for an M-matrix A we have $p = q$, see [12], [11].

(2.29) *Definition* A vector b is said to be *(strictly) positive* ($b \gg 0$) if all its entries are positive.

(2.30) *Definition* Let H be a set of vertices in $R(A)$, and let $h = |H|$. A set of vectors $\{x^i, i \in H\}$ is said to be an *H-preferred set* (for A) if

$$\left.\begin{array}{lll} x_j^i \gg 0 & \text{if} & i \in \Delta_{R(A)}(i) \\ x_j^i = 0 & \text{if} & j \notin \Delta_{R(A)}(i) \end{array}\right\} \quad i \in H, \quad j \in \langle r \rangle,$$

and

$$-Ax^i = \sum_{i \in H} c_{ik} x^k, \qquad i \in H,$$

where the c_{ik} satisfy

$$\left.\begin{array}{ll} c_{ik} > 0 & \text{if } \; k \in \Delta_{R(A)}(i) \backslash \{i\} \\ c_{ik} = 0 & \text{otherwise} \end{array}\right\} \quad i \in H.$$

(2.31) *Definition* Let H be a set of vertices in $R(A)$. An H-preferred set that forms a basis for a vector space V is called an *H-preferred basis* for V. An S-preferred basis for $E(A)$ (if exists) is called a *preferred basis* for A.

(2.32) *Remark* By the Preferred Basis Theorem (see paper [6] and the references there), if A is an M-matrix then there exists a preferred basis for $E(A)$.

3. COMBINATORIAL BASES

(3.1) *Definition* Let A be an M-matrix, and let i be a singular vertex in $R(A)$.

(i) A vector x in $E(A)$ is said to be an *i-combinatorial* vector if $\text{supp}(x) \subseteq \Delta_{R(A)}(i)$.
(ii) An i-combinatorial vector x is said to be a *proper i-combinatorial* vector if $x_i \neq 0$.
(iii) A vector in $E(A)$ is said to be a *combinatorial [proper combinatorial]* vector if it is an i-combinatorial [proper i-combinatorial] vector for some singular vertex i.

We remark that an i-combinatorial vector x is defined in [5] to be a *weak i-combinatorial extension.*

(3.2) *Definition* Let A be an M-matrix, and let T be a set of singular vertices in $R(A)$.

(i) A set $\{x^i : i \in T\}$ of vectors in $E(A)$ is said to be a *T-combinatorial set* if x^i is an i-combinatorial vector, $i \in T$.
(ii) A set $\{x^i : i \in T\}$ of vectors in $E(A)$ is said to be a proper *T-combinatorial set* if x^i is a proper i-combinatorial vector, $i \in T$.
(iii) A set of vectors in $E(A)$ is said to be a *combinatorial [proper combinatorial]* set of vectors if it is a T-combinatorial [proper T-combinatorial] set of vectors for some set T of singular vertices.

(3.3) *Remark* Observe that every weakly preferred basis for $E(A)$, as defined in [5], is a proper S-combinatorial set of vectors.

(3.4) *Observation* Let x be an i-combinatorial vector. If x is a proper i-combinatorial vector then we have level(x) = level(i). Otherwise we have level(x) < level(i).

(3.5) *Observation* Let x be a proper i-combinatorial vector. Then the vector x_i is a nonzero vector in $E(A_{ii}) = N(A_{ii})$. Therefore, since $N(A_{ii})$ is one dimensional, it follows that x_i is a nonzero scalar multiple of the unique unit (length 1) positive nullvector of A_{ii}.

(3.6) PROPOSITION *Every proper combinatorial vector is a peak vector.*

Proof Let x be a proper i-combinatorial vector. Let $\mathcal{B}$ be a preferred basis for $E(A)$. By Remark (3.3), there exists a proper i-combinatorial vector y in $\mathcal{B}$. Denote by k the level of i. By observation (3.5) we can find a scalar c such that the vector $z_i = (x + cy)_i = 0$, and hence the vector $z = x + cy$ is an i-combinatorial vector but not a proper i-combinatorial vector. By Observation (3.4), the level of z is less than k, and by Corollary (4.17) in [8], height(z) < k. Since height(y) = k, it follows that height(x) = height($z - cy$) = k = level(x). ■

(3.7) COROLLARY *Every proper combinatorial set of vectors is a peak set.*

The following elementary lemma is proven as Lemma (3.1) in [7].

(3.8) LEMMA *For every vector x we have* $\operatorname{supp}(Ax) \subseteq \Delta_{R(A)}(\operatorname{supp}(x))$.

(3.9) PROPOSITION *Let T be a set of singular vertices in $R(A)$, let $\mathcal{T} = \{x^i : i \in T\}$ be a proper T-combinatorial set of vectors, let c_i, $i \in T$, be nonzero scalars, and let*

$$y = \sum_{i\in T} c_i x^i.$$

Then:

(i) $y_j \neq 0$ for every $j \in \operatorname{top}(T)$.
(ii) level(y) = max{level(x^i) : $i \in T$}.
(iii) $(Ay)_j = 0$ for every $j \in \operatorname{top}(T)$.
(iv) level(Ay) < level(y).

Proof
(i) Clearly, $y_j = c_j x^j_j \neq 0$ for every $j \in \operatorname{top}(T)$.
(ii) follows from (i).
(iii) follows from Observation (3.5).
(iv) follows from Lemma (3.8) and (iii). ■

(3.10) THEOREM *Every proper combinatorial set of vectors is linearly independent.*

Proof Let $\mathcal{T}$ be a proper T-combinatorial set of vectors. It follows from Proposition (3.9) that every nontrivial linear combination of elements of $\mathcal{T}$ is not equal to zero. Our claim follows. ■

(3.11) *Definition* A basis for $E(A)$ which is a combinatorial [proper combinatorial] set of vectors is said to be a *combinatorial [proper combinatorial] basis* for $E(A)$.

(3.12) PROPOSITION *Every proper combinatorial set of vectors can be completed to a proper combinatorial basis for* $E(A)$.

Proof Let $T \subseteq S$, and let $\mathcal{T}$ be a proper T-combinatorial set of vectors. By Lemma (3.2) in [5], for every $i \in S$ there exists a proper i-combinatorial vector x^i in $E(A)$. Let $\mathcal{S}$ be a proper S-combinatorial set of such vectors. Observe that the union $\mathcal{B}$ of $\mathcal{T}$ and the proper $(S \backslash T)$-combinatorial subset of $\mathcal{S}$ is a proper S-combinatorial set of vectors. Since the cardinality of $\mathcal{B}$ is $m(A)$, it follows from Theorem (3.10) that $\mathcal{B}$ is the required basis. ■

(3.13) PROPOSITION *Every combinatorial basis for* $E(A)$ *is an S-combinatorial set of vectors.*

Proof Let $\mathcal{B}$ be a combinatorial basis for $E(A)$. By Definition (3.11), $\mathcal{B}$ is a T-combinatorial set of vectors for some $T \subseteq S$. By Definition (3.2), the cardinality of $\mathcal{B}$ is $|T|$. Since the cardinality of a basis for $E(A)$ is $m(A) = |S|$, it now follows that $T = S$. ■

(3.14) THEOREM *Let* $\mathcal{B} = \{x^i : i \in S\}$ *be a combinatorial basis for* $E(A)$. *Then* x^i *is a proper i-combinatorial vector,* $i \in S$.

Proof Let $i \in S$ be of minimal level k such that the i-combinatorial vector x^i is not a proper i-combinatorial vector. It follows from Observation (3.4) that

$$\sum_{j=1}^{k-1} \lambda_j(A) < \sum_{j=1}^{k-1} \lambda_j(\mathcal{B}),$$

which contradicts (4.27) in [8]. ■

(3.15) COROLLARY *A combinatorial basis* $\{x^i : i \in S\}$ *for* $E(A)$ *is a proper combinatorial basis.*

(3.16) *Remark* It follows from Theorem (3.14) and Corollary (3.15) that combinatorial bases and proper combinatorial bases are the same. Therefore, in general we use only the term "combinatorial basis".

(3.17) COROLLARY *Every combinatorial basis for* $E(A)$ *is a peak level basis.*

Proof Let $\mathcal{B}$ be a combinatorial basis for $E(A)$. It follows from Theorem (3.14) and Corollary (3.7) that $\mathcal{B}$ is a peak basis. It follows from Theorem (3.14) and Observation (3.4) that $\mathcal{B}$ is a level basis. ■

(3.18) PROPOSITION *Let* $\mathcal{B} = \{x^i : i \in S\}$ *be a combinatorial basis for* $E(A)$, *and let* $x \in E(A)$. *The coefficients* c_i *in the expression*

$$x = \sum_{i \in S} c_i x^i$$

satisfy $c_i = 0$ *whenever* $i \notin \Delta_{R(A)}(\mathrm{supp}(x))$.

Proof Let $T = \{i \in S : c_i \neq 0\}$. By Proposition (3.9.i), top(T) $\subseteq$ supp(x). Hence, $T \subseteq \Delta_{R(A)}(\text{top}(T)) \subseteq \Delta_{R(A)}(\text{supp}(x))$, and the result follows. ■

(3.19) THEOREM *Let $\mathcal{B} = \{x^i : i \in S\}$ be a combinatorial basis for $E(A)$, and let $j \in S$. The coefficients c_i in the expression*

$$Ax^j = \sum_{i \in T} c_i x^i$$

satisfy $c_i = 0$ whenever $i = j$ or $i \notin \Delta_{R(A)}(j)$.

Proof Let $T = \{i \in S : c_i \neq 0\}$. By Proposition (3.18) we have $T \subseteq \text{supp}(Ax^j)$. Since, by Lemma (3.8), we have $\text{supp}(Ax^j) \subseteq \Delta_{R(A)}(\text{supp}(x^j)) = \Delta_{R(A)}(j)$, it now follows that $T \subseteq \Delta_{R(A)}(j)$. We now show that $j \notin T$. Assume that $j \in T$. Then, since $T \subseteq \Delta_{R(A)}(j)$, it follows that $j \in$ top(T). By Proposition (3.9.i) it implies that $(Ax^j)_j \neq 0$. However, it follows from Proposition (3.9.iii) that $(Ax^j)_j = 0$. This contradiction yields that $j \notin T$. ■

(3.20) *Remark* In view of Remark (3.3), Theorem (3.19) proves that for M-matrices, combinatorial bases and weakly preferred bases coincide.

We continue with the definition of induced matrices, as defined in Definition (7.1) in [8]. Induced matrices occur in [11] under the name of S-matrices. We shall use this definition in the sequel.

(3.21) *Definition* Let A be an $n \times n$ matrix and let $\mathcal{B} = \{x^1, \ldots, x^{m(A)}\}$ be a basis for $E(A)$.

(i) We define the corresponding *basis matrix* to be the $n \times m(A)$ matrix whose columns are $x^1, \ldots, x^{m(A)}$. We normally denote this matrix by B.

(ii) Clearly there exists a unique $m(A) \times m(A)$ matrix C such that $AB = BC$. We call this matrix the *induced matrix for A by $\mathcal{B}$*, and we denote it by $C(A, \mathcal{B})$.

(3.22) *Observation* (see Observation (6.6) in [8]) Let $(\lambda_1, \ldots, \lambda_p)$ be the level characteristic of an M-matrix A, and let $\mathcal{B}$ be a level basis for A. We partition C into a $p \times p$ block matrix where the ith diagonal block is a $\lambda_{p+1-i} \times \lambda_{p+1-i}$ matrix. By Lemma (4.13) in [8], for every nonzero element x of $E(A)$ we have level(Ax) < level(x). Therefore, C in its block form appears as

$$C = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ C_{p-1,p} & 0 & & 0 \\ & & \ddots & \\ \vdots & \ddots & 0 & \vdots \\ C_{1p} & \cdots & C_{12} & 0 \end{pmatrix}.$$

Observe that our indexing of blocks is unusual. However, it is natural for this problem and it is consistent with the indexing used in [11] and [8]. It follows from the definition of a preferred basis that if $\mathcal{B}$ is a preferred basis then the blocks $C_{12}, \ldots, C_{p-1,p}$ have no zero columns.

We conclude the section with a corollary that follows immediately from Theorem (3.19) and from the definition of induced matrices.

(3.23) COROLLARY *Let B be a combinatorial basis for $E(A)$, and let $C = C(A,B)$. Then $G(C)$ is a subgraph of $S(C)$, which is a subgraph of $S(A)$ (after relabelling of vertices).*

4. THE HALL CONDITION

In this section we show the relation of the Hall Marriage Condition to the concept of a well structured graph as defined in [1], and to an equivalent condition in [10].

We first state Hall's marriage Theorem essentially as it is found in [2, p. 155].

(4.1) THEOREM *Let $E_1,\ldots,E_h$ be subsets of a given set E. Then the following are equivalent:*

(i) *We have*

$$\left|\bigcup_{i\in\alpha} E_i\right| \geq |\alpha|, \qquad \textit{for all} \quad \alpha \subseteq \langle h\rangle. \tag{4.2}$$

(ii) *There exist distinct elements $e_1,\ldots,e_h$ of E such that $e_i \in E_i$, $i \in \langle h\rangle$.*

The condition (4.2) is often referred to as the Hall Marriage Condition. We refer to the equivalent condition (ii) as the SDR *(system of distinct representatives)* Condition.

(4.3) COROLLARY *Let Q be an $m \times n$ matrix, and define the sets E_j, $j \in \langle n\rangle$, by $E_j = \{i \in \langle m\rangle : q_{ij} \neq 0\}$. Then the following are equivalent:*

(i) *There exists a nonnegative matrix C of full column rank which has the same zero pattern as Q.*
(ii) *There exists a matrix C of full column rank which has the same zero pattern as Q.*
(iii) *The sets $E_1,\ldots,E_n$ satisfy the Hall Marriage Condition.*

Proof

(i) ⇒(ii) is obvious.

(ii) ⇒(iii) Let C be a matrix satisfying (ii). Then C has a nonsingular $n \times n$ submatrix, which implies that there exist distinct $e_1,\ldots,e_n$ in $\langle m\rangle$ such that $c_{e_jj} \neq 0$. Hence also $q_{e_jj} \neq 0$, and the sets $E_1,\ldots,E_n$ satisfy the SDR Condition. Our claim follows by Theorem (4.1).

(iii) ⇒(i) If the sets $E_1,\ldots,E_n$ satisfy the Hall Marriage Condition, then by Theorem (4.1) we can find distinct $e_1,\ldots,e_n$ in $\langle m\rangle$ such that $q_{e_jj} \neq 0$. We set $c_{e_jj} = 1$, $j \in \langle n\rangle$, we set $c_{ij} = \epsilon > 0$ whenever $q_{ij} \neq 0$ and $i \neq e_j$, $i \in \langle m\rangle$, $j \in \langle n\rangle$, and we set all other entries of C equal to 0. If ϵ is sufficiently small, then it is clear that C has a nonsingular $n \times n$ submatrix, and hence C has full column rank. ■

(4.4) *Definition* Let S be an acyclic graph. A chain $(i_1,\ldots,i_t)$ is called an *anchored chain* if the level of i_k is k, $k \in \langle t \rangle$.

(4.5) *Definition* Let S be an acyclic graph.

(i) A set κ of chains in S is said to be a *chain decomposition* of S if each vertex of S belongs to exactly one chain in κ.
(ii) A chain decomposition κ of S is said to be an *anchored chain decomposition* of S if every chain in κ is anchored.
(iii) S is said to be *well structured* if there exists an anchored chain decomposition of S.

We comment that the term "well structured" is essentially due to [1]. An anchored chain decomposition of S is called there a *covering strategy* for S.

In view of Definition (2.11), the following proposition is clear.

(4.6) PROPOSITION *Let G be a subgraph of a G'. If G is well structured then G' is well structured. Furthermore, every anchored chain decomposition of G is an anchored chain decomposition of G'.*

(4.7) THEOREM *Let S be an acyclic graph with levels $L_1,\ldots,L_q$, and let L_k' be a subset of L_k, $k \in \langle q-1 \rangle$. The following are equivalent:*

(i) *The sets $E_i' = \Delta(i) \cap L_k'$, $i \in L_{k+1}$, satisfy the Hall Marriage Condition for all $k \in \langle q-1 \rangle$.*
(ii) *S is well structured, and there exists an anchored chain decomposition κ for S such that all the elements in $L_k \backslash L_k'$, $k \in \langle q-1 \rangle$, are final elements of chains in κ.*

Proof

(i) $\Rightarrow$(ii) It follows from (i) that the sets $E_i = \Delta(i) \cap L_k$, $i \in L_{k+1}$, satisfy the Hall Marriage Condition, for all $k \in \langle q-1 \rangle$. We prove our assertion by induction on q. For $q = 1$ there is nothing to prove. Assume our claim holds for $q < h$, $h > 1$, and let $q = h$. Let S' be the acyclic graph obtained from S by removing the level L_q. By the inductive assumption there exists an anchored chain decomposition κ' for S' such that all the elements in $L_k \backslash L_k'$, $k \in \langle q-2 \rangle$, are final elements of chains in κ'. Clearly, all the elements in L_{q-1} are final elements of chains in κ'. Since the sets $E_i' = \Delta(i) \cap L_{q-1}'$, $i \in L_q$, satisfy the Hall Marriage Condition, it follows from Theorem (4.1) that we can find distinct predecessors in L_{q-1}' for the elements of L_q. Accordingly, we append the elements of L_q to the corresponding chains in κ' and we obtain an anchored chain decomposition κ for S which satisfies the conditions in (ii). By Definition (4.5), S is well structured.

(ii) $\Rightarrow$(i) This implication follows very easily by induction on the number q of levels of S, by use of the equivalence of the Hall Marriage Condition and the SDR Condition. The details are omitted. ■

(4.8) *Definition* Let $\mu = (\mu_1,\ldots,\mu_t)$ be a non-increasing sequence of positive integers. Consider the diagram formed by t columns of stars, such that the jth column (from the left) has μ_j stars. The sequence μ^* *dual* to μ is defined as the sequence of row lengths of the diagram (read upwards).

The following observation and proposition are well known.

(4.9) *Observation* Two equivalent definitions of μ^* are the following:

(i) Let $\mu = (\mu_1, \ldots, \mu_t)$ be a non-increasing sequence of positive integers, and let $s = \mu_1$. The sequence $\mu^* = (\mu_1^*, \ldots, \mu_s^*)$ *dual* to μ is defined by

$$\mu_k^* = \max\{i \in \langle t \rangle : \mu_i \geq k\}, \qquad k \in \langle s \rangle.$$

(ii) Let $\mu = (\mu_1, \ldots, \mu_t)$ be a non-increasing sequence of positive integers. The sequence μ^* *dual* to μ is the non-increasing sequence of positive integers contained in $\langle t \rangle$, where for every $k \in \langle t \rangle$, the number of elements of μ^* that are equal to k is equal to the difference $\mu_k - \mu_{k+1}$ (where μ_{t+1} is defined to be 0).

(4.10) PROPOSITION *Let $\mu^* = (\mu_1^*, \ldots, \mu_s^*)$ be the sequence dual to $\mu = (\mu_1, \ldots, \mu_t)$. Then*

(i) $\mu_1 + \cdots + \mu_t = \mu_1^* + \cdots + \mu_s^*$.
(ii) $(\mu^*)^* = \mu$.

(4.11) *Definition* The *length signature* $h(\kappa)$ of a set κ of chains in S is defined to be the sequence of the lengths of the chains in κ, ordered in a non-increasing order.

(4.12) THEOREM *Let S be an acyclic graph with level characteristic $\lambda = (\lambda_1, \ldots, \lambda_q)$, and let κ be a chain decomposition of S. Then the following are equivalent:*

(i) *κ is an anchored chain decomposition of S.*
(ii) *λ is a non-increasing sequence, and $h(\kappa) = \lambda^*$.*

Proof

(i) $\Rightarrow$(ii) If κ is an anchored chain decomposition of S then, in view of Definition (4.8), it is easy to verify that $h(\kappa)^* = \lambda$. Hence, λ is non-increasing. By Proposition (4.10.ii) it now follows that $h(\kappa) = \lambda^*$.

(ii) $\Rightarrow$(i) Clearly all chains of length q are anchored. Since $h(\kappa) = \lambda^*$, it follows that the number of such chains is λ_q. Therefore, the other chains form a chain decomposition for an acyclic graph with level characteristic $\lambda' = (\lambda_1 - \lambda_q, \ldots, \lambda_k - \lambda_q)$, where k is the greatest index such that $\lambda_k > \lambda_q$. Since $\lambda = h(\kappa)^*$, it follows from Definition (4.8) that λ' is the dual of the sequence obtained by eliminating the first λ_q elements of $h(\kappa)$. Our assertion now follows using induction. ■

(4.13) THEOREM *Let S be an acyclic graph with levels $L_1, \ldots, L_q$. Then the following are equivalent.*

(i) *S is well structured.*
(ii) *The sets $E_i = \Delta(i) \cap L_k$, $i \in L_{k+1}$, satisfy the Hall Marriage Condition, for all $k \in \langle q-1 \rangle$.*

(iii) *The level characteristic λ of S is a non-increasing sequence, and there exists a chain decomposition κ of S such that $h(\kappa) = \lambda^*$.*

Proof

(i) $\Leftrightarrow$(ii) follows from Theorem (4.7) with $L_k^t = L_k$, $k \in \langle q-1 \rangle$.

(i) $\Rightarrow$(iii) By Definition (4.5), (i) means that there exists an anchored chain decomposition of S, and (iii) follows by Theorem (4.12).

(iii) $\Rightarrow$(i) Let κ be a chain decomposition of S snch that $h(\kappa) = \lambda^*$. By Theorem (4.12), κ is an anchored chain decomposition of S, and by Definition (4.5), S is well structured. ■

We note that the equivalence of conditions (ii) and (iii) is already proven in [10]. Thus, Theorem (4.13) also follows from our Theorem (4.12) and Theorem 4.4 in [10], without the use of Theorem (4.7). However, the general case of Theorem (4.7) in full strength will be applied in the sequel, and it does not follow from Theorem (4.13).

5. JORDAN BASES DERIVED FROM PROPER COMBINATORIAL SETS

In this section we apply the graph theoretic tools developed in the previous section in order to obtain a necessary condition on the tops of the chains in a Jordan basis, provided the set of tops is a proper combinatorial set of vectors.

(5.1) *Definition* Let W_1 and W_2 be subspaces of a vector space V. We say that a vector z in V is in W_1 *modulo* W_2 if z can be written as $z = x + y$, where $x \in W_1$ and $y \in W_2$.

(5.2) LEMMA *Let $y^1, \ldots, y^m$ be elements of a Jordan basis $\mathcal{B}$ for A, all of same height t, all tops of chains. Then no nontrivial linear combination of $y^1, \ldots, y^m$ is in* Range(A) *modulo* $N(A^{t-1})$.

Proof Assume that there exists a nontrivial linear combination y of $y^1, \ldots, y^m$ which is in Range(A) modulo $N(A^{t-1})$. Then $y = Ax + w$, where $w \in N(A^{t-1})$. Since $\mathcal{B}$ is a height basis, it follows from Proposition (3.14) in [8] that the expression of w as a linear combination of elements of $\mathcal{B}$ does not involve elements of height greater than or equal to t, and therefore it involves none of $y^1, \ldots, y^m$. Note that $x \in E(A)$. So x can be expressed as a linear combination of elements of $\mathcal{B}$. Consequently, Ax is a linear combination of elements of $\mathcal{B}$, none of which is a top of a chain. Therefore, the expression of Ax involves none of $y^1, \ldots, y^m$. Hence, the expression of $Ax + w$ involves none of $y^1, \ldots, y^m$, which contradicts the equality $y = Ax + w$. ■

(5.3) PROPOSITION *Let $\mathcal{B}$ be a height basis for an M-matrix A, and let t be a positive integer, $1 < t \leq p$. Let $x^1, \ldots, x^m$ be the elements of $\mathcal{B}$ with height t, and let $y^1, \ldots, y^k$ be the elements of $\mathcal{B}$ with height $t-1$. Furthermore, assume that $y^{m+1}, \ldots, y^k$ are tops of chains in some Jordan basis $\mathcal{J}$ for $E(A)$. Suppose that*

$$Ax^i = \sum_{j=1}^{k} c_{ji} y^j + w_i, \qquad \textit{where} \quad w_i \in N(A^{t-2}), \qquad i \in \langle m \rangle.$$

Then the matrix $C = (c_{ij})_1^m$ is nonsingular.

Proof Suppose that C is singular and let d be a nonzero m-vector such that $Cd = 0$. Define the n-vector x by

$$x = \sum_{i=1}^{m} d_i x^i.$$

Then $x \neq 0$. Also, by Proposition (3.14) in [8], $x \notin N(A^{t-1})$ and hence

$$Ax \notin N(A^{t-2}). \tag{5.4}$$

Note that

$$Ax = \sum_{i=1}^{m}\sum_{j=1}^{k} d_i c_{ji} y^j + \nu, \qquad \text{where} \quad \nu \in N(A^{t-2}). \tag{5.5}$$

Since $Cd = 0$ it now follows from (5.5) that

$$Ax = \sum_{i=1}^{m}\sum_{j=m+1}^{k} d_i c_{ji} y^j + \nu, \qquad \text{where} \quad \nu \in N(A^{t-2}). \tag{5.6}$$

By (5.4) it now follows that

$$\sum_{i=1}^{m}\sum_{j=m+1}^{k} d_i c_{ji} y^j$$

is a nontrivial linear combination of $y^{m+1}, \ldots, y^k$, and hence (5.6) is a contradiction to Lemma (5.2). ■

We continue with the definition of Jordan set derived from a given set of vectors in $E(A)$.

(5.7) *Definition* Let A be a matrix.

(i) Let x be a vector in $E(A)$, and let $t = \text{height}(x)$. The Jordan chain $(x, Ax, \ldots, A^{t-2}x, A^{t-1}x)$ is said to be the *Jordan chain derived* from x. The vector $A^{t-1}x$ is said to be the *fundament* of x.

(ii) Let $\mathcal{S}$ be a set of vectors in $E(A)$. The multi-set which consists of the union of the Jordan chains derived from the elements of $\mathcal{S}$ is said to be the *Jordan set derived* from $\mathcal{S}$. The multi-set which consists of the fundaments of the elements of $\mathcal{S}$ is said to be the *fundament of* $\mathcal{S}$.

(iii) Let $\mathcal{S}$ be a set of vectors in $E(A)$, and let $\mathcal{J}$ be the Jordan set derived from $\mathcal{S}$. If $\mathcal{J}$ is a (Jordan) basis for A then we say that $\mathcal{J}$ is the *Jordan basis for A derived* from $\mathcal{S}$.

(5.8) *Observation* It is easily shown that the Jordan set derived from $\mathcal{S}$ is linearly independent if and only if its fundament is linearly independent. A related result is proved in Lemma 2.1 in [1].

(5.9) THEOREM *Let A be an M-matrix and assume that $\eta(A) = \lambda(A)$. Let T be a set of singular vertices. If there exists a Jordan basis $\mathcal{J}$ for A derived from a proper T-combinatorial set of vectors, then there exists an anchored chain decomposition κ of $S(A)$ such that T is the set of final vertices of the chains in κ.*

Proof Let $\mathcal{J}$ be a Jordan basis derived from a proper T-combinatorial set $\mathcal{T}$. Let $k \in \langle p \rangle$, and let T_k be the set of k-level vertices in T. Since $\mathcal{T}$ is the set of the tops of the chains in $\mathcal{J}$, and since $\eta(A) = \lambda(A)$, it follows that the cardinality of T_k is $\eta_k - \eta_{k+1} = \lambda_k - \lambda_{k+1}$, where $\eta_{p+1} = \lambda_{p+1} = 0$. Let $L'_k = L_k \backslash T_k$. Without loss of generality we assume L'_k is indexed by the first λ_{k+1} elements of L_k. By Proposition (3.12), $\mathcal{T}$ can be completed to a combinatorial basis $\mathcal{B}$. Let C be the induced matrix $C(A,\mathcal{B})$. By Proposition (5.3), the first λ_{k+1} rows of $C_{k,k+1}$ form a full column rank matrix for all $k \in \langle p-1 \rangle$. This implies, by Corollary (4.3), that the sets $E_i = \Delta_{G(C)}(i) \cap L'_k$, $i \in L_{k+1}$, satisfy the Hall Marriage Condition, for all $k \in \langle p-1 \rangle$. By Theorem (4.7), there exists an anchored chain decomposition κ for $G(C)$ such that the elements in $T_k = L_k \backslash L'_k$, $k \in \langle p \rangle$, are final elements of chains in κ.

We must still show that T is the set of *all* final vertices in the chains in κ, and that κ is an anchored chain decomposition for $S(A)$. The first statement follows since the number of the Jordan chains in $\mathcal{J}$ is λ_1, and the number of chains in κ is λ_1. The second statement follows by Proposition (4.6), since by Proposition (3.23), $G(C)$ is a subgraph of $S(A)$. ∎

The converse of Theorem (5.9) is false in general. In the following example we give an M-matrix A with $\eta(A) = \lambda(A)$, and a set T of singular vertices, such that there exists no Jordan basis derived from a proper T-combinatorial set of vectors, although there exists an anchored chain decomposition κ of $S(A)$ such that T is the set of final vertices of the chains in κ.

(5.10) *Example* Let

$$A = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ -1 & -1 & 0 & 0 & 0 \\ -1 & -1 & 0 & 0 & 0 \\ -1 & -2 & 0 & 0 & 0 \end{pmatrix}.$$

We have $\eta(A) = \lambda(A) = (3,2)$. It is easy to see that $(3,1)$, $(4,2)$ and (5) form an anchored chain decomposition κ of $S(A)$, where the set of final vertices of chains in κ is $T = \{1,2,5\}$. However, every proper 5-combinatorial vector is of the form $[0\ 0\ 0\ 0\ c]^T$, where $c \neq 0$. Since this vector is in Range(A), it follows from Lemma (5.2) that it cannot be a top of a Jordan chain in a Jordan basis.

6. EQUALITY OF THE HEIGHT AND THE LEVEL CHARACTERISTICS

In this section we add twenty three statements equivalent to the conditions in Theorem (8.1) in [8]. The section in concluded with an affirmative answer to a conjecture in [1].

(6.1) *Definition* Let A be a square matrix and let $\mathcal{B}$ be a height basis for $E(A)$. A Jordan basis for A that is derived from a subset of $\mathcal{B}$ is called a *Jordan basis linked to* $\mathcal{B}$.

(6.2) *Observation* Let $\mathcal{J}$ be a Jordan basis linked to a height basis $\mathcal{B}$, and let $\mathcal{S}$ be the subset of $\mathcal{B}$ such that $\mathcal{J}$ is derived from $\mathcal{S}$. Then $\mathcal{S}$ is the set of the tops of the Jordan chains in $\mathcal{J}$, and hence $\eta_k(\mathcal{S}) = \eta_{k+1}(A) - \eta_k(A)$, $k \in \langle p-1 \rangle$, and $\eta_p(\mathcal{S}) = \eta_p(A)$.

(6.3) *Remark* Jordan bases linked to height bases are defined in Definition (6.7) in [8]. Proposition (6.1) in [8] proves that for every height basis $\mathcal{B}$, there exists a Jordan basis $\mathcal{J}$ that is linked to $\mathcal{B}$. The proof of that proposition describes the construction of such a $\mathcal{J}$.

(6.4) *Definition* Let $\mathcal{S}$ be a set of vectors in $E(A)$. We define the *level sum* of $\mathcal{S}$ to be the sum of the levels of the elements of $\mathcal{S}$.

(6.5) PROPOSITION *If $x \in E(A)$ is a peak vector then Ax is a peak vector.*

Proof By two results in [8], Lemma (4.13) and Corollary (4.17), we have level(x) $-1 =$ height$(x) - 1 =$ height$(Ax) \leq$ level$(Ax) <$ level(x), which yields that height $(Ax) =$ level(Ax). ■

We now come to the main result of the section, which adds 23 Conditions to Theorem (8.1) in [8].

(6.6) THEOREM *Let A be an M-matrix. The following are equivalent:*

1. $\eta(A) = \lambda(A)$.
2. *Every vector in $E(A)$ is a peak vector.*
3. *Every basis for $E(A)$ is a peak basis.*
4. *Every height basis for $E(A)$ has a peak subset $\mathcal{S}$ with level sum $m(A)$ such that the fundament of $\mathcal{S}$ is linearly independent.*
5. *Every height basis for $E(A)$ has a peak subset $\mathcal{S}$ such that the Jordan set derived from $\mathcal{S}$ is a Jordan basis for A.*
6. *Some height basis for $E(A)$ has a peak subset $\mathcal{S}$ with level sum $m(A)$ such that the fundament of $\mathcal{S}$ is linearly independent.*
7. *Some height basis for $E(A)$ has a peak subset $\mathcal{S}$ such that the Jordan set derived from $\mathcal{S}$ is a Jordan basis for A.*
8. *Every level basis for $E(A)$ has a peak subset $\mathcal{S}$ with level sum $m(A)$ such that the fundament of $\mathcal{S}$ is linearly independent.*
9. *Every level basis for $E(A)$ has a peak subset $\mathcal{S}$ such that the Jordan set derived from $\mathcal{S}$ is a Jordan basis for A.*
10. *Every level basis for $E(A)$ has a subset $\mathcal{S}$ with level sum $m(A)$ such that the fundament of $\mathcal{S}$ is linearly independent.*
11. *Every level basis for $E(A)$ has a subset $\mathcal{S}$ such that the Jordan set derived from $\mathcal{S}$ is a Jordan basis for A.*
12. *Every combinatorial basis for $E(A)$ has a subset $\mathcal{S}$ with level sum $m(A)$ such that the fundament of $\mathcal{S}$ is linearly independent.*

13. *Every combinatorial basis for $E(A)$ has a subset S such that the Jordan set derived from S is a Jordan basis for A.*
14. *Some combinatorial basis for $E(A)$ has a subset S with level sum $m(A)$ such that the fundament of S is linearly independent.*
15. *Some combinatorial basis for $E(A)$ has a subset S such that the Jordan set derived from S is a Jordan basis for A.*
16. *Some level basis for $E(A)$ has a peak subset S with level sum $m(A)$ such that the fundament of S is linearly independent.*
17. *Some level basis for $E(A)$ has a peak subset S such that the Jordan set derived from S is a Jordan basis for A.*
18. *Some peak basis for $E(A)$ has a subset S with level sum $m(A)$ such that the fundament of S is linearly independent.*
19. *Some peak basis for $E(A)$ has a subset S such that the Jordan set derived from S is a Jordan basis for A.*
20. *There exists a peak set S with level sum $m(A)$ such that the fundament of S is linearly independent.*
21. *There exists a Jordan basis for A which is derived from a peak subset of $E(A)$.*
22. *Some height basis for $E(A)$ is a peak basis.*
23. *Every height basis for $E(A)$ is a level basis for $E(A)$.*
24. *Every level basis for $E(A)$ is a height basis for $E(A)$.*
25. *Some preferred basis for $E(A)$ is a height basis for $E(A)$.*
26. *Some combinatorial basis for $E(A)$ is a height basis for $E(A)$.*
27. *There exists a Jordan basis for A which is derived from a proper combinatorial set of vectors.*
28. *There exists a nonnegative height-level basis for $E(A)$.*
29. *There exists a nonnegative height basis for $E(A)$.*
30. *There exists a nonnegative Jordan basis for $-A$.*
31. *For all j, $j \in \langle p \rangle$, there exists a nonnegative basis for $N(A^j)$.*
32. *For every level basis $\mathcal{B}$ for $E(A)$ with induced matrix $C = C(A,\mathcal{B})$, the block $C_{k,k+1}$ has full column rank for all $k \in \langle p-1 \rangle$.*
33. *There exists a level basis $\mathcal{B}$ for $E(A)$ with induced matrix $C = C(A,\mathcal{B})$, such that for all $k \in \langle p-1 \rangle$ the block $C_{k,k+1}$ has full column rank.*
34. *For every combinatorial basis $\mathcal{B}$ for $E(A)$, there exists a proper T-combinatorial subset of $\mathcal{B}$ with linearly independent fundament, where T is the set of final vertices of the chains in some anchored chain decomposition of $S(A)$.*
35. *There exists a proper T-combinatorial set of vectors with linearly independent fundament, where T is the set of final vertices of the chains in some anchored chain decomposition of $S(A)$.*
36. *There exists a proper T-combinatorial set of vectors with linearly independent fundament, where the sum of the levels of the vertices in T is $m(A)$.*

Proof The equivalence of Conditions (1), (2), (3), (22), (23), (24), (25), (28), (29), (30), (31), (32), and (33) is proven in Theorem (8.1) in [8]. Therefore, it is enough to prove the implications:

$$(4) \Leftrightarrow (5); \quad (6) \Leftrightarrow (7); \quad (8) \Leftrightarrow (9); \quad (16) \Leftrightarrow (17); \quad (18) \Leftrightarrow (19).$$
$$(3) \Rightarrow (4) \Rightarrow (6) \Rightarrow (20) \Rightarrow (21) \Rightarrow (22),$$
$$(4)\&(24) \Rightarrow (8) \Rightarrow (10) \Rightarrow (12) \Rightarrow (14) \Rightarrow (16) \Rightarrow (20),$$
$$(9) \Rightarrow (11) \Rightarrow (13) \Rightarrow (15) \Rightarrow (17),$$
$$(14) \Rightarrow (18) \Rightarrow (20),$$
$$(25) \Rightarrow (26) \Rightarrow (27) \Rightarrow (21),$$
$$(13) \Rightarrow (34) \Rightarrow (35) \Rightarrow (36) \Rightarrow (20),$$

$(4) \Leftrightarrow (5)$; $(6) \Leftrightarrow (7)$; $(8) \Leftrightarrow (9)$; $(16) \Leftrightarrow (17)$; $(18) \Leftrightarrow (19)$. All these equivalences follow from Observation (5.8).

$(3) \Rightarrow (4)$. Let $\mathcal{B}$ be a height basis for $E(A)$, and let $\mathcal{J}$ be a Jordan basis linked to $\mathcal{B}$. Since by (3) $\mathcal{B}$ is a peak basis, it follows from Observation (6.2) that the set S of the tops of the chains in $\mathcal{J}$ satisfies the conditions of (4).

$(4) \Rightarrow (6) \Rightarrow (20)$ is trivial.

$(20) \Rightarrow (21)$. Let S satisfy the conditions in (20). By Observation (5.8), the Jordan set $\mathcal{J}$ derived from S is a linearly independent set. Furthermore, since $\mathcal{J}$ is derived from the peak set S with level sum $m(A)$, the cardinality of $\mathcal{J}$ is $m(A)$, and hence $\mathcal{J}$ is a Jordan basis for A.

$(21) \Rightarrow (22)$. Let $\mathcal{J}$ be a Jordan basis for A which is derived from a peak subset of $E(A)$. By Proposition (6.5), $\mathcal{J}$ is a peak basis, and (22) follows.

$(4)\&(24) \Rightarrow (8) \Rightarrow (10)$ are all immediate.

$(10) \Rightarrow (12)$ follows since every combinatorial basis is a level basis.

$(12) \Rightarrow (14)$ is trivial.

$(14) \Rightarrow (16)$ follows since every combinatorial basis is a peak level basis.

$(16) \Rightarrow (20)$ is immediate.

$(9) \Rightarrow (11)$ is trivial.

$(11) \Rightarrow (13)$ follows since every combinatorial basis is a level basis.

$(13) \Rightarrow (15)$ is trivial.

$(15) \Rightarrow (17)$ follows since every combinatorial basis is a peak level basis.

$(14) \Rightarrow (18)$ follows since every combinatorial basis is a peak basis.

$(18) \Rightarrow (20)$ is trivial.

$(25) \Rightarrow (26)$ is trivial since a preferred basis is a combinatorial basis.

$(26) \Rightarrow (27)$. Let $\mathcal{B}$ be a combinatorial basis for A which is a height basis. Clearly, every Jordan basis linked to $\mathcal{B}$ satisfies (27).

$(27) \Rightarrow (21)$ follows since every proper combinatorial set is a peak set.

$(13) \Rightarrow (34)$. Let $\mathcal{B}$ be a combinatorial basis for $E(A)$, and let S be a proper T-combinatorial subset of $\mathcal{B}$ such that the Jordan set derived from S is a Jordan basis for A. Then the fundament of S is linearly independent. Furthermore, by Theorem (5.9) there exists an anchored chain decomposition κ of $S(A)$ such that

T is the set of final vertices of the chain in κ.

(34) $\Rightarrow$ (35) is trivial.
(35) $\Rightarrow$ (36). Clearly, if T is the set of final vertices of the chains in some chain decomposition of $S(A)$ then the sum of the levels of the vertices in T is $m(A)$.
(36) $\Rightarrow$ (20) follows since a proper combinatorial set is a peak set, and since the level of a proper i-combinatorial vector is equal to the level of i. ■

(6.7) *Remark* The following four conditions hold for every M-matrix A, and hence they are not equivalent to the conditions in Theorem (6.6).

(a) Every height basis for $E(A)$ has a subset S such that the Jordan set derived from S is a Jordan basis for A.
(b) Some height basis for $E(A)$ is a level basis for $E(A)$.
(c) Some level basis for $E(A)$ has a subset S such that the Jordan set derived from S is a Jordan basis for A.
(d) Every combinatorial basis for $E(A)$ is a level basis for $E(A)$.

Condition (a) holds for every M-matrix A, since by Proposition (6.1) in [8], for every height basis B for $E(A)$ there exists a Jordan basis linked to B.

Condition (b) holds for every M-matrix, as proven in Corollary (5.6) in [8].

It follows from Conditions (a) and (b) that Condition (c) holds for every M-matrix.

Condition (d) is proven in Corollary (3.17).

(6.8) *Remark* By Theorem (6.6), the following condition follows from Condition (1).

(e) Some level basis for $E(A)$ has a subset S with level sum $m(A)$ such that the fundament of S is linearly independent.

However, we do not have (e) $\Rightarrow$(1) in general, as demonstrated by the matrix

$$A = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -1 & -1 & 0 \end{pmatrix}.$$

Here we have $\lambda(A) = (1,2)$ and $\eta(A) = (2,1)$. The vectors $[1\ 0\ 0]^T$, $[1\ -1\ 0]^T$, and $[0\ 0\ 1]^T$ form a level basis B for $E(A)$. The last two vectors have level sum 3 ($= m(A)$) and have a linearly independent fundament. Yet, $\lambda(A) \neq \eta(A)$.

(6.9) *Remark* The following condition implies Condition (19) in Theorem (6.6).

(f) Every peak basis for $E(A)$ has a subset S such that the Jordan set derived from S is a Jordan basis for A.

However, Condition (f) is not implied in general by the conditions in Theorem (6.6), as follows from Theorem (6.12) below.

(6.10) PROPOSITION *Every basis for $E(A)$ has a subset S of η_p, but no more, p-height vectors, such that the fundament of S is linearly independent.*

Proof Clearly, every set S of p-height vectors in $E(A)$ is linearly independent modulo $N(A^{p-1})$ if and only if the fundament of S is linearly independent. Let B be a basis for $E(A)$. Since the p-height vectors in B span $E(A)$ modulo $N(A^{p-1})$, and since the dimension of $E(A)$ modulo $N(A^{p-1})$ is η_p, it now follows that we can find a set S of η_p, but no more, p-height vectors such that the fundament of S is linearly independent. ■

(6.11) LEMMA *There exists a peak basis B for $E(A)$ such that all the elements of B are of height p.*

Proof Take a preferred basis for $E(A)$ and then add one of the p-level vectors to all the others to obtain a new basis B for $E(A)$. Then B contains only nonnegative vectors, and hence, by Corollary (4.11) in [8], B is a peak basis. Also, all the vectors in B are of level p. ■

(6.12) THEOREM *Let A be an M-matrix, and let $\overline{m} = m(A)/p$. The following are equivalent:*

- (i) Every peak basis for $E(A)$ has a subset S such that the Jordan set derived from S is a Jordan basis for A.
- (ii) Some peak basis B for $E(A)$, such that all the elements of B are of height p, has a subset S such that the Jordan set derived from S is a Jordan basis for A.
- (iii) $\eta_1 = \overline{m}$.
- (iv) $\eta(A) = \lambda(A) = (\overline{m}, \overline{m}, \ldots, \overline{m})$.
- (v) $\eta_p = \overline{m}$.

Proof

- (i) ⇒(ii) is trivial, in view of Lemma (6.11).
- (ii) ⇒(iii) Let B be a peak basis for $E(A)$, such that all the elements of B are of height p. If there exists a subset S of B such that the Jordan set $\mathcal{J}$ derived from S is a Jordan basis for A, then clearly the number η_1 of Jordan chains in $\mathcal{J}$ is equal to $m(A)/p = \overline{m}$.
- (iii) ⇒(iv) Let $\eta_1 = \overline{m}$. Since $\eta(A)$ is a non-increasing sequence, and since $\eta_1 + \cdots + \eta_p = m(A) = p\overline{m}$, it follows that $\eta(A) = (\overline{m}, \overline{m}, \ldots, \overline{m})$. By Theorem (3.7) in [3] we have $\max\{\lambda_i : i \in \langle p \rangle\} \leq \overline{m}$. Since $\lambda_1 + \cdots + \lambda_p = m(A) = p\overline{m}$, it follows that $\lambda(A) = (\overline{m}, \overline{m}, \ldots, \overline{m})$.
- (iv) ⇒(v) is trivial.
- (v) ⇒(i) Assume that $\eta_p = \overline{m}$. Let B be a peak basis for $E(A)$. By Proposition (6.10) let S be a set of $\overline{m}$ p-height vectors in B such that the fundament of S is linearly independent. Since the level sum of S is $\overline{m}p = m(A)$, it follows from Observation (5.8) that the Jordan set derived from S is a Jordan basis for A. ■

We now explain the relation of Theorem (6.6) (and Theorem (5.9) used in the proof of Theorem (6.6)) to the results in [1].

(6.13) *Remark* The paper [1] investigates Rothblum bases, which are a special case of combinatorial bases. Using our terminology, Theorem 3.4 in [1] can be stated as:

(i) *If there exists an anchored chain decomposition of $S(A)$ such that for some Rothblum basis $\mathcal{R}$ for $E(A)$, the fundament of the subset of $\mathcal{R}$ that corresponds to the final vertices of the chains in κ is linearly independent, then there exists a nonnegative Jordan basis for $-A$.*

(ii) *If there exists a nonnegative Jordan basis for $-A$, then there exists a set T of singular vertices with level sum $m(A)$ and a proper T-combinatorial set $\mathcal{T}$ of Rothblum vectors with linearly independent fundament.*

The converse of part (i) is also conjectured in [1].

Observe that part (i) follows immediately from the more general implication (35) $\Rightarrow$(30) in Theorem (6.6). Part (ii), as well as the converse of part (i), follows from the stronger result (30) $\Rightarrow$(34) in Theorem (6.6).

Let S be a transitive acyclic graph. In [11] the authors prove necessary and sufficient conditions on S such that all M-matrices A with $S(A) = S$ satisfy $\eta(A) = \lambda(A)$, see also [14]. We conclude the paper with a companion result.

(6.14) THEOREM *Let S be a transitive acyclic graph. Then the following are equivalent:*

(i) *There exists an M-matrix A with $S(A) = S$ such that $\eta(A) = \lambda(A)$.*

(ii) *The graph S is well structured.*

(iii) *The sets $E_i = \Delta(i) \cap L_k$, $i \in L_{k+1}$, satisfy the Hall Marriage Condition for all $k \in \langle q-1 \rangle$.*

Proof

(i) $\Rightarrow$(ii) By the implication (1) $\Rightarrow$(35) in Theorem (6.6), if (i) holds then there exists an anchored chain decomposition of S.

(ii) $\Leftrightarrow$(iii) follows from Theorem (4.7) with $L'_k = L_k$.

(iii) $\Rightarrow$(i) Let S be a well structured graph with level $L_1, \ldots, L_p$. Let Q be a matrix such that $G(Q)$ is equal to S with its loops removed. Then Q is a strictly lower triangular matrix. Let $|L_k| = \lambda_k$, $k \in \langle p \rangle$, and partition Q in the same manner as the matrix C in Observation (3.22). Let $k \in \langle q-1 \rangle$ and let F be the matrix $Q_{k,k+1}$. Define

$$E_j = \{i \in \langle \lambda_k \rangle : f_{ij} \neq 0\} = \Delta(j) \cap L_k, \qquad j \in L_{k+1}.$$

Since S is well structured, it follows by Theorem (4.7) with $L'_k = L_k$ that the sets E_j, $j \in L_{k+1}$, satisfy the Hall Marriage Condition. By Corollary (4.3) there exists a nonnegative matrix H_k, which has the same zero pattern as $Q_{k,k+1}$, such that H_k is of full column rank. Let A be any nonpositive matrix partitioned conformably with Q such that $G(A) = G(Q)$ and $A_{k,k+1} = -H_k$ for all $k \in \langle p-1 \rangle$. Observe that A is an M-matrix, that the standard basis B of unit vectors is a level basis for A, and that $C(A,B) = A$. By the implication (33) $\Rightarrow$(1) in Theorem (6.6) we obtain (i). ■

We note that the equivalence of conditions (i) and (iii) is already proven as Theorem (3.2) in [10]. We have provided a proof for the sake of completeness, and as an application of Theorem (6.6).

As a corollary we obtain the following affirmative answer to the conjecture that concludes [1]. The corollary follows immediately from Theorems (6.6) and (6.14) above.

(6.15) COROLLARY *Let A be an M-matrix. If there exists a nonnegative Jordan basis for $-A$ then $S(A)$ is well structured.*

Acknowledgments

We wish to acknowledge discussions with Philip Kavanagh at an initial stage of our investigations. In particular, he showed us different proofs of the conjectures in [1] proved above.

References

[1] R. Bru and M. Neumann, Nonnegative Jordan bases, *Lin. Multilin. Alg.* **23** (1988), 95–109.
[2] R. A. Brualdi, *Introductory Combinatorics*, North-Holland, 1977.
[3] D. Hershkowitz, A majorization relation between the height and the level characteristics, *Lin. Alg. Appl.* **125** (1989), 97–101.
[4] D. Hershkowitz, U. G. Rothblum and H. Schneider, Characterizations and classifications of M-matrices using generalized nullspaces, *Lin. Alg. Appl.* **109** (1988), 59–69.
[5] D. Hershkowitz, U. G. Rothblum and H. Schneider, The combinatorial structure of the generalized nullspace of a block triangular matrix, *Lin. Alg. Appl.* **116** (1989), 9–26.
[6] D. Hershkowitz and H. Schneider, On the generalized nullspace of M-matrices and Z-matrices, *Lin. Alg. Appl.* **106** (1988), 5–23.
[7] D. Hershkowitz and H. Schneider, Solutions of Z-matrix equations, *Lin. Alg. Appl.* **106** (1988), 25–38.
[8] D. Hershkowitz and H. Schneider, Height bases, level bases, and the equality of the height and the level characteristics of an M-matrix, *Lin. Multilin. Alg.* **25** (1989), 149–171.
[9] W. C. Huang, On the singular graph and Jordan diagram of strictly lower triangular matrices and M-matrices, *Lin. Alg. Appl.* **143** (1991), 145–169.
[10] D. J. Richman, The singular graph of lower triangular, nilpotent matrices, *Lin. Multilin. Alg.* **6** (1978), 37–49.
[11] D. J. Richman and H. Schneider, On the singular graph and the Weyr characteristie of an M-matrix, *Aequ. Math.* **17** (1978), 208–234.
[12] U. G. Rothblum, Algebraic eigenspaces of non-negative matrices, *Lin. Alg. Appl.* **12** (1975), 281–292.
[13] H. Schneider, The elementary divisors associated with 0 of a singular M-matrix, *Proc. Edinburgh Math. Soc. (2)* **19** (1956), 108–122.
[14] H. Schneider, The influence of the marked reduced graph of a nonnegative matrix on the Jordan form and related properties: A survey, *Lin. Alg. Appl.* **84** (1986), 161–189.