Allowable Spectral Perturbations for *ZME*-Matrices*‡

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ABSTRACT

A *ZME*-matrix is a matrix A all of whose positive integer powers are Z-matrices, and whose odd powers are irreducible. We find a combinatorial partial order on the spectral idempotents of a *ZME*-matrix A which determines the allowable spectral perturbations B for which B is again a *ZME*-matrix. We apply this result to show that under certain restrictions, the product of two *ZME*-matrices is a *ZME*-matrix.

1. INTRODUCTION

In two recent papers [2, 4], Friedland, Hershkowitz, and Schneider characterized certain classes of Z- and M-matrices all of whose positive-integer powers are again Z- or M-matrices. In this paper, we further examine the structure of these matrices. In particular, we find a combinatorial structure for the eigenspace projections of such matrices, and we use this

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ordering to determine the allowable spectral perturbations which preserve class membership. Further, we show that under certain restrictions, class membership is preserved under multiplication of commuting matrices within the class. Additionally, we state a conjecture and an open question.

A *Z-matrix* is a real, square matrix with all of its off-diagonal entries nonpositive. An *M-matrix* is a Z-matrix for which all of the eigenvalues have nonnegative real parts. The matrix A is a *ZM-matrix* (*MM-matrix*) if all of its positive powers are Z-matrices (M-matrices). The matrix A is a *ZMA-matrix* (*MMA-matrix*) if A is a ZM-matrix (MM-matrix) and all positive powers of A are irreducible. The matrix A is a *ZMO-matrix* if A is a ZM-matrix, all of the odd positive powers of A are irreducible, and all of the even positive powers of A are completely reducible but not irreducible. The matrix A is a *ZME-matrix* if it is either a ZMA-matrix or a ZMO-matrix. It is apparent that the classes of MMA-, ZMA-, and ZMO-matrices are contained in the class of ZME-matrices.

In their paper [2], Friedland, Hershkowitz, and Schneider present a characterization for the class of ZME-matrices and its subclasses in terms of an operation which they call iinflation. They show that an $n \times n$ ZME-matrix A has a special type of spectral decomposition:

$$A = \sum_{j=1}^{k} \alpha_j E_j, \tag{1.1}$$

where the spectrum of A is real and satisfies $\alpha_1 < \alpha_2 < \cdots < \alpha_k$, and where $E_1, E_2, \dots, E_k$ are the spectral projectors. Further, they show that the sequence $E_1, \dots, E_k$ corresponds to an "inflation sequence." We will use the inflation sequences of a ZME-matrix to study the structure of the spectral projectors, and to deduce certain results on spectral perturbation.

What follows is a section-by-section summary of the principal results.

The subsections of Section 2 contain the definitions and results from the paper [2] by Friedland, Hershkowitz, and Schneider which will be needed in the subsequent chapters. Most notable among these are the operation of inflation, the inflation theorem (Theorem 2.6.3) and its corollary (which is a spectral decomposition theorem for ZME-matrices), and an allowable spectral perturbation theorem which is here called the original slide-around theorem (Theorem 2.9.1).

Beginning with Section 3, we focus on the set $\mathscr{E}$ of spectral projectors for a fixed ZME-matrix. In this section, we derive a simple combinatorial partial order $\preccurlyeq\cdot$ on the elements of $\mathscr{E}$, and we relate it to the operation of inflation. In particular, we show that if E and F are in $\mathscr{E}$, and if $E \prec\cdot F$ in $\mathscr{E}$, then E precedes F for every possible inflation sequence for the set of projectors $\mathscr{E}$ (Lemma 3.4). We also investigate the structure of $\mathscr{E}$ under $\prec\cdot$.

There is a natural correspondence between the partially ordered set $(\mathscr{E}, \preccurlyeq\cdot)$ and its directed comparability graph $\mathscr{L}(\mathscr{E})$. In Section 4, we present the graph-theoretic results which will be needed in the remainder of the paper. The basic properties of $\mathscr{L}(\mathscr{E})$ are derived, and a special subgraph, the covering graph $\mathscr{G}(\mathscr{E})$ of $\mathscr{L}(\mathscr{E})$ is introduced. This sparser subgraph is shown to have a "tap-rooted" structure (Theorem 4.8).

In Sections 5 through 8, we apply the results of Sections 2 through 4.

In Section 5, we prove the alternative inflation-sequences theorem (Theorem 5.6), which states that every ordering of the nodes of $\mathscr{L}(\mathscr{E})$ which is consistent with the ordering on $\mathscr{L}(\mathscr{E})$ gives rise to an inflation sequence for $\mathscr{E}$.

In Section 6, we relate the partial order $\prec\cdot$ in $\mathscr{F}$ (a complete set of inflation-generated projectors) to the partial order $\prec\cdot$ in $\mathscr{E}$ (an inflation-generated, rank-one refinement of $\mathscr{F}$) (Theorem 6.3). It is then shown which complete sets of projectors possessing $\mathscr{E}$ as a refinement can be inflation-generated (Theorem 6.5).

In Section 7, the results of the preceding sections are used to prove two generalizations of the original slide-around theorem in [2], our Theorem 2.9.1. These generalizations are our allowable spectral perturbation theorems, hereafter called *slide-around theorems*. The strict-inequality slide-around theorem (Theorem 7.1) determines all α_i such that the matrix A given by (1.1) is a *ZME*-matrix, under the assumption that the α_i are pairwise distinct. The weak-inequality slide-around theorem (Theorem 6.2) addresses the case when the α_i are permitted to coincide.

In Section 8, we show that the product of two commuting *ZME*-matrices is again a *ZME*-matrix, provided that at least one of the matrices is a *ZMA*-matrix and provided that they have a common complete set of projectors possessing an inflation-generated refinement. In particular, this latter condition holds if one of the matrices has distinct eigenvalues. The question of whether two commuting *ZME*-matrices must be a *ZME*-matrix is left open.

The reader who is interested in further papers on the properties of *ZM*- and *MM*-matrices should consult the papers by one of the authors [2, 4], or the other [6–13], or by M. Fiedler [14].

2. PRELIMINARIES

Throughout this paper, $\mathscr{M}_{m,n}(\mathscr{F})$ will be the set of all $m \times n$ matrices over the set $\mathscr{F}$. Although many of the results presented can be extended to other algebraic structures, in this paper $\mathscr{F}$ will always be either $\mathbb{C}$ or $\mathbb{R}$. If $m = n$, then $\mathscr{M}_{n,n}(\mathscr{F})$ will be denoted by $\mathscr{M}_n(\mathscr{F})$. If A is in $\mathscr{M}_n(\mathscr{F})$, then

A is said to have *order* n. The set of $1 \times n$ matrices over $\mathcal{F}$ will be denoted as $\mathcal{F}^n$, and the term vector will always mean row vector. A *strictly nonzero matrix* (*strictly nonzero vector*) will be a matrix (vector) each of whose entries is nonzero. A *strictly positive matrix* (*strictly positive vector*) will be a real matrix (vector) each of whose entries is positive. Matrices will always be denoted with capital letters; and if A is a matrix, then its i, j entry may be denoted by either A_{ij} or a_{ij}. A power of a square matrix will always mean an integer power of that matrix. If A is in $\mathcal{M}_n(\mathcal{F})$ for some field $\mathcal{F}$, then $A^0 = I_n$, the $n \times n$ identity.

Let A be in $\mathcal{M}_n(\mathbb{C})$. The matrix A is *reducible* if there is an $n \times n$ permutation matrix P such that

$$PAP^t = \begin{bmatrix} B_1 & B_2 \\ 0 & B_3 \end{bmatrix}$$

where the matrices B_1 and B_3 are square matrices. If no such permutation matrix P exists, then A is *irreducible*. If A is either irreducible or permutation-similar to a direct sum of irreducible matrices, then A is *completely reducible*. There is a well-known graph-theoretic characterization of these properties (see [1] for example).

2.1. *Inflation*

In this subsection, the concept of inflation, which was introduced in [2], is discussed.

Let m and n be positive integers with $m \leqslant n$. An *m-partition of n* is a partition of the set $\{1,2,\ldots,n\}$ into an ordered collection of m nonempty, disjoint sets such that the elements within each set are arranged in ascending order. When there is no confusion as to which m-partiton of n is being used, the partition will be denoted by $P_{m,n}$.

Throughout Section 2, the following conventions are assumed:

(1) The letters m, $\hat{m}$, n, and $\hat{n}$ are positive integers with $m \leqslant n$ and $\hat{m} \leqslant \hat{n}$.

(2) The set Π is an m-partition of n given by $B_1, B_2, \ldots, B_m$.

(3) The set $\hat{\Pi}$ is an $\hat{m}$-partition of $\hat{n}$ given by $\hat{B}_1, \hat{B}_2, \ldots, \hat{B}_{\hat{m}}$.

Let U be in $\mathcal{M}_{n,\hat{n}}(\mathbb{C})$. The partition pair $(\Pi, \hat{\Pi})$ induces a block partitioning of the matrix U as follows: For $1 \leqslant i \leqslant m$ and $1 \leqslant j \leqslant \hat{m}$, the i, j block of U consists of all entries $U_{\alpha\beta}$ such that α is in B_i and β is in $\hat{B}_j$. Denote the i, j block of U by $U_{\langle i,j \rangle}$. If $m = 1$, then $\Pi = \{\{1\}\}$, and the $1, j$ block of U will be denoted by $U_{\langle j \rangle}$.

The partition pair (Π, Π) induces a symmetric partitioning for each matrix in $\mathscr{M}_{n,n}(\mathscr{F})$ which is called the *block partition induced by* Π. The integer m is called the *block order of the partition* Π, or simply the *block order* when there is no confusion as to the partition. If A is in $\mathscr{M}_n(\mathscr{F})$ and is partitioned by Π, then A is said to have *block order m*. If $A_{\langle i,j\rangle} = 0$ for $i \neq j$, then A is a *block-diagonal matrix*. The partition block $B_i \times B_i$ and the matrix block $A_{\langle i,i\rangle}$ are each called a *trivial block* if $|B_i| = 1$. If $m = n$, then the block partition induced by Π is called a *trivial block partition*.

Throughout this paper, the following convention will be employed: When block-partitioned matrices and vectors are displayed, they will be displayed for convenience as if the partition sets consisted of consecutive integers.

Suppose that A is in $\mathscr{M}_{n,n}(\mathscr{F})$, and that A has been symmetrically partitioned into blocks. Then the corresponding partition Π of $\{1,2,\ldots,n\}$ is unique. If only the block structure of the partition is given, then Π is determined up to a permutation of the partition sets. Thus if A is the matrix whose block structure is given by

$$\left[\begin{array}{c|cc} * & * & * \\ \hline * & * & * \\ * & * & * \end{array}\right],$$

then there are two choices for Π: $B_1 = \{1\}$ and $B_2 = \{2,3\}$, and $B_1 = \{2,3\}$ and $B_2 = \{1\}$.

Let A be in $\mathscr{M}_{m,\hat{m}}(\mathbb{C})$. Let U be in $\mathscr{M}_{n,\hat{n}}(\mathbb{C})$. The *inflation matrix of A by U with respect to the partition pair* $(\Pi, \hat{\Pi})$ is the $n \times \hat{n}$ matrix denoted by $A \times\times U$ which is defined as follows: For each α in $\{1,2,\ldots,n\}$ and each β in $\{1,2,\ldots,\hat{n}\}$, there exist unique indices r and s such that $\alpha \in B_r$ and $\beta \in \hat{B}_s$; let $(A \times\times U)_{\alpha\beta} = a_{rs}U_{\alpha\beta}$. Equivalently, in the block partition induced by the partition pair $(\Pi, \hat{\Pi})$, $(A \times\times U)_{\langle r,s\rangle} = a_{rs}U_{\langle r,s\rangle}$ for each r and s. When the partitions are clear, $A \times\times U$ will be called *A inflated by U*. The operation denoted by $\times\times$ is called *inflation*.

If $m = \hat{m}$, $n = \hat{n}$, and $\Pi = \hat{\Pi}$, then this definition reduces to the definition of inflation given in [2, Definition 4.1] and $A \times\times U$ is called the *inflation matrix of A by U with respect to the partition* Π.

If $m = n = 1$, then $\Pi = \{\{1\}\}$, and $A \times\times U$ is called the *inflation vector of A by U with respect to the partition* $\hat{\Pi}$.

The next lemma follows immediately from the definition of inflation.

LEMMA 2.1.1. *Let A and B be in* $\mathscr{M}_{m,\hat{m}}(\mathbb{C})$. *Let U be in* $\mathscr{M}_{n,\hat{n}}(\mathbb{C})$. *Let s be in* $\mathbb{C}$. *Then with respect to the partition pair* $(\Pi, \hat{\Pi})$

(i) $(A + B) \times\times U = (A \times\times U) + (B \times\times U)$,
(ii) $(sA) \times\times U = s(A \times\times U)$.

THEOREM 2.1.2 (Associativity of inflation). *Let p and $\hat{p}$ be positive integers such that $n \leqslant p$ and $\hat{n} \leqslant \hat{p}$. Let Ω be an n-partition of p given by the ordered collection of sets $C_1, C_2, \dots, C_n$. Let $\hat{\Omega}$ be an $\hat{n}$-partition of $\hat{p}$ given by $\hat{C}_1, \hat{C}_2, \dots, \hat{C}_n$. Let A be in $\mathscr{M}_{m,\hat{m}}(\mathbb{C})$. Let U be in $\mathscr{M}_{n,\hat{n}}(\mathbb{C})$. Let V be in $\mathscr{M}_{p,\hat{p}}(\mathbb{C})$. Then there exist an m-partition Γ of p and an $\hat{m}$-partition $\hat{\Gamma}$ of $\hat{p}$ such that*

$$(A \times\times U) \times\times V = A \times\times (U \times\times V).$$

Further, Γ is given by the ordered collection of sets $D_1, D_2, \dots, D_m$ where

$$D_i = \bigcup_{j \in B_i} C_j;$$

and $\hat{\Gamma}$ is given by the ordered collection of sets $\hat{D}_1, \hat{D}_2, \dots, \hat{D}_m$ where

$$\hat{D}_i = \bigcup_{j \in \hat{B}_i} \hat{C}_j.$$

Proof. See [6, Theorem 2.3]. ∎

2.2. *Inflators*

Suppose that $n \geqslant 2$. Let U be in $\mathscr{M}_{n,n}(\mathbb{R})$. The matrix U is called an *inflator* (*with respect to* Π) if there exist vectors u and $\tilde{u}$ in $\mathbb{R}^n$ which are partitioned by Π such that the following conditions hold:

(i) u and $\tilde{u}$ are strictly positive vectors,
(ii) For $1 \leqslant i, j \leqslant m$, $U_{\langle i,j \rangle} = [u_{\langle i \rangle}]^t [\tilde{u}_{\langle j \rangle}]$,
(iii) For $1 \leqslant i \leqslant m$, $u_{\langle i \rangle} [\tilde{u}_{\langle i \rangle}]^t = 1$.

The pair of vectors u and $\tilde{u}$ is called a *generating pair for the inflator* U. The matrix U is called a *normalized inflator* if u and $\tilde{u}$ can be chosen so that they also satisfy a fourth condition:

(iv) For $1 \leqslant i \leqslant m$, $u_{\langle i \rangle}[u_{\langle i \rangle}]^t = \tilde{u}_{\langle i \rangle}[\tilde{u}_{\langle i \rangle}]^t$.

Observe that $U = u^t[\tilde{u}]$. (These conditions are Definition 4.3 of [2].)

The 1×1 matrix $U = [0]$ is called the *inflator associated with the unique 1-partition* 1.

LEMMA 2.2.1. *Let $m \leqslant n$ be positive integers with $n \geqslant 2$. Let Π be an m-partition of n. Let U be an inflator associated with Π. Suppose that U is* *symmetric. Then U is normalized. Further, there is a strictly positive, partitioned vector u such that $U = u^t u$ and such that $u_i u_i^t = 1$ for $1 \leqslant i \leqslant m$.*

Proof. Since U is an inflator, U has a generating pair of vectors: $U = u^t v$. Since U is symmetric, $U_{\langle i,j\rangle} = [U_{\langle j,i\rangle}]^t$ for all i and j. Thus $u_i^t v_j = v_i^t u_j$. Fix i and j with $i \neq j$. Let $\lambda = u_j u_j^t$. Then $\lambda > 0$. By condition (iii), $u_j v_j^t = 1$. Thus $\lambda v_j = (u_j u_j^t) v_j = u_j(u_j^t v_j) = u_j(v_j^t u_j) = (u_j v_j^t) u_j = (1) u_j$. Similarly, if $\sigma = u_i u_i^t$, then $\sigma v_i = u_i$. So $(\sigma v_i)^t v_j = v_i^t(\lambda v_j)$. By the positivity of v, $\sigma = \lambda$. Thus $u = \lambda v$, and $\lambda = u_j u_j^t$ is independent of the choice of j. Let $\hat{u} = (\sqrt{\lambda})^{-1} u$. Let $\hat{v} = (\sqrt{\lambda}) v$. Then $\hat{u}_i^t \hat{v}_j = u_i^t v_j = U_{\langle i,j\rangle}$ for each i and j. Also, $\hat{u}_i \hat{v}_i^t = u_i v_i^t = 1$ for each i. Thus $\hat{u}$ and $\hat{v}$ are strictly positive, partitioned vectors which satisfy conditions (ii) and (iii) of the definition of an inflator. Finally, $\hat{u} = (\sqrt{\lambda})^{-1} u = (\sqrt{\lambda})^{-1} \lambda v = (\sqrt{\lambda}) v = \hat{v}$. Thus U is a normalized, symmetric inflator. Relabel $\hat{u}$ as u. ■

LEMMA 2.2.2. *Let $n \geqslant m$ be positive integers with $n \geqslant 2$. Let Π be an m-partition of n given by $B_1, B_2, \ldots, B_m$. Let U be an inflator for Π. Then U is strictly positive. Further,*

(i) *If $|B_i| = 1$, then $U_{\langle i,i\rangle} = [1]$.*
(ii) *If $|B_i| > 1$, then $0 < [U_{\langle i,i\rangle}]_{jj} < 1$ for $1 \leqslant j \leqslant |B_i|$.*
(iii) *For $1 \leqslant i, j, k \leqslant m$, $U_{\langle i,j\rangle} U_{\langle j,k\rangle} = U_{\langle i,k\rangle}$.*
(iv) *For $1 \leqslant i \leqslant m$, $U_{\langle i,i\rangle}$ is an irreducible, idempotent matrix of rank one.*
(v) *For all matrices A and B in $\mathcal{M}_m(\mathbb{C})$, $(A \times\times U)(B \times\times U) = (AB) \times\times U$.*
(vi) *For each matrix A in $\mathcal{M}_m(\mathbb{C})$,* $\operatorname{rank}(A \times\times U) = \operatorname{rank} A$.
(vii) *$(B \times\times U)_{\langle i,j\rangle} = 0$ for some i and j implies $b_{ij} = 0$.*

Proof. Results (i) and (ii) are clear from the definition of an inflator. For (iii)–(vi), see [2, Section 4]. To prove (vii) note that $U \gg 0$. ■

THEOREM 2.2.3. *Let p be a positive integer with $n \leqslant p$. Let Ω be an n-partition of p given as in the statement of Theorem 2.1.2. Let Γ be the m-partition of p derived from Π and Ω as given in Theorem 2.1.2. Let U in $\mathcal{M}_{n,n}(\mathbb{C})$ be an inflator associated with Π, and let u and $\hat{u}$ be a generating pair for U. Let V in $\mathcal{M}_{p,p}(\mathbb{C})$ be an inflator associated with Ω, and let v and $\hat{v}$ be a generating pair for V. Let $W = U \times\times V$ with respect to Ω. Then W is an inflator associated with Γ, and W has generating pair $u \times\times v$ and $\hat{u} \times\times \hat{v}$.*

Proof. See [6, Theorem 3.2]. ■

It should be noted that it has *not* been shown that $U \times\times V$ is a *normalized* inflator if U and V are normalized inflators, and indeed, the normality of U and V is not always sufficient to imply W is normal.

2.3. *The Matrix G(U)*

Let U be an inflator associated with the m-partition Π of n. For $n > 1$, define the matrix $G(U)$ by $G(U) = I_n - (I_m \times\times U)$. For $n = 1$, define $G(U)$ to be the 1×1 identity matrix. Suppose that $G(U)$ is rank k. Then U is called a *rank-k inflator*.

CONVENTION (Direct sums and block-diagonal matrices). Suppose that A is in $\mathcal{M}_n(\mathcal{F})$ for some field $\mathcal{F}$. Suppose that A is a block-diagonal matrix for Π. For $1 \leqslant i \leqslant m$, let $b_i = |B_i|$. Then there is an $n \times n$ permutation matrix P such that

$$PAP^t = \bigoplus_{i=1}^{m} U_i$$

where each U_i is $b_i \times b_i$. For each i, let

$$\hat{U}_i = \left[\bigoplus_{j=1}^{i-1} 0 \cdot I_{b_j}\right] \oplus U_i \oplus \left[\bigoplus_{j=i+1}^{m} 0 \cdot I_{b_j}\right].$$

Then PAP^t is the internal direct sum of the matrices $\hat{U}_i$. That is, $PAP^t = \sum_{i=1}^m \hat{U}_i$. So $A = \sum_{i=1}^m P^t \hat{U}_i P$. Since membership in the classes of Z- and M-matrices as well as irreducibility is preserved by permutation similarities, and since the internal direct-sum representation is very unwieldy, A will be written as $A = \oplus_{i=1}^k U_i$ with the understanding that a permutation similarity may be implied, and with the understanding from the context that operations on U_i may actually be operations on $P^t \hat{U}_i P$. *This convention will be in effect throughout this paper.*

EXAMPLE. Let A be the matrix

$$A = \begin{bmatrix} 3 & 0 & 2 & 0 & 1 \\ 0 & 5 & 0 & 4 & 0 \\ 2 & 0 & 1 & 0 & 2 \\ 0 & 4 & 0 & 5 & 0 \\ 1 & 0 & 2 & 0 & 3 \end{bmatrix}.$$

Then A can be written according to the preceding convention as

$$A = \begin{bmatrix} 3 & 2 & 1 \\ 2 & 1 & 2 \\ 1 & 2 & 3 \end{bmatrix} \oplus \begin{bmatrix} 5 & 4 \\ 4 & 5 \end{bmatrix}.$$

REMARK. In light of the preceding convention, $G(U)$ defined above can be written as

$$G(U) = I_n - \bigoplus_{i=1}^{m} U_{\langle i, i \rangle} = \bigoplus_{i=1}^{m} \left[I_{b_i} - U_{\langle i, i \rangle} \right]. \tag{2.3.1}$$

That is, $G(U)$ is (permutation similar to) a block-diagonal matrix with $G_{\langle i, i \rangle}$ equal to $I_{b_i} - U_{\langle i, \rangle}$ for each i.

LEMMA 2.3.2. *Let $n \geqslant m$ be positive integers with $n \geqslant 2$. Let Π be an m-partition of n given by $B_1, B_2, \ldots, B_m$. Let U be an inflator associated with Π. Then the following properties hold:*

(i) *If $|B_i| = 1$, then $G(U)_{\langle i, i \rangle} = [0]$.*
(ii) *If $|B_i| > 1$, then $G(U)_{\langle i, i \rangle}$ has a strictly positive diagonal, and each off-diagonal entry of $G(U)_{\langle i, i \rangle}$ is negative.*
(iii) *$G(U)$ is an idempotent M-matrix which is completely reducible with index of reducibility m.*
(iv) *The rank of $G(U)$ is $n - m$.*
(v) *For each i, $G(U)_{\langle i, i \rangle}$ is an idempotent, singular, irreducible M-matrix for which zero is a simple eigenvalue.*
(vi) *$G(U)$ is diagonalizable.*
(vii) *There is a unique normalized inflator $\hat{U}$ associated with Π such that $G(U) = G(\hat{U})$.*
(viii) *Suppose that $G(U) = G(V)$, where V is an inflator associated with some m'-partition Π' of n'. Then $\Pi' = \Pi$.*
(ix) *Suppose that N is in $\mathcal{M}_n(\mathbb{C})$. Then $NG(U) = G(U)N = 0$ if and only if $N = A \times\times U$ for some A in $\mathcal{M}_m(\mathbb{C})$.*

Proof. Properties (i) and (ii) follow immediately from Lemma 2.2.2. Properties (iii)–(v) are in Lemma 5.5 of [2]. Since $G(U)$ is idempotent, its minimal polynomial has linear factors. Thus $G(U)$ is similar to a diagonal matrix; hence (vi) holds. Properties (vii) and (viii) are Lemma 4.16 of [2]. Property (ix) is Lemma 4.23 of [2]. ∎

REMARK. If $n = 1$, then $G(U) = [1]$, which is an idempotent, irreducible, nonsingular M-martix.

LEMMA 2.3.3. *Let G be in $\mathcal{M}_n(\mathbb{R})$. Suppose that $G = \oplus_{i=1}^{m} G_i$, where the submatrices G_i are idempotent, singular, irreducible M-matrices. Then there exists an m-partition Π of n which is unique up to a permutation of the partition set labels, and there exists a unique, normalized inflator U associated with Π such that $G = G(U)$ and such that $G_i = G(U)_{\langle i,i \rangle}$ for $1 \leqslant i \leqslant m$.*

Proof. See [2, Lemma 5.5]. ∎

2.4. *Inflation Sequences and Complete Sets of Projectors*

It will be necessary to distinguish between sets of matrices and sequences of matrices. Consequently the following notation is adopted:

NOTATION. The script letters $\mathcal{E}$ and $\mathcal{F}$ will always be sets of matrices, usually sets of projectors. If $\mathcal{E}$ has k elements, then they will be routinely labeled as E_i for $1 \leqslant i \leqslant k$. The notation $\{U_i\}_{i=1}^{k}$ will always denote a sequence of matrices, usually inflators. By convention, every set of k matrices will have k pairwise distinct elements.

Let $n_0, n_1, n_2, \ldots, n_k$ be a sequence of integers such that $n_0 = 0$ and $1 = n_1 < n_2 < \cdots < n_k = n$. For $1 < i \leqslant k$, let $P_{i-1,i}$ be an n_{i-1}-partition of n_i. Let $U_1 = [0]$, the 1×1 zero matrix. For $1 < i \leqslant k$, let U_i be an inflator associated with $P_{i-1,i}$. The sequence $\{U_i\}_{i=1}^{k}$ is called an *inflation sequence.* If each of the inflators U_i is normalized for $1 < i \leqslant k$, then the sequence is called a *normalized inflation sequence.*

LEMMA 2.4.1. *Let $\{U_i\}_{i=1}^{k}$ be an inflation sequence. For h with $1 \leqslant h \leqslant k$, let $W_h = U_h \times\times \cdots \times\times U_k$, and let $T_h = G(U_h) \times\times U_{h+1} \times\times \cdots \times\times U_k$. Then W_h is a strictly positive matrix for $h \geqslant 2$, and the diagonal entries of T_h are nonnegative for each h. Further, for $h < k$, the matrix T_h has nonnegative diagonal blocks when given the partition of W_{h+1}, hence also when given the partition of U_k, since the partition of U_k subdivides that of W_{h+1}.*

Proof. Since U_i is strictly positive for $i \geqslant 2$, W_h is strictly positive for $h \geqslant 2$. By Lemma 2.3.2, $G(U_h)$ has a nonnegative diagonal for each h. If $h = k$, then the result for T_h is clear. If $h < k$, then $T_h = G(U_h) \times\times W_{h+1}$. Since the diagonal entries of $G(U_h)$ are nonnegative, and since W_{h+1} is strictly positive, T_h has nonnegative diagonal *blocks* in the block partition of W_{h+1}. Clearly, T_h has a nonnegative diagonal. ■

Let $\mathscr{E}$ be a set of k pairwise distinct matrices in $\mathscr{M}_n(\mathbb{C})$, where $n \geqslant k$. Label the elements of $\mathscr{E}$ as E_i for $1 \leqslant i \leqslant k$. The set $\mathscr{E}$ is a *set of projectors* if

(i) $E_i \neq 0$ for $1 \leqslant i \leqslant k$,
(ii) $E_i E_j = \delta_{ij} E_i$ for $1 \leqslant i, j \leqslant k$.

If $\mathscr{E}$ is a set of projectors, each matrix E_i is a *projector*. The set $\mathscr{E}$ is a *complete set of projectors* if in addition to satisfying conditions (i) and (ii),

$$\sum_{i=1}^{k} E_i = I_n.$$

If E is a projector, then fix(E) will denote the range of E. That is, fix(E) is the subspace upon which E acts as the identity.

LEMMA 2.4.2. *Let A be in $\mathscr{M}_n(\mathbb{C})$. Suppose that A is diagonalizable and that the distinct eigenvalues of A are $\lambda_1, \lambda_2, \dots, \lambda_k$. Then there is a complete set of projectors $\mathscr{E} = \{E_i : 1 \leqslant i \leqslant k\}$ such that A has spectral decomposition given by* (1.1). *Conversely, if A can be expressed as in* (1.1) *where the λ_i are distinct complex numbers and where $\{E_i : 1 \leqslant i \leqslant k\}$ is a complete set of projectors, then A is diagonalizable and the eigenvalues of A are precisely the λ_i.*

Proof. This is a standard result. See [3, Vol. I, p. 41]. ■

NOTATION. If $\{U_i\}_{i=1}^k$ is an inflation sequence, we will adopt the convention that $G(U_i) \times\times U_{i+1} \times\times \cdots \times\times U_k = G(U_k)$ when $i = k$. For $1 \leqslant i \leqslant k$, let $E_i = G(U_i) \times\times U_{i+1} \times\times \cdots \times\times U_k$. Let $\mathscr{E}$ denote the set $\mathscr{E} = \{E_i : 1 \leqslant i \leqslant k\}$.

LEMMA 2.4.3. *Let $\{U_i\}_{i=1}^k$ be an inflation sequence. For $1 \leqslant i \leqslant k$, the $n \times n$ matrix E_i in $\mathscr{E}$ is an idempotent matrix of rank $n_i - n_{i-1}$. Further, $E_i E_j = 0$ whenever $i \neq j$. Consequently $\mathscr{E}$ is a complete set of projectors.*

Proof. Since inflation is associative by Theorem 2.1.2, and since the inflation product of inflators is an inflator by Theorem 2.2.3, it follows that $E_i = G(U_i) \times\times [U_{i+1} \times\times \cdots \times\times U_k]$ is an idempotent matrix with the given rank by Lemmas 2.2.2(iv) and 2.3.2(iv).

Without loss of generality suppose that $i<j$. Let $F_i=G(U_i)\times\times U_{i+1}\times\times\cdots\times\times U_{j-1}$. If $j<k$, let $W=U_{j+1}\times\times\cdots\times\times U_k$. Then $E_i=F_i\times\times U_j\times\times W$, and $E_j=G(U_j)\times\times W$. If $j=k$, then suppress the symbols "$\times\times W$" in the following arguments. By Theorem 2.2.3, W is an inflator; hence by Lemma 2.2.2(ii), $E_iE_j=[(F_i\times\times U_j)G(U_j)]\times\times W$. By Lemma 2.3.2(ix), $(F_i\times\times U_j)G(U_j)=0$. Consequently, $E_iE_j=0$. A similar argument shows that $E_jE_i=0$. Since the elements of $\mathscr{E}$ are pairwise orthogonal idempotents, the rank of their sum is the sum of their ranks. Since the sequence of ranks is a telescoping sequence, the rank of the sum is $n_k-n_0=n$. Since the sum of pairwise orthogonal idempotents is an idempotent, the sum in the statement of the corollary is an $n\times n$ idempotent of rank n. There is only one such matrix: I_n. ■

A complete set of projectors $\mathscr{E}$ is called a complete set of *inflation-generated* projectors if there exists an inflation sequence $\{U_p\}_{p=1}^k$ such that $\mathscr{E}=\{G(U_p)\times\times U_{p+1}\times\times\cdots\times\times U_k : 1\leqslant p\leqslant k\}$. Note that if the elements of $\mathscr{E}$ are labeled *a priori* as E_i for $1\leqslant i\leqslant k$, this does not imply $E_i=G(U_i)\times\times U_{i+1}\times\times\cdots\times\times U_k$ for any value of i. Rather, it implies that there exists a permutation σ on the set $\{1,2,\ldots,k\}$ such that $E_{\sigma(i)}=G(U_i)\times\times\cdots\times\times U_k$ for all i.

THEOREM 2.4.4. *Let $\mathscr{E}$ be a complete set of inflation-generated projectors. Then there is a normalized inflation sequence for $\mathscr{E}$.*

Proof. The principal steps in the proof consist of iteratively constructing a normalized inflation sequence from the original inflation sequence, and then verifying that the normalized inflation sequence generates the same set of projectors as the original sequence does. The verification that the sets of projectors are the same is straightforward, and the full details of the proof are in [7].

Suppose that u is an inflator with respect to a partition Π with generating pair u and $\hat{u}$. Since u and $\hat{u}$ are strictly nonzero vectors, there exist m unique, positive numbers λ_i which satisfy

$$u_{\langle i\rangle}[u_{\langle i\rangle}]^*=(\lambda_i)^2\hat{u}_{\langle i\rangle}[\hat{u}_{\langle i\rangle}]^*$$

for $1\leqslant i\leqslant m$. Let $D(U)$ be the nonsingular, $m\times m$ diagonal matrix

$$D(U)=\operatorname{diag}(\lambda_1,\lambda_2,\ldots,\lambda_m).$$

$D(U)$ is called the normalizer of U. Let V be the matrix defined by

$$V = \left[D(U) \times\times I_n\right]^{-1} U \left[D(U) \times\times I_n\right].$$

V is called the normalization of U. It is a straightforward computation to verify that V is an inflator such that $G(V) = G(U)$; and that if $A \times\times U$ is defined, then $A \times\times U = [D(U)]A[D(U)]^{-1} \times\times V$.

Suppose that $\{U_i\}_{i=1}^{k}$ is an inflation sequence which is not a normalized inflation sequence. The following algorithm constructs the desired normalized inflation sequence $\{V_i\}_{i=1}^{k}$:

(1) Let $D^{(k)}$ be the normalizer of U_k.
(2) Let V_k be the normalization of U_k.
(3) For $i = k-1, k-2, \ldots, 2$, let $D^{(i)}$ be the normalizer of the matrix $D^{(i+1)}U_i[D^{(i+1)}]^{-1}$.
(4) For $i = k-1, k-2, \ldots, 2$, let V_i be the normalization of the matrix $D^{(i+1)}U_i[D^{(i+1)}]^{-1}$.
(5) *Let* $V_1 = U_1$, *the* 1×1 *zero matrix.* ■

LEMMA 2.4.5. *Let $\mathscr{E}$ be a complete set of $n \times n$ inflation-generated projectors. Let $\{U_i\}_{i=1}^{k}$ be an inflation sequence for $\mathscr{E}$. Let α and β be fixed indices with $1 \leqslant \alpha, \beta \leqslant n$. Then there exists a unique sequence of entries $\{u_i\}_{i=1}^{k}$ with u_i from U_i for each i such that for every E in $\mathscr{E}$, $E_{\alpha\beta}$ can be expressed in terms of the u_i. In particular, if $E = G(U_h) \times\times U_{h+1} \times\times \cdots \times\times U_k$, then*

$$E_{\alpha\beta} = \begin{cases} (1-u_h)u_{h+1}\cdots u_k & \text{if } E_{\alpha\beta} > 0, \\ 0 \cdot u_{h+1}\cdots u_k & \text{if } E_{\alpha\beta} = 0, \\ -u_h U_{h+1}\cdots u_k & \text{if } E_{\alpha\beta} < 0. \end{cases}$$

Proof. The existence and uniqueness of the sequence $\{u_i\}_{i=1}^{k}$ is a consequence of the definition of inflation and of the construction of $\mathscr{E}$ from $\{U_i\}_{i=1}^{k}$. Since $U_h \times\times \cdots \times\times U_k$ is strictly positive, and since the structure of $G(U_h)$ is given by Lemma 2.3.2, the structure of $E_{\alpha\beta}$ is as specified. ■

2.5. *The Compression Theorem*

A natural operation on a set of projectors is that of *compression*, the summation of several projectors into a single, higher-rank projector.

LEMMA 2.5.1. *Let p be a positive integer with $n \leqslant p$. Let Ω be an n-partition of p given as in the statement of Theorem 2.1.2. Let Γ be the m-partition of p derived from Π and Ω as given in Theorem 2.1.2. Let U in $\mathscr{M}_{n,n}(\mathbb{C})$ be an inflator associated with Π. Let V in $\mathscr{M}_{p,p}(\mathbb{C})$ be an inflator associated with Ω. Let $W = U \times\times V$ with respect to Ω. Then the inflator W satisfies*:

(i) $G(W) = G(U) \times\times V + G(V)$,
(ii) $\operatorname{rank}[G(W)] = \operatorname{rank}[G(U)] + \operatorname{rank}[G(V)]$.

Proof of (ii). From Theorem 2.2.3, W is an inflator associated with an m-partition of p. By Lemma 2.3.2, $\operatorname{rank}[G(W)] = p - m$, $\operatorname{rank}[G(U)] = n - m$, and $\operatorname{rank}[G(V)] = p - n$. ∎

Proof of (i). Throughout, the notation from Theorem 2.2.3 will be adopted. Thus the partition Π for U is expressed in terms of sets B_i, the partition Γ for V is expressed in terms of sets C_i, and the partition Ω for W is expressed in terms of sets D_i (with $d_i = |D_i|$ for each i). Blocks with respect to Ω will be denoted by $\langle\ ,\ \rangle$, while blocks with respect to Γ will be denoted by $\langle\ ,\ \rangle^*$. Let u and $\hat{u}$ be a generating pair for U; v and $\hat{v}$ for V; and w and $\hat{w}$ for W. Observe that

$$[G(W)]_{\langle i,j\rangle} = \begin{cases} 0 & \text{if} \quad i \neq j, \\ I_{d_i} - W_{\langle i,i\rangle} & \text{if} \quad i = j. \end{cases}$$

Since the block partitioning of V subpartitions the block partitioning of W, and since $[G(V)]_{\langle i,j\rangle^*} = 0$ if $i \neq j$,

$$[G(V)]_{\langle i,j\rangle} = \begin{cases} 0 & \text{if} \quad i \neq j, \\ \bigoplus_{\alpha \in B_i} [G(V)]_{\langle \alpha,\alpha\rangle^*} & \text{if} \quad i = j. \end{cases} \tag{2.5.2}$$

Additionally, $[G(U) \times\times V]_{\langle i,j\rangle}$ subpartitions into blocks of the form

$$[G(U) \times\times V]_{\langle \alpha,\beta\rangle^*} = [G(U)_{\alpha\beta}] V_{\langle \alpha,\beta\rangle^*}$$

for each α in B_i and each β in B_j. Since $G(U)_{\alpha\beta} = 0$ unless $i = j$,

$$[G(U) \times\times V]_{\langle i,j\rangle} = 0 \qquad \text{if} \quad i \neq j.$$

Thus, in proving (i), it suffices to show that for $1 \leqslant i \leqslant m$,

$$[G(W)]_{\langle i,i\rangle} = [G(U) \times\times V]_{\langle i,i\rangle} + [G(V)]_{\langle i,i\rangle}.$$

Fix i. Subpartition the $\langle i,i\rangle$ block into $\langle r,s\rangle^*$ blocks where $1 \leqslant r, s \leqslant |B_i|$. It suffices to prove that for each r and s,

$$(2.5.3) \qquad [G(W)]_{\langle r,s\rangle^*} = [G(U) \times\times V]_{\langle r,s\rangle^*} + [G(V)]_{\langle r,s\rangle^*}.$$

First, suppose that $r = s$. Since $W = w^t[\hat{w}]$, Equation (2.5.2) becomes

$$I_{d_i} - [w_{\langle i\rangle}]^t [\hat{w}_{\langle i\rangle}].$$

Thus

$$[G(W)]_{\langle r,r\rangle^*} = I_{b_r} - [u_r \cdot v_{\langle r\rangle}]^t [\hat{u}_r \cdot \hat{v}_{\langle r\rangle}] = I_{b_r} - [u_r \hat{u}_r] \cdot V_{\langle r,r\rangle^*}.$$

The $\langle r,r\rangle^*$ subblock of $[G(U) \times\times V]$ is

$$[G(U) \times\times V]_{\langle r,r\rangle^*} = G(U)_{rr} \cdot V_{\langle r,r\rangle^*} = (1 - u_r \hat{u}_r) V_{\langle r,r\rangle^*}.$$

Finally, the $\langle r,r\rangle^*$ subblock of $G(V)$ is

$$[G(V)]_{\langle r,r\rangle^*} = I_{b_r} - V_{\langle r,r\rangle^*}.$$

Clearly, (2.5.3) holds when $r = s$.

Suppose that $r \neq s$. Note that $G(V)_{\langle r,s\rangle^*} = 0$ is immediate from the definition of $G(V)$. Since $r \neq s$, the $\langle r,s\rangle^*$ subblock is an off-diagonal subblock. Thus

$$[G(W)]_{\langle r,s\rangle^*} = -W_{\langle r,s\rangle^*} = -[u_r \cdot v_{\langle r\rangle}]^t [\hat{u}_s \cdot \hat{v}_{\langle s\rangle}] = -[u_r \hat{u}_s] \cdot V_{\langle r,s\rangle^*}.$$

The $\langle r,s\rangle^*$ subblock of $G(U) \times\times V$ is

$$[G(U) \times\times V]_{\langle r,s\rangle^*} = G(U)_{rs} \cdot V_{\langle r,s\rangle^*} = -[u_r \hat{u}_s] \cdot V_{\langle r,s\rangle^*}.$$

Thus (2.5.3) holds when $r \neq s$. ■

THEOREM 2.5.4 (The compression theorem). *Let $\mathscr{E}$ be a complete set of inflation-generated projectors with $k = |\mathscr{E}| \geqslant 3$. Let $\{U_i\}_{i=1}^{k}$ be an inflation sequence for $\mathscr{E}$. Label the elements of $\mathscr{E}$ so that $E_i = G(U_i) \times\times \cdots \times\times U_k$ for each i. Suppose that j satisfies $2 \leqslant j \leqslant k-1$. Let $\mathscr{E}' = (\mathscr{E} \setminus \{E_j, E_{j+1}\}) \cup \{E_j + E_{j+1}\}$. Then $\mathscr{E}'$ is a complete set of inflation-generated projectors with inflation sequence $\{V_i\}_{i=1}^{k-1}$, where*

$$V_i = \begin{cases} U_i & \text{if } i < j, \\ U_j \times\times U_{j+1} & \text{if } i = j, \\ U_{i+1} & \text{if } i > j. \end{cases}$$

Finally,

$$\operatorname{rank}\left[E_j + E_{j+1}\right] = \operatorname{rank}\left[E_j\right] + \operatorname{rank}\left[E_{j+1}\right].$$

Proof. Clearly $\mathscr{E}'$ is a complete set of projectors. By Lemma 2.5.1, V_j is an inflator associated with a k-partition of m, where k is the block order of U_j and m is the order of U_{j+1}. Since $V_{j-1} = U_{j-1}$, the order of V_{j-1} is k. Since $V_{j+1} = U_{j+2}$, the block order of V_{j+1} is m. Thus $\{V_i\}_{i=1}^{k-1}$ is an inflation sequence. By the associativity of $\times\times$, $E_i = G(V_i) \times\times V_{i+1} \times\times \cdots \times\times V_{k-1}$ for $i < j$, and $E_{i+1} = G(V_i) \times\times V_{i+1} \times\times \cdots \times\times V_{k-1}$ for $i > j$. Finally, by Lemma 2.5.1,

$$E_j + E_{j+1} = G(V_j) \times\times V_{j+1} \times\times \cdots \times\times V_{k-1}.$$

The rank statement is a consequence of the pairwise orthogonality of the E_i in $\mathscr{E}$. ■

2.6. *Inflation-Generated Matrices*

The matrix A in $\mathscr{M}_{n,n}(\mathbb{C})$ is called an *inflation-generated matrix* if A can be expressed as in (1.1) such that the set $\{E_i : 1 \leqslant i \leqslant k\}$ is inflation-generated.

THEOREM 2.6.1. *Let A be an $n \times n$ ZME-matrix with distinct eigenvalues $\alpha_1, \alpha_2, \ldots, \alpha_k$. Then the eigenvalues of A are real, and without loss of generality, $\alpha_1 < \alpha_2 < \cdots < \alpha_k$. The eigenvalue α_1 is simple with $|\alpha_1| \leqslant \alpha_2$. Further,*

(i) $\alpha_1 = -\alpha_2$ *if and only if A is a ZMO-matrix,*
(ii) $\alpha_1 > -\alpha_2$ *if and only if A is a ZMA-matrix,*
(iii) $0 \leqslant \alpha_1$ *if and only if A is an MMA-matrix.*

Proof. See [2, Lemma 3.1; Theorems 3.6., 3.7, 3.9.]. ■

THEOREM 2.6.2. *Let A be a ZM-matrix. If some positive power of A is irreducible, and some other positive power is reducible, then A is a ZMO-matrix. If some positive, even power of A is irreducible, then A is a ZMA-matrix. If some positive, odd power of A is an M-matrix, and if some positive power of A is irreducible, then A is an MMA-matrix.*

Proof. See [2, Theorems 3.6, 3.7, 3.9]. ■

THEOREM 2.6.3 (The inflation theorem). *Let $k \leqslant n$ be positive integers with $n \geqslant 2$. Let A be in $\mathscr{M}_n(\mathbb{R})$. Let λ_p be in $\mathbb{R}$ for $1 \leqslant p \leqslant k$ such that $\lambda_1 < \lambda_2 < \cdots < \lambda_k$ and $|\lambda_1| \leqslant \lambda_2$. Then the following are equivalent:*

(i) *A is a ZME-matrix with distinct eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_k$;*

(ii) *there exist uniquely: a sequence of integers n_p for $1 \leqslant p \leqslant k$ such that $1 = n_1 < \cdots < n_k = n$, a sequence of n_{p-1}-partitions P_{n_{p-1}, n_p} of n_p for $2 \leqslant p \leqslant k$, and a sequence of normalized inflators U_p associated with P_{n_{p-1}, n_p} for $2 \leqslant p \leqslant k$, such that A can be expressed as in* (1.1), *where $E_i = G(U_i) \times\times U_{i+1} \times\times \cdots \times\times U_k$ for $1 \leqslant i \leqslant k$.*

Proof. See [2, Theorem 6.18, Corollary 6.25]. ■

COROLLARY 2.6.4. *Suppose that condition* (ii) *of Theorem* 2.6.3 *holds. Let $\mathscr{E}$ be the set $\{E_p\colon 1 \leqslant p \leqslant k\}$ whose elements are defined as in Theorem* 2.6.3. *Then $\mathscr{E}$ is a complete set of projectors, and* (1.1) *is the spectral decomposition for A.*

Proof. See [2, Corollary 6.25]. ■

If the sequence $\{U_p\}_{p=2}^k$ is associated with a ZME-matrix A through Theorem 2.6.3, then $\{U_p\}_{p=1}^k$ is called an *inflation sequence for* A. Note that by Theorem 2.6.3, every ZME-matrix has a normalized inflation sequence.

LEMMA 2.6.5. *Let c be a fixed, positive, real number. For each real number ϵ with $c > \epsilon > 0$, let $\{\alpha_1(\epsilon), \alpha_2(\epsilon), \dots, \alpha_k(\epsilon)\}$ be a set of k distinct real numbers such that $|\alpha_1(\epsilon)| \leqslant \alpha_i(\epsilon)$ for $i \geqslant 2$. For each ϵ, define the matrix $A(\epsilon)$ by*

$$A(\epsilon) = \sum_{i=1}^{k} \alpha_i(\epsilon) E_i.$$

Suppose that $\lim_{\epsilon \to 0^+} \alpha_i(\epsilon) = \alpha_i$ *for each* i, *and that* $\alpha_1 < \alpha_i$ *for* $i \geqslant 2$. *Let* $A = \sum_{i=1}^{k} \alpha_i E_i$. *If* $A(\epsilon)$ *is a ZME-matrix for all* ϵ *with* $c > \epsilon > 0$, *then* A *is a ZME-matrix.*

Proof. Since $A = \lim_{\epsilon \to 0^+} A(\epsilon)$, it follows that $A^n = \lim_{\epsilon \to 0^+} [A(\epsilon)]^n$ for all positive intengers n. Since the limit of a sequence of Z-matrices is a Z-matrix, it follows that A^n is a Z-matrix for all positive integers n. Thus A is a *ZM*-matrix. For each ϵ with $c > \epsilon > 0$, the *ZME*-matrix $A(\epsilon)$ has simple, minimal eigenvalue $\alpha_1(\epsilon)$. Consequently, E_1 must be the unique strictly positive element of $\mathscr{E}$ and E_1 is of rank one. Thus E_1, and hence A, has strictly positive row and column eigenvectors. Noting that the Z-matrix A can be expressed as $sI - B$ for some real number s and some nonnegative matrix B, it follows that B has strictly positive row and column eigenvectors. Since $\alpha_1 = s - \rho(B)$, where $\rho(B)$ is the spectral radius of B, the simplicity of α_1 implies the simplicity of $\rho(B)$ in the spectrum of B. By Corollary 3.3.15 of [1, p. 42], B is irreducible. Thus A is an irreducible *ZM*-matrix. Now apply Theorem 2.6.2. ■

2.7. *An Example*

The following is an example of an inflation sequence and a complete set of inflation-generated projectors for a *ZME*-matrix.

Let $\Pi_2 = \{\{1,2\}\}$. Let $\Pi_3 = \{\{1,2\},\{3\}\}$. Let $\Pi_4 = \{\{1\},\{2\},\{3,4\}\}$. Let $\{U_p\}_{p=1}^{4}$ be the following normalized inflation sequence corresponding to this sequence of partitions:

$$U_1 = [0],$$

$$U_2 = \frac{1}{3}\begin{bmatrix} 2 & \sqrt{2} \\ \sqrt{2} & 1 \end{bmatrix},$$

$$U_3 = \frac{1}{2}\left[\begin{array}{cc|c} 1 & 1 & \sqrt{2} \\ 1 & 1 & \sqrt{2} \\ \hline \sqrt{2} & \sqrt{2} & 2 \end{array}\right],$$

$$U_4 = \frac{1}{2}\left[\begin{array}{c|c|cc} 2 & 2 & \sqrt{2} & \sqrt{2} \\ \hline 2 & 2 & \sqrt{2} & \sqrt{2} \\ \hline \sqrt{2} & \sqrt{2} & 1 & 1 \\ \sqrt{2} & \sqrt{2} & 1 & 1 \end{array}\right].$$

The corresponding projectors are

$$E_1 = G(U_1) \times\times U_2 \times\times U_3 \times\times U_4 = \frac{1}{6}\begin{bmatrix} 2 & 2 & \sqrt{2} & \sqrt{2} \\ 2 & 2 & \sqrt{2} & \sqrt{2} \\ \sqrt{2} & \sqrt{2} & 1 & 1 \\ \sqrt{2} & \sqrt{2} & 1 & 1 \end{bmatrix},$$

$$E_2 = G(U_2) \times\times U_3 \times\times U_4 = \frac{1}{6}\begin{bmatrix} 1 & 1 & -\sqrt{2} & -\sqrt{2} \\ 1 & 1 & -\sqrt{2} & -\sqrt{2} \\ -\sqrt{2} & -\sqrt{2} & 2 & 2 \\ -\sqrt{2} & -\sqrt{2} & 2 & 2 \end{bmatrix},$$

$$E_3 = G(U_3) \times\times U_4 = \frac{1}{2}\begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

$$E_4 = G(U_4) = \frac{1}{2}\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix}.$$

The set $\{E_p : 1 \leqslant p \leqslant 4\}$ is a complete set of inflation-generated projectors. Let α_1, α_2, α_3, and α_4 be distinct real numbers with $\alpha_1 < \alpha_2 < \alpha_3 < \alpha_4$ and $|\alpha_1| \leqslant \alpha_2$. Then $A = \Sigma_{p=1}^{4} \alpha_p E_p$ is a *ZME*-matrix. As a particular example,

$$A = 0E_1 + 3E_2 + 6E_3 + 8E_4 = \frac{1}{2}\begin{bmatrix} 7 & -5 & -\sqrt{2} & -\sqrt{2} \\ -5 & 7 & -\sqrt{2} & -\sqrt{2} \\ -\sqrt{2} & -\sqrt{2} & 10 & -6 \\ -\sqrt{2} & -\sqrt{2} & -6 & 10 \end{bmatrix}$$

is a singular *MMA*-matrix with spectrum $\{0, 3, 6, 8\}$. ■

2.8. *Decomposition*

In this section, it is shown that a complete set of inflation-generated projectors has an inflation-generated refinement consisting of rank-one projectors.

LEMMA 2.8.1. *Suppose that* $\{U_i\}_{i=1}^{k}$ *is an inflation sequence with corresponding complete set of inflation-generated projectors* $\mathscr{F}$. *Suppose that for some* j, $G(U_j)$ *has rank* r *with* $r \geqslant 2$. *Then there exist inflators* $V_1, V_2, \ldots, V_r$ *such that* $U_j = V_1 \times\times V_2 \times\times \cdots \times\times V_r$, *and such that* $G(V_i)$ *is rank one for each* i. *Let* $\{W_i\}_{i=1}^{k+r-1}$ *be the sequence such that*

$$W_i = \begin{cases} U_i & \text{if } 1 \leqslant i < j, \\ V_{i-j+1} & \text{if } j \leqslant i < j+r, \\ U_{i-r+1} & \text{if } j+r \leqslant i \leqslant k+r-1. \end{cases}$$

Then $\{W\}_{i=1}^{k+r-1}$ *is an inflation sequence. Further, the corresponding complete set of inflation-generated projectors is*

$$\mathscr{E} = \left[\mathscr{F} \setminus \left\{G(W_j) \times\times W_{j+1} \times\times \cdots \times\times W_k\right\}\right]$$
$$\cup \left\{G(V_i) \times\times \cdots \times\times V_r \times\times W_{j+1} \times\times \cdots \times\times W_k : 1 \leqslant i \leqslant r\right\}.$$

Proof. This follows from Theorem 5.2 of [8] once it has been shown that the V_i are strictly positive. Since U_j is strictly positive, it follows that U_j has a generating pair of strictly positive vectors. By examining the proof of Theorem 4.1 of [8], it follows that the inflators $V_1, V_2, \ldots, V_r$ are strictly positive. ∎

Let $\mathscr{F}$ be a complete set of projectors with $|\mathscr{F}| = m$. Let $\mathscr{E}$ be a complete set of projectors with $|\mathscr{E}| = n$. Suppose that

$$\mathscr{F} = \left\{ \sum_{i \in B_j} E_i : 1 \leqslant i \leqslant m \right\},$$

where the elements of $\mathscr{E}$ are labeled as $E_1, E_2, \ldots, E_n$ and where the sets $B_1, B_2, \ldots, B_m$ form an m-partition of n. Then $\mathscr{E}$ is called a *refinement of* $\mathscr{F}$. Further, if each element of $\mathscr{E}$ is a rank-one projector, then $\mathscr{E}$ is called a *rank-one refinement of* $\mathscr{F}$.

THEOREM 2.8.2 (The decomposition theorem). *Let* $\mathscr{F}$ *be a complete set of* $n \times n$ *inflation-generated projectors with* $|\mathscr{F}| = m$. *Then there exists a complete set of inflation-generated projectors* $\mathscr{E}$ *which is a rank-one refinement of* $\mathscr{F}$.

Proof. If $n = m$, let $\mathscr{E} = \mathscr{F}$. If $m < n$, then this follows from repeated applications of Lemma 2.8.1 to the set $\mathscr{F}$. ∎

2.9. The Original Slide-Around Theorem

The following theorme is the first of several theorems which discuss how perturbation of the eigenvalues of a *ZME*-matrix affects the property of being a *ZME*-matrix.

THEOREM 2.9.1 (The original slide-around theorem). *Let A be in* $\mathscr{M}_n(\mathbb{R})$. *Suppose that A is a ZME-matrix with distinct eigenvalues* $\alpha_1, \alpha_2, \dots, \alpha_k$ *such that* $\alpha_1 < \alpha_2 < \cdots < \alpha_k$ *and* $|\alpha_1| \leqslant \alpha_2$. *Suppose that A has spectral decomposition given by* (1.1). *Let* $\beta_1, \beta_2, \dots, \beta_k$ *be real numbers satisfying* $\beta_1 < \beta_2 < \cdots < \beta_k$ *and* $|\beta_1| \leqslant \beta_2$. *Let* $B = \sum_{p=1}^{k} \beta_p E_p$. *Then*:

(i) *B is a ZMO-matrix if and only if* $\beta_1 = -\beta_2$;
(ii) *B is a ZMA-matrix if and only if* $|\beta_1| < \beta_2$;
(iii) *B is an MMA-matrix if and only if* $0 \leqslant \beta_1$.

Proof. See [2, Corollary 6.28]. ■

It is important to observe that Theorems 2.6.3 and 2.9.1 do not rule out the possibility that there exist real numbers $\gamma_1, \gamma_2, \dots, \gamma_k$ which do not satisfy $\gamma_1 < \gamma_2 < \cdots < \gamma_k$, but for which $C = \sum_{p=1}^{k} \gamma_p E_p$ is a *ZME*-matrix. Indeed, in the example of the preceding subsection, A is still a *ZME*-matrix when $\alpha_2 \leqslant \alpha_4 \leqslant \alpha_3$.

3. THE COMBINATORIAL PARTIAL ORDER $\preccurlyeq$.

Let E and F be in $\mathscr{M}_n(\mathbb{R})$. Denote the entries of E by e_{pq} and the entries of F by f_{pq}. The matrices E and F are *comparable* (*with respect to* $\prec$), denoted by $E \prec F$, if there exist indices i and j such that $e_{ij} > 0$ and $f_{ij} < 0$. Define E and F to be *noncomparable* (*with respect to* $\prec$), denoted E *and* F $\prec$-NC, if $e_{ij} f_{ij} \geqslant 0$ for all i and j. Clearly E and F are $\prec$-NC if and only if neither of $E \prec F$ and $F \prec E$ hold. Finally, $E \preccurlyeq F$ means one of $E = F$ and $E \prec F$ holds.

EXAMPLE. Let

$$E = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \quad \text{and} \quad F = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}.$$

Then $E \prec F$ and $F \prec E$ both hold.

Let α be in $\mathbb{R}$. Define $\operatorname{sign}\alpha$ by

$$\operatorname{sign}\alpha = \begin{cases} 1 & \text{if } \alpha > 0, \\ 0 & \text{if } \alpha = 0, \\ -1 & \text{if } \alpha < 0. \end{cases}$$

If A is in $\mathscr{M}_n(\mathbb{R})$ and $\operatorname{sign} a_{ij} = \operatorname{sign} a_{11}$ for $1 \leqslant i, j \leqslant n$, then define $\operatorname{sign} A$ to be $\operatorname{sign} a_{11}$.

LEMMA 3.1. *Let E and F be in $\mathscr{M}_m(\mathbb{R})$. Let U be an inflator associated with an m-partition $P_{m,n}$ of n for some $n \geqslant m$ with $n \geqslant 2$. Then:*

(i) *$E \prec F$ if and only if $(E \times\times U) \prec (F \times\times U)$, and*

(ii) *E and F are $\prec$-NC if and only if $(E \times\times U)$ and $(F \times\times U)$ are $\prec$-NC.*

Proof of (i). Since U is an inflator with $n \geqslant 2$, U is strictly positive. Thus for each i and j, $\operatorname{sign} U_{\langle i,j\rangle} = 1$. Then for each α in $\mathbb{R}$, $\operatorname{sign}(\alpha U_{\langle i,j\rangle}) = \operatorname{sign}\alpha$. In particular, $e_{ij} > 0$ if and only if $\operatorname{sign}(e_{ij}U_{\langle i,j\rangle}) = 1$, and $f_{ij} < 0$ if and only if $\operatorname{sign}(f_{ij}U_{\langle i,j\rangle}) = -1$. Clearly $E \prec F$ implies $E \times\times U \prec F \times\times U$. Conversely $E \times\times U \prec F \times\times U$. implies there are indices r and s such that $(E \times\times U)_{rs} > 0$ and $(F \times\times U)_{rs} < 0$. Then there are indices i and j such that the rs entry is in the i, j block of the partition induced by $P_{m,n}$. Thus $\operatorname{sign}(e_{ij}U_{\langle i,j\rangle}) = \operatorname{sign}((E \times\times U)_{rs}) = 1$. Similarly, $\operatorname{sign}(f_{ij}U_{\langle i,j\rangle}) = -1$. Thus $E \prec F$. ■

Proof of (ii). This is a consequence of (i): The matrices E and F are $\prec$-NC precisely when neither $E \prec F$ nor $F \prec E$ holds. The matrices $E \times\times U$ and $F \times\times U$ are $\prec$-NC precisely when neither $E \times\times U \prec F \times\times U$ nor $F \times\times U \prec E \times\times U$ holds. Now use (i). ■

Let $\mathscr{E}$ be a complete set of inflation-generated projectors with $|\mathscr{E}| = k$. Let $\{U_i\}_{i=1}^k$ be an inflation sequence for $\mathscr{E}$. Suppose that $E = G(U_r) \times\times \cdots \times\times U_k$ and that $F = G(U_s) \times\times \cdots \times\times U_k$. If $r < s$, then E *arises before* F (*for the given inflation sequence*). If E arises before F for every inflation sequence for $\mathscr{E}$, then E *precedes* F.

Let A be in $\mathscr{M}_n(\mathbb{R})$, and denote the entries of A by a_{ij}. The *support of* A, denoted $\operatorname{supp}(A)$, is the subset of indices $\operatorname{supp}(A) = \{(i,j) : a_{ij} \neq 0\}$. The *positive support of* A, denoted $\operatorname{possupp}(A)$, is the subset $\{(i,j) : a_{ij} > 0\}$. The *negative support of* A, denoted $\operatorname{negsupp}(A)$, is the subset $\{(i,j) : a_{ij} < 0\}$. Two matrices A and B in $\mathscr{M}_n(\mathbb{R})$ have *nonoverlapping supports* if

$\text{supp}(A) \cap \text{supp}(B) = \varnothing$. Equivalently, A and B have nonoverlapping supports if and only if $a_{ij}b_{ij} = 0$ for all i and j.

Let $\mathscr{R}$ be a binary relation on a nonempty set S. The relation R is *weakly transitive on* S if, for all x, y, and z in S, xRy and yRz together imply either xRz or else x and z are noncomparable with respect to R.

THEOREM 3.2. *Let $\mathscr{E}$ be a complete set of inflation-generated projectors. Let E and F be in $\mathscr{E}$. Then the following hold:*

(i) *$E \prec F$ implies E precedes F,*

(ii) *$\prec$ is an antisymmetric and weakly transitive relation on $\mathscr{E}$,*

(iii) *E and F are $\prec$-NC if and only if E and F have nonoverlapping supports.*

Proof of (i). Suppose $E \prec F$. Assume that E does not precede F. Since clearly $E \neq F$, there is an inflation sequence for $\mathscr{E}$ for which F arises before E. Let $\{U_i\}_{i=1}^{k}$ be such a sequence. Then there exist r and s with $1 \leqslant s < r \leqslant k$ such that $E = G(U_r) \times\times \cdots \times\times U_k$ and $F = G(U_s) \times\times \cdots \times\times U_k$. Since $E \prec F$, there are indices i and j such that $e_{ij} > 0$ and $f_{ij} < 0$. By Lemma 2.4.1, f_{ij} cannot be a diagonal entry of F, hence $i \neq j$. Since $e_{ij} > 0$ and $i \neq j$, E cannot be an M-matrix. Since $G(U_k)$ is an M-matrix by Lemma 2.3.2, $r \neq k$. So $s < r < k$. Let $W = U_{r+1} \times\times \cdots \times\times U_k$. Then $E = G(U_r) \times\times W$ and $F = G(U_s) \times\times \cdots \times\times U_r \times\times W$. By Lemma 3.1, $E \prec F$ implies $G(U_r) \prec G(U_s) \times\times \cdots \times\times U_r$. Then there exist indices p and q such that $[G(U_r)]_{pq} > 0$ and $[G(U_s) \times\times \cdots \times\times U_r]_{pq} < 0$. Since $G(U_r)$ is an M-matrix by Lemma 2.3.2, $p = q$. Then $[G(U_s) \times\times \cdots \times\times U_r]_{pp} < 0$, contradicting Lemma 2.4.1. ∎

Proof of (ii). Suppose that E and F are in $\mathscr{E}$ and that $E \prec F$. By (i), E must precede F. Assume that $F \prec E$ also holds. Then by (i), F must precede E, a contradiction. Thus $\prec$ is antisymmetric.

Suppose that E, F, and H are in $\mathscr{E}$ and that both $E \prec F$ and $F \prec H$ hold. By (i), E precedes F, and F precedes H, so E precedes H. By (i), $H \prec E$ cannot hold. Thus either $E \prec H$ or else E and H are $\prec$-NC. ∎

Proof of (iii). Suppose that $E \neq F$. Clearly, if E and F have nonoverlapping supports, then E and F are $\prec$-NC. Conversely, suppose that E and F have overlapping supports. Choose an inflation sequence for $\mathscr{E}$. Then there are indices r and s such that $E = G(U_r) \times\times \cdots \times\times U_k$ and $F = G(U_s) \times\times \cdots \times U_k$. Since $E \neq F$, $r \neq s$. Without loss of generality, $r < s$. Let $W = U_{s+1} \times\times \cdots \times\times U_k$, let $\hat{E} = G(U_r) \times\times \cdots \times\times U_s$, and let $\hat{F} = G(U_s)$.

Then $E = \hat{E} \times\times W$ and $F = \hat{F} \times\times W$. (Delete W if $s = k$.) Assume that E and F are $\prec$-NC. Then $\hat{E}$ and $\hat{F}$ are $\prec$-NC by Lemma 3.1. Let i and j be indices such that $\hat{e}_{ij}\hat{f}_{ij} \neq 0$. Then $\hat{e}_{ij}\hat{f}_{ij} > 0$. Since the nonzero entries of $\hat{F} = G(U_s)$ must be in the nontrivial diagonal blocks, there is a partition subset B_α belonging to the natural partition of U_s such that B_α contains at least two elements and such that $\hat{f}_{ij}$ is in $\hat{F}_{\langle\alpha,\alpha\rangle}$. Let μ and ν be in B_α such that $\mu \neq \nu$. Then $\hat{f}_{\mu\nu} < 0$ by Lemma 2.3.2. Since $\hat{e}_{ij}$ and $\hat{e}_{\mu\nu}$ are entries of $\hat{E}_{\langle\alpha,\alpha\rangle} = [G(U_r)\times\times \cdots \times\times U_{s-1}]_{\alpha\alpha}[U_s]_{\langle\alpha,\alpha\rangle}$ and since $\hat{e}_{ij} > 0$, it follows that $\hat{e}_{\mu\nu} > 0$. Thus $\hat{e}_{\mu\nu}\hat{f}_{\mu\nu} < 0$, so $\hat{E} \prec \hat{F}$, a contradiction. ∎

There exist complete sets of inflation-generated projectors on which $\prec$ is not transitive; thus conclusion (ii) of the preceding theorem cannot be strengthened without further hypotheses. The following example of such a set is minimal both with respect to the order of the projectors and to the cardinality of set.

EXAMPLE 3.3 (Nontransitivity of $\prec$). Let $\{U_i\}_{i=1}^5$ be the normalized inflation sequence

$$U_1 = [0], \qquad U_2 = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \qquad U_3 = \frac{1}{2}\left[\begin{array}{cc|c} 1 & 1 & \sqrt{2} \\ 1 & 1 & \sqrt{2} \\ \hline \sqrt{2} & \sqrt{2} & 2 \end{array}\right],$$

$$U_4 = \frac{1}{2}\left[\begin{array}{cc|c|cc} 1 & 1 & \sqrt{2} & 1 & 1 \\ 1 & 1 & \sqrt{2} & 1 & 1 \\ \hline \sqrt{2} & \sqrt{2} & 2 & \sqrt{2} & \sqrt{2} \\ \hline 1 & 1 & \sqrt{2} & 1 & 1 \\ 1 & 1 & \sqrt{2} & 1 & 1 \end{array}\right],$$

$$U_5 = \frac{1}{2}\left[\begin{array}{c|c|c|c|c|c} 2 & 2 & 2 & 2 & \sqrt{2} & \sqrt{2} \\ \hline 2 & 2 & 2 & 2 & \sqrt{2} & \sqrt{2} \\ \hline 2 & 2 & 2 & 2 & \sqrt{2} & \sqrt{2} \\ \hline 2 & 2 & 2 & 2 & \sqrt{2} & \sqrt{2} \\ \hline \sqrt{2} & \sqrt{2} & \sqrt{2} & \sqrt{2} & 1 & 1 \\ \sqrt{2} & \sqrt{2} & \sqrt{2} & \sqrt{2} & 1 & 1 \end{array}\right].$$

Let $\mathscr{E} = \{E_i : 1 \leqslant i \leqslant 5\}$ be the complete set of projectors generated by the inflators such that E_i corresponds to U_i for each i. Then

$$E_1 = \frac{1}{8}\begin{bmatrix} 1 & 1 & \sqrt{2} & \sqrt{2} & 1 & 1 \\ 1 & 1 & \sqrt{2} & \sqrt{2} & 1 & 1 \\ \sqrt{2} & \sqrt{2} & 2 & 2 & \sqrt{2} & \sqrt{2} \\ \sqrt{2} & \sqrt{2} & 2 & 2 & \sqrt{2} & \sqrt{2} \\ 1 & 1 & \sqrt{2} & \sqrt{2} & 1 & 1 \\ 1 & 1 & \sqrt{2} & \sqrt{2} & 1 & 1 \end{bmatrix},$$

$$E_2 = \frac{1}{8}\begin{bmatrix} 1 & 1 & \sqrt{2} & -\sqrt{2} & -1 & -1 \\ 1 & 1 & \sqrt{2} & -\sqrt{2} & -1 & -1 \\ \sqrt{2} & \sqrt{2} & 2 & -2 & -\sqrt{2} & -\sqrt{2} \\ -\sqrt{2} & -\sqrt{2} & -2 & 2 & \sqrt{2} & \sqrt{2} \\ -1 & -1 & -\sqrt{2} & -\sqrt{2} & 1 & 1 \\ -1 & -1 & -\sqrt{2} & \sqrt{2} & 1 & 1 \end{bmatrix},$$

$$E_3 = \frac{1}{4}\begin{bmatrix} 1 & 1 & -\sqrt{2} \\ 1 & 1 & -\sqrt{2} \\ -\sqrt{2} & -\sqrt{2} & 2 \end{bmatrix} \oplus \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

$$E_4 = \frac{1}{4}\begin{bmatrix} 2 & -2 & 0 \\ -2 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \oplus \begin{bmatrix} 2 & -\sqrt{2} & -\sqrt{2} \\ -\sqrt{2} & 1 & 1 \\ -\sqrt{2} & 1 & 1 \end{bmatrix},$$

$$E_5 = \frac{1}{2}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \oplus \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{bmatrix}.$$

Notice that $E_1 \prec E_2 \prec E_4 \prec E_5$ and $E_1 \prec E_2 \prec E_3 \prec E_4$, but that $E_3 \prec E_5$ fails to hold.

By Theorem 3.2, $\preccurlyeq$ is a weakly transitive partial order on complete sets of inflation-generated projectors. It remains to produce a true partial order on complete sets of inflation-generated projectors. This will be done by taking the transitive closure of $\preccurlyeq$.

Let $\mathscr{E}$ be a complete set of inflation-generated projectors. Define $\prec\cdot$ on $\mathscr{E}$ as follows: For each E and F in $\mathscr{E}$, $E \prec\cdot F$ *in* $\mathscr{E}$ if at least one of the following holds:

(1) $E \prec F$, or

(2) There is a positive integer r and elements $H_1, H_2, \dots, H_r$ in $\mathscr{E}$ such that $E \prec H_1$, $H_1 \prec H_2, \dots, H_{r-1} \prec H_r$, and $H_r \prec F$.

Define $E \preccurlyeq\cdot F$ *in* $\mathscr{E}$ to mean that one of $E = F$ and $E \prec\cdot F$ in $\mathscr{E}$ holds.

LEMMA 3.4. *Let $\mathscr{E}$ be a complete set of inflation-generated projectors. Then for all E and F in $\mathscr{E}$:*

(i) $E \prec\cdot F$ implies E precedes F;
(ii) $\preccurlyeq\cdot$ is the transitive closure of $\preccurlyeq$ on $\mathscr{E}$;
(iii) $\preccurlyeq\cdot$ is a partial order on $\mathscr{E}$.

Proof. The proof of (i) is immediate from the definition of $\prec\cdot$ and Theorem 3.2. The proof of (ii) is immediate from the fact that condition (2) of the preceding definition says that $\prec\cdot$ is the transitive closure of $\prec$. To prove (iii), it is sufficient to show that $\preccurlyeq\cdot$ is transitive, reflexive, and antisymmetric on $\mathscr{E}$. Transitivity and reflexivity on $\mathscr{E}$ are clear. Suppose that $E \preccurlyeq\cdot F$ and $F \preccurlyeq\cdot E$ hold for some E and F in $\mathscr{E}$. Assume that $E \neq F$. By (ii), E precedes F, and F precedes E. This is clearly a contradiction. Thus $\preccurlyeq\cdot$ is antisymmetric on $\mathscr{E}$. ∎

LEMMA 3.5. *Let $\{U_i\}_{i=1}^{k}$ be an inflation sequence with $k \geqslant 2$. Let $\mathscr{E}$ be the complete set of inflation-generated projectors given by $\mathscr{E} = \{E_i : E_i = G(U_i) \times\times \cdots \times\times U_k$ for $1 \leqslant i \leqslant k\}$. Suppose that for some $t \geqslant 2$, U_t has a unique, nontrivial, diagonal block. Then $E_i \prec E_t$ implies* supp(E_t) *is contained in* possupp(E_i).

Proof. Since $E_i \prec E_t$, it follows from Theorem 3.2 that $i < t$. Let $F = G(U_i) \times\times \cdots \times\times U_t$ and let $F' = G(U_t)$. If $t < k$, let $W = U_{t+1} \times\times \cdots \times\times U_k$. Then by Lemma 3.2, $F \prec F'$. Let α be the index of the unique nontrivial block of F'. Then F' is the direct sum of the strictly nonzero matrix $I - (U_t)_{\langle\alpha,\alpha\rangle}$ and as many 1×1 zero matrices as are necessary for the sum to be of the order of F'. Since $F \prec F'$, it follows that $F_{\langle\alpha,\alpha\rangle}$ must contain a positive entry. Since $F_{\langle\alpha,\alpha\rangle} = [G(U_i) \times\times \cdots \times\times U_{t+1}]_{\alpha\alpha}(U_t)_{\langle\alpha,\alpha\rangle}$, and since U_t is strictly positive, it follows that $F_{\langle\alpha,\alpha\rangle}$ is strictly positive. Thus the support of F' is contained in the positive support of F. Since W is strictly positive, the desired result holds. ∎

The following result provides sufficient conditions for when $\prec$ is itself transitive.

THEOREM 3.6. *Let $\{U_i\}_{i=1}^k$ be an inflation sequence with $k \geqslant 2$. Let $\mathscr{E}$ be the complete set of inflation-generated projectors given by $\mathscr{E} = \{E_i : E_i = G(U_i) \times\times \cdots \times\times U_k$ for $1 \leqslant i \leqslant k\}$. Suppose that U_i has a unique, nontrivial, diagonal block for each i with $2 \leqslant i \leqslant k$. Then for all α and β, $E_\alpha \prec E_\beta$ if and only if $E_\alpha \prec\cdot E_\beta$ in $\mathscr{E}$.*

Proof. Suppose that $\alpha \neq \beta$. Clearly $E_\alpha \prec E_\beta$ implies $E_\alpha \prec\cdot E_\beta$. Suppose that $E_\alpha \prec\cdot E_\beta$. Assume that $E_\alpha \prec E_\beta$ does not hold. Since E_1 is strictly positive, and hence $E_1 \prec E_i$ for each $i \geqslant 2$, it follows that $\alpha \geqslant 2$. Since E_α and E_β are $\prec$-NC but $E_\alpha \prec\cdot E_\beta$, there exist integers $i_0 = \alpha$, $i_1, \ldots, i_{n-1}$, $i_N = \beta$ such that $E_\alpha \prec E_{i_1} \prec \cdots \prec E_{i_{n-1}} \prec E_\beta$. By Theorem 3.2, $\alpha < i_1 < \cdots < i_n < \beta$. Since $i_0 = \alpha \geqslant 2$, U_{i_j} has a unique nontrivial diagonal block for each j with $0 \leqslant j \leqslant n$. Applying Lemma 3.5 to each pair in the sequence, $\operatorname{supp}(E_{i_{j+1}}) \subseteq \operatorname{supp}(E_{i_j})$ for $0 \leqslant j < n$. Since set containment is transitive, the support of E_α contains the support of E_β. This contradicts Theorem 3.2. Thus $E_\alpha \prec E_\beta$. ∎

THEOREM 3.7. *Let $k \geqslant 2$. Let $\mathscr{E}$ be a complete set of inflation-generated projectors with $|\mathscr{E}| = k$. Then $\mathscr{E}$ has two distinguished elements E_1 and E_2 such that:*

(i) *E_1 is the unique strictly positive element of $\mathscr{E}$,*
(ii) *E_2 is the unique element of $\mathscr{E} \setminus \{E_1\}$ which has no zero entries.*

Further, E_1 and E_2 satisfy

$$E_1 \prec H \qquad \text{for each } H \text{ in } \mathscr{E} \setminus \{E_1\},$$

$$E_2 \prec H \qquad \text{for each } H \text{ in } \mathscr{E} \setminus \{E_1, E_2\}.$$

For every inflation sequence $\{U_i\}_{i=1}^k$ for $\mathscr{E}$,

$$E_1 = G(U_1) \times\times \cdots \times\times U_k,$$

$$E_2 = G(U_2) \times\times \cdots \times\times U_k.$$

Finally, the element E_1 is the unique minimal element of $\mathscr{E}$ both $\preccurlyeq$ and $\preccurlyeq\cdot$; and the element E_2 is the unique minimal element of $\mathscr{E} \setminus \{E_1\}$ with respect to both $\preccurlyeq$ and $\preccurlyeq\cdot$.

Proof. First the existence of E_1 is shown. Let $\{U_i\}_{i=1}^k$ be any inflation sequence for $\mathscr{E}$. Since $G(U_1) \times\times \cdots \times\times U_k$ is in $\mathscr{E}$ and since $G(U_1) = [1]$,

$G(U_1)\times\times \cdots \times\times U_k$ is strictly positive. Next, for $i \geqslant 2$, $G(U_i)$ is an M-matrix which has at least one negative off-diagonal entry by Lemma 2.3.2 and the fact that $n_j < n_{j+1}$ for each j in the partition sequence corresponding to the inflation sequence. Hence for $i \geqslant 2$, $G(U_i)\times\times U_{i+1}\times\times \cdots \times\times U_k$ has at least one negative entry. Thus $\mathscr{E}$ has a unique strictly positive element, call it E_1, and $E_1 = G(U_1)\times\times U_2 \times\times \cdots \times\times U_k$ for every choice of inflation sequence. Since E_1 is strictily positive and since every other element of $\mathscr{E}$ has a negative entry, $E_1 \prec H$ for every H in $\mathscr{E} \setminus \{E_1\}$.

Now the existence of E_2 is shown. Let $\{U_i\}_{i=1}^{k}$ be an inflation sequence for $\mathscr{E}$. From the definition of an inflation sequence, the partition corresponding to U_2 is a 1-partition of n_2 for some integer $n_2 \geqslant 2$. Thus by Lemma 2.3.2, $G(U_2)$ has no zero entries and it does have off-diagonal entries. Thus $G(U_2)$ is an M-matrix with a strictly positive diagonal. Suppose that $i \geqslant 3$. Then $G(U_2)\times\times U_3 \times\times \cdots \times\times U_i$ has no zero entries. Further, as noted in Lemma 2.4.1, $G(U_2)\times\times U_3 \times\times \cdots \times\times U_i$ has nonnegative diagonal blocks in the partition of U_i. Together, these results imply that $G(U_2)\times\times U_3 \times\times \cdots \times\times U_i$ has strictly positive diagonal blocks in the partition of U_i. From the definition of an inflation sequence, the partition corresponding to U_i contains at least two sets, and at least one of those sets contains more than a single element. Thus by Lemma 2.3.2, $G(U_i)$ is a block-diagonal matrix with at least two diagonal blocks, at least one of which is nontrivial. Consequently, $G(U_i)$ has zero entries, hence $G(U_i)\times\times U_{i+1}\times\times \cdots \times\times U_k$ has zero entries. Thus there is a unique element of $\mathscr{E} \setminus \{E_1\}$ which has no zero entries, call it E_2. Since the inflation sequence used was chosen arbitrarily, it has been shown that $E_2 = G(U_2)\times\times U_3 \times\times \cdots \times\times U_k$ for every inflation sequence for $\mathscr{E}$. It remains to check the ordering assertion. If $k = 2$, then the claim is trivially true, since $\mathscr{E} = \{E_1, E_2\}$. Suppose that $k > 2$. From the above, $G(U_i)$ is an M-matrix with a nontrivial diagonal block; hence it has a diagonal block with an off-diagonal negative entry. Also from the above, the diagonal block of $G(U_2)\times\times \cdots \times\times U_i$ is strictly positive. Thus $G(U_2)\times\times \cdots \times\times U_i \prec G(U_i)$ for $3 \leqslant i \leqslant k$. Then by Lemma 3.1, $E_2 \prec H$ for every H in $\mathscr{E} \setminus \{E_1, E_2\}$. ∎

4. $\mathscr{L}(\mathscr{E})$ AND $\mathscr{G}(\mathscr{E})$

Corresponding to a partially ordered set and its order relation, there is a natural directed-graph structure. In this section, the necessary definitions and results from the theory of directed graphs are presented, and then the directed graph corresponding to the order $\preccurlyeq\cdot$ on a complete set $\mathscr{E}$ of inflation-generated projectors is studied. It will be shown that a certain, much sparser graph, the directed covering graph (Hasse diagram), contains all of the essential properties of the full graph, and that certain results concerning the structure of $\mathscr{E}$ are more effectively stated in terms of this sparser graph.

CONVENTION 4.1. The term "graph" will mean a directed graph having no multiple directed edges and no loops. That is, a graph G consists of a finite set of nodes (called vertices by some authors) and a finite set of directed arcs (called directed edges by some authors) joining certain nodes. The graph G may have at most one arc from node i to node j for each ordered pair of nodes i and j. That is, G does not have multiple arcs as a directed graph. No arc may begin and end at the same node. That is, the graph G cannot have loops.

CONVENTION 4.2. All terms which refer to collections of arcs will be understood to involve *directed* arcs unless otherwise explicitly stated. In particular, the term "cycle" will always mean a directed cycle.

NOTATION. If G is a directed graph, i.e. a "graph", then UG will denote G considered as an undirected graph. It should be noted that although G cannot contain multiple directed arcs, UG may contain multiple undirected arcs (occurring in pairs).

Let G be a graph. Let e be an arc in G. Suppose that e goes from node x to node y. Then e *exits* x and *enters* y. The node x is the *initial node of* e, and the node y is the *terminal node of* e. The arc e *meets* the nodes x and y. The arc e will be denoted (x, y).

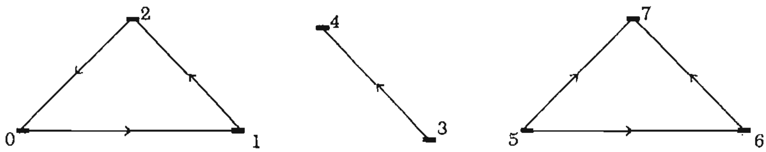
Let H be an undirected graph with no loops and no multiple arcs. If for each pair of distinct nodes x and y in H, there is a positive integer k and a sequence of pairwise distinct nodes $x = v_0, \ldots, v_k = y$ such that there is an undirected arc in H between v_{i-1} and v_i for $1 \leqslant i \leqslant k$, then H is *connected*. Let G be a graph. If UG is connected, then G is *connected*.

Let H be an undirected graph. An *undirected path* P is a sequence of nodes $v_0, \ldots, v_k$ such that for $1 \leqslant i \leqslant k$, there is an undirected arc between v_{i-1} and v_i in H, and such that for $i \neq j$, $v_i = v_j$ implies either $i = 0$ and $j = k$, or else $i = k$ and $j = 0$. (Some authors use the term "simple path.") If P is an undirected path in H for which $k > 1$ and $v_0 = v_k$, then P is an *undirected cycle*. (Some authors use the terms "closed simple path" and "simple cycle".)

Let G be a graph. A *path* is a sequence of nodes $v_0, \ldots, v_k$ such that (v_{i-1}, v_i) is an arc for $1 \leqslant i \leqslant k$, and such that $v_i = v_j$ with $i < j$ implies $i = 0$ and $j = k$. The *path arcs* (*for* P) are the arcs (v_{i-1}, v_i) for $1 \leqslant i \leqslant k$. If P is a path in G for which $k > 1$ and $v_0 = v_k$, then P is a *cycle*. A *maximal path* is a path in G whose node set does not form a proper subset of the node set of any other path in G.

It is apparent that every path is finite and that every path is contained in a maximal path.

EXAMPLE. The graph G_1 has five maximal paths and three cycles. The maximal paths are $P_0 = \{3,4\}$, $P_1 = \{5,6,7\}$, and $P_i = \{i, i+1, i+2 \pmod 3\}$ for $i = 2,3,4$. The cycles are $C_i = \{i, i+1, i+2, i+3 \pmod 3\}$ for $i = 0,1,2$.

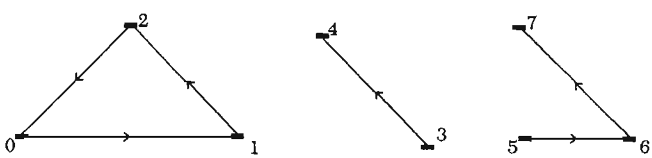


G_1

Let x be a node of a graph G. The *in degree of* x, denoted $d_i(x)$, is the number of arcs entering node x. The *outdegree of* x, denoted $d_o(x)$, is the number of arcs exiting node x. The node x is a *maximal node* if $d_o(x) = 0$. The node x is a *minimal node* if $d_i(x) = 0$.

Let (x, y) be an arc in a graph G. The arc (x, y) is a *covering arc* (*in* G) if there exists no node w (instinct from x and y) in G such that there is a path in G from x to y passing through w. The *skeleton of* G (called the covering subgraph by some authors) is the subgraph of G consisting of all of the nodes of G and all of the arcs of G which are covering arcs in G. If the graph G is its own skeleton, G is called a *skeleton*.

EXAMPLE. The graph G_1 in the preceding example has skeleton



LEMMA 4.3. *Let G be a graph with at least one node. Suppose that G contains no cycles. Then:*

(i) *G contains at least one minimal node and one maximal node.*

(ii) *Every maximal path in G begins at a minimal node and ends at a maximal node.*

(iii) *For every pair of distinct nodes x and y, at most one of the edges (x, y) and (y, x) is in G.*

(iv) *The covering arcs in G are precisely those arcs which are path arcs for at least one maximal path in G.*

(v) *The arc set for the skeleton of G is the union of the path arcs sets for the maximal paths in G.*

Proof. These are standard results in the theory of directed graphs. Note that the hypotheses do not preclude the existence of *un*directed cycles. ■

Let G be a graph with at least two nodes. Suppose that there are two nodes x and y satisfying:

(i) x is a minimal node,
(ii) (x, y) is the unique arc exiting node x,
(iii) (x, y) is the unique arc entering node y,
(iv) if w is a node distinct from x and y, then there is a path from y to w.

Then G is called a *tap-rooted graph*. The node x is called the *root of* G. The node y is called the *stem*.

LEMMA 4.4. *Suppose that G is a tap-rooted graph. Then G has a unique root and a unique stem.*

Proof. Let x be a root and let y be a corresponding stem. Suppose that w is another root. Then by condition (iv), there is a path from y to w, contradicting the fact that w is a minimal node. Thus x is the unique root. Since x has outdegree one, y is uniquely determined. ■

EXAMPLE. The graph G_2 is a tap-rooted graph with root x and stem y.

G_2

The *transitive closure of a graph* G is the graph obtained from G by adding all arcs of the form (x, y) such that x and y are distinct nodes of G and such that there is a path in G from x to y but such that (x, y) is not an arc of G. The graph G is *transitively closed* if whenever (x, y) and (y, z) are arcs in G with $x \neq z$, the arc (x, z) is in G. (This is "essentially graph transitive" in [2].)

LEMMA 4.5. *The following properties hold for the transitive closure of a graph:*

(i) *A graph contains no cycles if and only if its transitive closure contains no cycles.*

(ii) *The transitive closure of the skeleton of a graph is the transitive closure of the graph.*

Proof. These are standard results from graph theory. ■

Let S be a finite, partially ordered set with a strict inequality order relation R. The *comparability graph of S with respect to R* is the graph whose nodes are bijectively mapped to the elements of S by a map σ, and which has an arc from node i to node j precisely when $\sigma(i) R \sigma(j)$ holds. Note that the partial-order graph is transitively closed. Note also that some authors use "comparability graph" to mean the *un*directed comparability graph.

NOTATION. Let $\mathscr{E}$ be a complete set of inflation-generated projectors. The comparability graph of $\mathscr{E}$ with respect to $\prec\cdot$ is denoted by $\mathscr{L}(\mathscr{E})$. The skeleton subgraph of $\mathscr{L}(\mathscr{E})$ is denoted by $\mathscr{G}(\mathscr{E})$. It is often convenient to assume that the nodes of $\mathscr{L}(\mathscr{E})$ and of $\mathscr{G}(\mathscr{E})$ share a common numbering with the projectors in $\mathscr{E}$, so that node i corresponds to E_i for each i.

Let E and F be distinct elements of a complete set $\mathscr{E}$ of inflation-generated projectors. Let v_E and v_F be the corresponding nodes of $\mathscr{L}(\mathscr{E})$. Then $E \prec\cdot F$ if and only if (v_E, v_F) is an arc in $\mathscr{L}(\mathscr{E})$.

LEMMA 4.6. *Let $\mathscr{E}$ be a complete set of inflation-generated projectors. The graphs $\mathscr{L}(\mathscr{E})$ and $\mathscr{G}(\mathscr{E})$ are connected and contain no directed cycles. Further, $\mathscr{L}(\mathscr{E})$ is the transitive closure of $\mathscr{G}(\mathscr{E})$.*

Proof. By Theorem 3.7, $\mathscr{E}$ contains an element E_1 such that $E_1 \prec\cdot E$ for every E in $\mathscr{E} \setminus \{E_1\}$. Let v_1 be the node of $\mathscr{L}(\mathscr{E})$ corresponding to E_1. Then there is an arc from v_1 to every other node. Thus $\mathscr{L}(\mathscr{E})$ is connected, and further, there is a maximal path from v_1 to every other node; hence $\mathscr{G}(\mathscr{E})$ is connected.

The remaining properties follow from the preceding lemmas. ■

Observe that $\mathscr{L}(\mathscr{E})$ and $\mathscr{G}(\mathscr{E})$ are equivalent in the sense that they both carry a complete description of the ordering $\prec\cdot$ for $\mathscr{E}$. Since $\mathscr{G}(\mathscr{E})$ has fewer arcs than $\mathscr{L}(\mathscr{E})$ when $|\mathscr{E}| > 2$, it is easier to work with $\mathscr{G}(\mathscr{E})$. Additionally, certain results (see Section 5) are more naturally expressed in terms of the structure of $\mathscr{G}(\mathscr{E})$.

Lemma 4.7. *Let $\mathscr{E}$ be a complete set of k inflation-generated projectors with $k \geqslant 2$. Label the projectors as E_i for $1 \leqslant i \leqslant k$, and label the nodes of $\mathscr{L}(\mathscr{E})$ so that node v_i corresponds to E_i for each i. If (v_i, v_j) is a covering arc, then $E_i \prec E_j$, and there does not exist an index h such that $E_i \prec\cdot E_h$ in and $E_h \prec\cdot E_j$ in $\mathscr{E}$.*

Proof. This is immediate from the definition of $\mathscr{G}(\mathscr{E})$ and $\prec\cdot$. ■

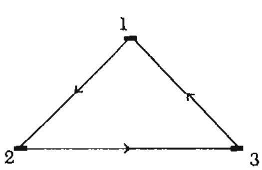
Theorem 4.8. *Let $\mathscr{E}$ be a complete set of inflation-generated projectors with $|\mathscr{E}| \geqslant 2$. Let the nodes v_1 and v_2 correspond to the projectors E_1 and E_2 of Theorem 3.7. Then every maximal path in $\mathscr{L}(\mathscr{E})$ begins with v_1 followed by v_2. Consequently, $\mathscr{G}(\mathscr{E})$ is a tap-rooted graph with root v_1 and stem v_2.*

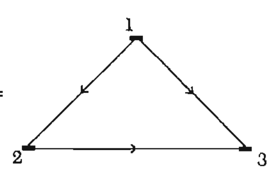
Proof. This a direct consequence of Theorem 3.7. ■

Lemma 4.9. *Let $\mathscr{E}$ be a complete set of inflation-generated projectors. For each pair of distinct nodes x and y, $\mathscr{G}(\mathscr{E})$ contains no undirected cycle arising from the arc (x, y) and a path from x to y. Further, $U\mathscr{G}(\mathscr{E})$ contains no undirected triangles.*

Proof. The nonexistence of such undirected cycles in $\mathscr{G}(\mathscr{E})$ follows from the definition of a skeleton.

If $\mathscr{G}(\mathscr{E})$ has at least one triangle when considered as an undirected graph, then $\mathscr{G}(\mathscr{E})$ has a subgraph isomorhpic to either

$T_1 =$ 

or $T_2 =$ 

Since $\mathscr{G}(\mathscr{E})$ contains no directed cycles, T_1 cannot occur. Since the arc (1,3) is not a covering arc in T_2, it cannot be a covering arc in $\mathscr{L}(\mathscr{E})$; thus T_2 cannot occur. ■

The preceding results indicate that if $\mathscr{E}$ is a complete set of inflation-generated projectors, then $\mathscr{G}(\mathscr{E})$ is a connected, tap-rooted graph which contains neither directed cycles nor undirected cycles formed from a path and an arc. That is, $\mathscr{G}(\mathscr{E})$ is a connected, tap-rooted skeleton which has no cycles. A natural question is whether the converse holds: If G is a connected, tap-rooted skeleton which contains no cycles, does there exist a complete set $\mathscr{E}$ of inflation-generated projectors such that $G = \mathscr{G}(\mathscr{E})$? The answer is affirmative (see [5, Chapters 4 and 7]).

5. ALTERNATIVE INFLATION SEQUENCES

In this section, the alternative-sequences theorem is proven. This theorem determines the set of normalized inflation sequences for a fixed, complete set of inflation-generated projectors.

LEMMA 5.1. *Let $\mathscr{E}$ be a complete set of inflation-generated projectors. Let E' be maximal in $\mathscr{E}$ with respect to $\prec\cdot$. Then:*

(i) *There exists a unique, normalized inflator U such that $E' = G(U)$.*
(ii) *If E is in $\mathscr{E} \setminus \{E'\}$, then $E = F \times\times U$ for some unique F.*
(iii) *E in $\mathscr{E}$ is maximal with respect to $\prec\cdot$ if and only if E is an M-matrix.*

Proof of (i). Let $\{U_i\}_{i=1}^{k}$ be an inflation sequence for $\mathscr{E}$. Then $E' = G(U_h) \times\times U_{h+1} \times\times \cdots \times\times U_k$ for some h. If $h = k$, then $E' = G(U_k)$, and by Lemma 2.3.2, there is a unique normalized inflator u such that $G(U) = G(U_k)$.

Suppose that $h < k$. Let $H = G(U_h)$, and label the entries of H by h_{ij}. Since $H = G(U_h)$, it follows that

$$H = \bigoplus_{j=1}^{s} H_{\langle j, j\rangle},$$

where s is the block order of U_h and where $\langle\ ,\ \rangle$ denotes blocks with respect to the natural partition of U_h. By Lemma 2.3.2, $H_{\langle j,j\rangle}$ is an irreducible M-matrix for each j; and when $H_{\langle j,j\rangle} \neq [0]$, $H_{\langle j,j\rangle}$ has no zero entries. Hence $h_{ij} \neq 0$ implies both $h_{ij}h_{ji} > 0$ and $h_{ii}h_{jj} > 0$. Let t be the order of H. Let $P \subseteq \{1,2,\ldots,t\}$ be the subset $P = \{i: h_{ii} \neq 0\}$. Then $h_{ij} \neq 0$ implies both i and j are in P.

Let $W = U_{h+1} \times\times \cdots \times\times U_k$. Then $E' = H \times\times W$. Let $\langle\langle\ ,\ \rangle\rangle$ denote the natural partitioning of W. Let m be the order of W. Assume that $W_{\langle\langle i,i\rangle\rangle} \neq [1]$ for some i in P. Then $W_{\langle\langle i,i\rangle\rangle}$ is a strictly positive submatrix which is at least 2×2. In particular, there is a first index r with $r > h$ such that h_{ii} is inflated by a nontrivial, diagonal block of U_r. Let $W' = U_{h+1} \times\times \cdots \times\times U_r$. Compare $H \times\times W'$ and $G(W')$. Let $\langle\langle\ ,\ \rangle\rangle'$ denote the natural partitioning of W'. Then $[H \times\times W']_{\langle\langle i,i\rangle\rangle'} = h_{ii} \cdot W'_{\langle\langle i,i\rangle\rangle'} \gg 0$. Since the partitioning of U_r subpartitions the partitioning of W', it follows that $G(U_r)$ has a nontrivial block inside $W'_{\langle\langle i,i\rangle\rangle'}$. Thus $H \times\times W' \prec G(U_r)$. Then $E' \prec [G(U_r) \times\times U_{r+1} \times\times \cdots \times\times U_k]$ by Lemma 3.1, which contradicts the maximality of E'. Thus $W_{\langle\langle i,i\rangle\rangle} = [1]$ whenever i is in P.

Suppose that i is in P. Then $W_{\langle\langle i,j\rangle\rangle}$ is a row vector and $W_{\langle\langle j,i\rangle\rangle}$ is a column vector. This follows from the way that the partitions are constructed. If both i and j are in P, then $W_{\langle\langle i,j\rangle\rangle} = [1]$. Recall that $E' = H \times\times W$, and that E' has blocks $h_{ij} \cdot W_{\langle\langle i,j\rangle\rangle}$. Also, $h_{ij} \neq 0$ if and only if both i and j are in P. Thus $h_{ij} \neq 0$ implies that the corresponding block in E' is $h_{ij} \cdot [1]$. Thus E' is formed from H by symmetrically inserting $m - t$ rows and $m - t$ columns of zeros. Thus E is permutation-similar to the direct sum of $H = G(U_h)$ and a zero matrix of order $m - t$. Since a zero matrix of order $m - t$ is equal to a direct sum of irreducible, idempotent, singular M-matrices. by Lemma 2.3.3, there exists a unique, normalized inflator U such that $E' = G(U)$. ∎

Proof of (ii). Suppose that E is in $\mathscr{E} \setminus \{E'\}$. Since $\mathscr{E}$ is a complete set of projectors, $EE' = E'E = 0$. Since $E' = G(U)$, Lemma 2.3.2 implies that there exists a matrix F such that $E = F \times\times U$. Since U is strictly positive, F is unique. ∎

Proof of (iii). If E is maximal in $\mathscr{E}$, then $E = G(U)$ for some inflator U by (i). The matrix $G(U)$ is an M-matrix. Conversely, suppose that E is an M-matrix. Then $E_{ij} \leqslant 0$ when $i \neq j$. Since E is in $\mathscr{E}$, E has a nonnegative diagonal by Lemma 2.4.1. Assume that E is not maximal, that is, $E \prec\cdot E''$ for some E'' in $\mathscr{E}$. Then there is an F in $\mathscr{E}$ such that $E \prec F$. Since F is in $\mathscr{E}$, F has a nonnegative diagonal. Since $E \prec F$, there are i and j such that $F_{ij} < 0$ (hence $i \neq j$) and $E_{ij} > 0$. This contradicts the fact that $E_{ij} \leqslant 0$ whenever $i \neq j$. ∎

NOTATION. Suppose that U is an inflator associated with a known partition Π such that Π is not the 1-partition of 1. Suppose that the matrix A can be expressed as $A = B \times\times U$ for some matrix B. Then the matrix B will be denoted by $A /\!/ U$. Observe that B is well defined, since U is strictly positive.

LEMMA 5.2: *Let $\mathscr{E}$ be a complete set of inflation-generated projectors with $|\mathscr{E}| \geqslant 2$. Let $E' = G(U)$ be a maximal element of $\mathscr{E}$ with respect to $\prec\cdot$. Let $\mathscr{F} = \{E /\!/ U : E \in \mathscr{E} \setminus \{E'\}\}$. Then $\mathscr{F}$ is a complete set of projectors with $|\mathscr{F}| = |\mathscr{E}| - 1$.*

Proof. Let m be the block order of E', and let n be the order of E'. Since u is strictly positive, $E /\!/ U = \hat{E} /\!/ U$ implies $E = \hat{E}$. Thus $|\mathscr{F}| = |\mathscr{E}| - 1$. Let $k = |\mathscr{F}|$. Note that the elements of $\mathscr{E} \setminus \{G(U)\}$ can be expressed as $F_i \times\times U$, where $1 \leqslant i < k$. Since $\mathscr{E}$ is a complete set of projectors,

$$I_n = \sum_{E \in \mathscr{E}} E.$$

Thus

$$I_n - E' = \sum_{E \in \mathscr{E} \setminus \{E'\}} E = \sum_{j=1}^{k-1} (F_j \times\times U) = \left[\sum_{j=1}^{k-1} F_j\right] \times\times U,$$

where the last equality is a consequence of Lemma 2.1.1. Also,

$$I_n - E' = I_n - G(U) = \bigoplus_{i=1}^{m} U_{\langle i,i \rangle}.$$

Hence, for $1 \leqslant \alpha, \beta \leqslant m$,

$$\left[\sum_{j=1}^{k-1} F_j\right]_{\alpha\beta} \cdot U_{\langle \alpha,\beta \rangle} = \delta_{ij} \cdot U_{\langle \alpha,\beta \rangle},$$

where δ_{ij} is the Kronecker delta. That is,

$$\left[\sum_{j=1}^{k-1} F_j\right] = I_m.$$

By Lemma 2.2.2, $[F_iF_j]\times\times U = [F_i \times\times U][F_j \times\times U]$. Since $F_i \times\times U$ and $F_j \times\times U$ are in $\mathscr{E}$, $[F_i \times\times U][F_j \times\times U] = \delta_{ij}[F_i \times\times U]$. Since U is strictly positive, $F_iF_j = \delta_{ij}F_i$. Thus the F_i are pairwise orthogonal idempotents which sum to the identity. Hence F is a complete set of projectors. ■

LEMMA 5.3. *Let $\mathscr{E}$ be a complete set of inflation-generated projectors with $|\mathscr{E}| \geqslant 2$. Let E' be a maximal element of $\mathscr{E}$ with respect to $\prec\cdot$. Let U be the unique, normalized inflator such that $E' = G(U)$. Let $\mathscr{F} = \{E//U : E \in \mathscr{E} \setminus \{E'\}\}$. Then $\mathscr{F}$ is a complete set of inflation-generated projectors.*

Proof. Since $\mathscr{E}$ is inflation-generated, here exists a normalized inflation sequence $\{U_i\}_{i=1}^{k}$ for $\mathscr{E}$ by Theorem 2.4.4. Label the elements of $\mathscr{E}$ so that E_i corresponds to U_i for each i. By the inflation theorem (Theorem 2.6.3), there exist real numbers $\lambda_i \geqslant 0$, for $1 \leqslant i \leqslant k$, such that

$$B = \sum_{i=1}^{k} \lambda_i E_i$$

is an MMA-matrix. (For example, let $\lambda_i = i$ for each i.) Choose a set of λ_i's so that B is an MMA-matrix. The projector E' corresponds to U_j for some j. Since E' is maximal with respect to $\prec\cdot$, it is an M-matrix by Lemma 5.1. Thus $[E']_{rs} \leqslant 0$ for $r \neq s$. For each $\alpha \geqslant \lambda_j$, define a new matrix B_α by

$$B_\alpha = \alpha \cdot E' + \sum_{i \neq j} \lambda_i E_i.$$

Then for each positive integer n,

$$(B_\alpha)^n = \alpha^n \cdot E' + \sum_{i \neq j} (\lambda_i)^n E_i.$$

Since $\alpha \geqslant \lambda_j \geqslant 0$, $\alpha^n \cdot [E']_{rs} \leqslant \lambda_j^n [E']_{rs} \leqslant 0$ for $r \neq s$ and n in $\mathbb{Z}_+$. Since B is a MMA-matrix, it follows that B_α is a ZM-matrix. By choosing α large enough that the nonzero entries of $\alpha E'$ dominate the corresponding entries of $\Sigma_{i \neq j}\lambda_i^n E_i$, the support of B_α will contain the support of B. Thus for large enough α, B_α will be an irreducible ZM-matrix, hence a ZME-matrix. Finally, $\operatorname{spec}(B_\alpha) = [\operatorname{spec}(B)/\{\lambda_j\}] \cup \{\alpha\} \subseteq \mathbb{R}_{+,0}$, so B_α is an MMA-matrix.

Now require that α also satisfy $\alpha > \lambda_i$ for all i. By the inflation theorem, there is a normalized inflation sequence $\{V_i\}_{i=1}^k$ for $\mathscr{E}$ such that $E' = G(V_k)$. By Lemma 2.3.3., $V_k = U$. Then $\mathscr{F}$ has inflation sequence $\{V_i\}_{i=1}^{k-1}$. By the preceding lemma, $\mathscr{F}$ is a complete set of projectors. ∎

Lemma 5.4. *Let $\mathscr{E}$ be a complete set of inflation-generated projectors with $|\mathscr{E}| \geqslant 2$. Let E' be a maximal element of $\mathscr{E}$ with respect to $\prec\cdot$. Let U be the unique, normalized inflator such that $E' = G(U)$. Let $\mathscr{F} = \{E /\!/ U : E \in \mathscr{E} \setminus \{E'\}\}$. Then the map "$/\!/U$" sending $\mathscr{E}$ to $\mathscr{F}$ preserves $\prec\cdot$. That is, suppose $E_1 = F_1 \times\times U$ and $E_2 = F_2 \times\times U$ are in $\mathscr{E} \setminus \{E'\}$; then $E_1 \prec\cdot E_2$ in $\mathscr{E}$ if and only if $F_1 \prec\cdot F_2$ in $\mathscr{F}$.*

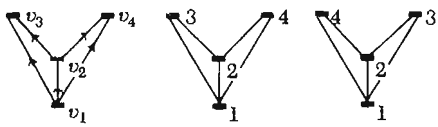
Proof. By the preceding lemma, $\mathscr{F}$ is a complete set of inflation-generated projectors. Thus the relation $\prec\cdot$ is a well-defined partial order on $\mathscr{F}$. Suppose that $E_1 \prec\cdot E_2$ in $\mathscr{E}$. Then either $E_1 \prec E_2$, or there is a sequence of projectors E_{i_j} in $\mathscr{E}$ which define $E_1 \prec\cdot E_2$. By the maximality of E', E' cannot be a member of such a sequence. Thus the sequence resides in $\mathscr{E} \setminus \{E'\}$. Thus each E_{i_j} can be expressed as $F_{i_j} \times\times U$. Apply Lemma 3.1 to $F_1 \times\times U$ and $F_2 \times\times U$, or to consecutive pairs of $F_{i_j} \times\times U$'s as necessary to arrive at $F_1 \prec\cdot F_2$. Since Lemma 3.1 reverses, a similar argument proves that $F_1 \prec\cdot F_2$ implies $E_1 \prec\cdot E_2$. ∎

Lemma 5.5. *Let $\mathscr{E}$ be a complete set of inflation-generated projectors with $|\mathscr{E}| \geqslant 2$. Let E' be a maximal element of $\mathscr{E}$ with respect to $\prec\cdot$. Let U be the unique, normalized inflator such that $E' = G(U)$. Let $\mathscr{F}$ be the complete set of inflation-generated projectors $\{E /\!/ U : E \in \mathscr{E} \setminus \{E'\}\}$. Then $\mathscr{L}(\mathscr{F})$ $[\mathscr{G}(\mathscr{F})]$ is obtained from $\mathscr{L}(\mathscr{E})$ $[\mathscr{G}(\mathscr{E})]$ by deleting the node of $\mathscr{L}(\mathscr{E})$ $[\mathscr{G}(\mathscr{E})]$ corresponding to E', and all arcs entering that node.*

Proof. This is immediate from Lemma 5.4 and the definitions of $\mathscr{F}\mathscr{L}$ and $\mathscr{G}$. ∎

Let L be a partial-order graph with respect to some partial-order relation R on a set S. Let $\hat{R}$ be an extension of R to a linear order on S. Then $\hat{R}$ is equivalent to a bijective function f mapping the set $\{1, 2, \ldots, k\}$ onto the set of nodes of L such that (α, β) is an arc in L implies $f^{-1}(\alpha) < f^{-1}(\beta)$. Such a map f will be called a *linear extension on L*.

EXAMPLE. Let L be the graph below, then L has exactly two linear extensions, f and g:



L f g

REMARK. Let f be a linear extension for a partial-order graph L with k nodes. Then along any path in L, the integers assigned by f^{-1} form an increasing sequence. Also, $f(k)$ must be a maximal node of L, and $f(1)$ must be a minimal node. Finally, it is clear that a map f is a linear extension on L if and only if it is also a *linear extension on the skeleton of L* in the sense that (α, β) is an arc in the skeleton of L only if $f^{-1}(\alpha) < f^{-1}(\beta)$.

Now that the necessary machinery has been established, the following theorem determines which inflation orderings are possible for a fixed, complete set of projectors which possesses a normalized inflation sequence.

THEOREM 5.6 (Alternative-inflation-sequences theorem). *Let $\mathscr{E}$ be a complete set of inflation-generated projectors. Let f be a linear extension on $\mathscr{L}(\mathscr{E})$. Then there exists an unique, normalized inflation sequence $\{U_i\}_{i=1}^k$ such that*:

(i) *$\mathscr{E}$ is generated by $\{U_i\}_{i=1}^k$,*
(ii) *$G(U_j) \times\times U_{j+1} \times\times \cdots \times\times U_k$ corresponds to node $f(j)$ for each j.*

That is, given any linear extension on $\mathscr{L}(\mathscr{E})$, the projectors in $\mathscr{E}$ can be generated in that order.

EXAMPLE. If $\mathscr{L}(\mathscr{E})$ is the graph L in the example above, and if the nodes of $\mathscr{L}(\mathscr{E})$ are labeled so that node v_i corresponds to E_i for each i, where each E_i corresponds to an inflator U_i in some inflation sequence for $\mathscr{E}$, then there are two distinct inflation orderings for $\mathscr{E}$:

(1) E_1, E_2, E_3, E_4 corresponding to f,
(2) E_1, E_2, E_4, E_3 corresponding to g.

Proof of Theorem 5.6. The proof proceeds by induction on $k = |\mathscr{E}|$. The result is clear if $k=1$, since $\mathscr{E} = \{[1]\}$ and $\mathscr{L}(\mathscr{E})$ consists of an isolated node v_1. There is only one linear extension on $\mathscr{L}(\mathscr{E})$, the map f sending 1 to v_1. There is only one normalized inflation sequence: $\{U_1 = [0]\}$.

Suppose that the theorem holds for $|\mathscr{E}| = k-1$. Suppose that $|\mathscr{E}| = k$. Let f be a linear extension on $\mathscr{L}(\mathscr{E})$. The node $f(k)$ is a maximal node in $\mathscr{L}(\mathscr{E})$. Let E' in $\mathscr{E}$ correspond to the node $f(k)$. Then E' is maximal with respect to $\prec\cdot$, and by Lemma 5.1, there exists a unique, normalized inflator U such that $E' = G(U)$. Let $U_k = U$. Let $\mathscr{F} = \{E/\!/U: E \in \mathscr{E} \setminus \{E'\}\}$. By Lemma 5.2, $\mathscr{F}$ is a complete set of projectors with $|\mathscr{F}| = k-1$. By Lemma 5.3, $\mathscr{F}$ is inflation-generated. By Theorem 2.4.4, $\mathscr{F}$ has a normalized inflation sequence. By Lemma 5.5, $\mathscr{L}(\mathscr{F})$ is derived from $\mathscr{L}(\mathscr{E})$ by deleting the node $f(k)$ and all arcs entering that node. Thus f restricted to the set $\{1,2,\dots, k-1\}$ is a linear extension on $\mathscr{L}(\mathscr{F})$. By the induction hypothesis, there is an unique, normalized inflation sequence $\{U_i\}_{i=1}^{k-1}$ for $\mathscr{F}$ such that $\mathscr{F}$ is generated by $\{U_i\}_{i=1}^{k-1}$, and such that $G(U_j) \times\times U_{j+1} \times\times \cdots \times\times U_{k-1}$ corresponds to the node $f(j)$ of $\mathscr{L}(\mathscr{F})$ for $1 \leqslant j \leqslant k-1$. Then $\mathscr{E}$ is generated by the unique, normalized inflation sequence $\{U_i\}_{i=1}^{k}$, and $G(U_j) \times\times U_{j+1} \times\times \cdots \times\times U_k$ corresponds to the node $f(j)$ of $\mathscr{L}(\mathscr{E})$ for $1 \leqslant j \leqslant k$. The induction is complete. ■

Corollary 5.7. *Let $\mathscr{E}$ be a complete set of inflation-generated projectors. Then the set of linear extensions on $\mathscr{L}(\mathscr{E})$ is in bijective correspondence with the set of all normalized inflation sequences for $\mathscr{E}$.*

Proof. From the preceding theorem, there exists a unique, normalized inflation sequence for $\mathscr{E}$ for each linear extension on $\mathscr{L}(\mathscr{E})$. Let φ be the map from the set of linear extensions on $\mathscr{L}(\mathscr{E})$ to the set of normalized inflation sequences for $\mathscr{E}$.

Suppose that $k = |\mathscr{E}|$ and that $\{U_i\}_{i=1}^{k}$ is a normalized inflation sequence for $\mathscr{E}$. Define the map f sending $\{1,2,\dots,k\}$ to the node set of $\mathscr{L}(\mathscr{E})$ by letting $f(i)$ be the node corresponding to $G(U_i) \times\times U_{i+1} \times\times \cdots \times\times U_k$ for each i. Suppose that $(f(i), f(j))$ is an arc in $\mathscr{L}(\mathscr{E})$. Then $G(U_i) \times\times U_{i+1} \times\times \cdots \times\times U_k \prec\cdot G(U_j) \times\times U_{j+1} \times\times \cdots \times\times U_k$ in $\mathscr{E}$. Hence Lemma 3.4 implies $i < j$. So $f^{-1}(f(i)) < f^{-1}f(j))$. Thus f is a linear extension on $\mathscr{L}(\mathscr{E})$. Let τ be this map which sends normalized inflation sequences for $\mathscr{E}$ to linear extensions on $\mathscr{L}(\mathscr{E})$.

Let τ be this map which sends normalized inflation sequences for $\mathscr{E}$ to linear extensions on $\mathscr{L}(\mathscr{E})$.

It is easily verified that τ and φ are inverses. Hence φ and τ are bijections. ■

EXAMPLE 5.8 (Example of Section 2.7 revisited). Let $\mathscr{E} = \{E_1, E_2, E_3, E_4\}$ be the complete set of inflation-generated projectors given in the example of section 2.7. Recall that $\mathscr{E}$ was generated by a sequence $\{U_i\}_{i=1}^4$ such that each U_i was symmetric (hence, normalized). The graph $\mathscr{L}(\mathscr{E})$ is the graph L given in the example immediately following the statement of Theorem 5.6. As noted there, there exist exactly two linear extensions on $\mathscr{L}(\mathscr{E})$, labeled f and g in that example. The extension f corresponds to the sequence $\{U_i\}_{i=1}^4$. The extension g corresponds to the normalized inflation sequence $\{V_i\}_{i=1}^4$ where

$$V_i = U_i \qquad \text{for} \quad i = 1,2,$$

$$V_3 = \frac{1}{2}\left[\begin{array}{c|cc} 2 & \sqrt{2} & \sqrt{2} \\ \hline \sqrt{2} & 1 & 1 \\ \sqrt{2} & 1 & 1 \end{array}\right],$$

$$V_4 = \frac{1}{2}\left[\begin{array}{cc|c|c} 1 & 1 & \sqrt{2} & \sqrt{2} \\ 1 & 1 & \sqrt{2} & \sqrt{2} \\ \hline \sqrt{2} & \sqrt{2} & 2 & 2 \\ \hline \sqrt{2} & \sqrt{2} & 2 & 2 \end{array}\right].$$

Thus,

$$\begin{aligned} E_1 &= G(U_1) \times\times U_2 \times\times U_3 \times\times U_4 = G(V_1) \times\times V_2 \times\times V_3 \times\times V_4, \\ E_2 &= G(U_2) \times\times U_3 \times\times U_4 = G(V_2) \times\times V_3 \times\times V_4, \\ E_3 &= G(U_3) \times\times U_4 = G(V_4), \\ E_4 &= G(U_4) = G(V_3) \times\times V_4. \end{aligned}$$

Using the alternate-inflation-sequences theorem, it is possible to strengthen Lemma 3.4. Recall that Lemma 3.4 states that $E \prec\cdot F$ in $\mathscr{E}$ implied that E preceded F. It is now shown that the converse holds when $\mathscr{E}$ is generated by a normalized inflation sequence.

COROLLARY 5.9. *Let $\mathscr{E}$ be a complete set of inflation-generated projectors. Suppose that E and E′ are in $\mathscr{E}$. Then $E \prec\cdot E'$ in $\mathscr{E}'$ is equivalent to E preceding E′.*

Proof. If $E \prec\cdot E'$ in $\mathscr{E}$, then E precedes E' for *every* inflation sequence generating $\mathscr{E}$ by Lemma 3.4. Conversely, suppose that $E \prec\cdot E'$ in $\mathscr{E}$ is false. If $E' \prec\cdot E$ in $\mathscr{E}$, then E' precedes E for every inflation sequence generating $\mathscr{E}$; hence E does not precede E'. So suppose that E and E' are noncomparable in $\mathscr{E}$ with respect to $\prec\cdot$. Let v and v' be the nodes of $\mathscr{L}(\mathscr{E})$ corresponding to E and E', respectively. Since there is no path from v to v' in $\mathscr{L}(\mathscr{E})$, and there is no path from v' to v in $\mathscr{L}(\mathscr{E})$, there exist two linear extensions f and g on $\mathscr{L}(\mathscr{E})$ such that $f^{-1}(v) < f^{-1}(v')$ and such that $g^{-1}(v') < g^{-1}(v)$. Applying Theorem 5.6 twice, once for f and once for g, produces two normalized inflation sequences for $\mathscr{E}$. In the first one, E arises before E'; in the second, E' arises before E. Thus E does not precede E'. ■

The following is a useful application of Theorem 5.6.

THEOREM 5.10. *Let $\mathscr{E}$ be a complete set of projectors. The following algorithm determines in a finite number of steps whether $\mathscr{E}$ is generated by a normalized inflation sequence, and if so, it produces such a sequence:*

Algorithm. *At each of steps* (ii) *through* (v), *a negative answer terminates the algorithm by implying that $\mathscr{E}$ is not generated by a normalized inflation sequence. Let $k = |\mathscr{E}|$.*

(i) *If $k > 1$, go to* (ii). *If $k = 1$, does $\mathscr{E} = \{[1]\}$? If yes, the inflation sequence is $\{U_1 = [0]\}$. Stop.*

(ii) *Does $\mathscr{E}$ have a strictly positive element? If yes, continue.*

(iii) *Form the weak partial-order graph with respect to $\prec$ for $\mathscr{E}$. Let L be the transitive closure of this graph. If L free of directed cycles? If yes, continue.*

(iv) *Is each element of $\mathscr{E}$ corresponding to a maximal node of L an M-matrix? If yes, continue.*

(v) *Choose an element E′ in $\mathscr{E}$ which is maximal in L. Is there a normalized inflator U such that $E' = G(U)$? If yes, let $U_k = U$. Continue.*

(vi) *Replace k by $k-1$. Replace $\mathscr{E}$ by $\{E /\!/ U : E \in \mathscr{E} \setminus \{E'\}\}$. Go to step* (i).

Proof. The proof is an easy consequence of Theorem 3.7 and the results of this section. ■

6. THE RELATION BETWEEN $\prec\!\cdot$ IN $\mathscr{E}$ AND IN $\mathscr{F}$

LEMMA 6.1. *Let $\mathscr{E}$ be a complete set of rank-one, inflation-generated projectors. Let $\{U_i\}_{i=1}^n$ be an inflation sequence for $\mathscr{E}$. Then for each i with $i \geqslant 2$, U_i has a unique nontrivial diagonal block, and that block is 2×2. If E and E′ are in $\mathscr{E}$, then $E \prec\!\cdot E'$ in $\mathscr{E}$ implies both $E \prec E'$ and* $\text{supp}(E') \subseteq \text{possupp}(E)$.

Proof. Let E be in $\mathscr{E}$ and let U be the corresponding inflator. Then $\text{rank}[E] = \text{rank}[G(U)]$ by Lemma 2.2.2. Recall from Lemma 2.3.2 that the rank of $G(U)$ is the difference between the order of U and the block order of U. Thus, if E is rank-one, then U must exactly one nontrivial diagonal block, and further, that block must be 2×2. Now apply Lemma 3.5 and Theorem 3.6. ■

LEMMA 6.2. *Let $\mathscr{F}$ be a complete set of inflation-generated projectors. Let $\mathscr{E}$ be a complete set of inflation-generated projectors which is a rank-one refinement of $\mathscr{F}$. Let E^+ be the unique strictly positive element of $\mathscr{E}$. Then E^+ is the unique strictly positive element of $\mathscr{F}$.*

Proof. Since $\mathscr{F}$ is inflation-generated, it has a unique strictly positive element, call it F^+. Let $\{U_i\}_{i=1}^k$ be an inflation sequence for $\mathscr{F}$. By Theorem 3.7, F^+ corresponds to U_1, so that $\text{rank}[F^+] = \text{rank}[G(U_1)] = 1$. Thus F^+ is in $\mathscr{E}$. Since $\mathscr{E}$ is inflation-generated, it has a unique strictly positive element. Thus $F^+ = E^+$. ■

THEOREM 6.3. *Let $\mathscr{F}$ be a complete set of inflation-generated projectors. Let $\mathscr{E}$ be a complete set of inflation-generated projectors which is a rank-one refinement of $\mathscr{F}$. Let E and E′ be in $\mathscr{E}$. Let F and F′ be the unique element(s) of $\mathscr{F}$ such that* $\text{fix}(E) \subseteq \text{fix}(F)$ *and* $\text{fix}(E') \subseteq \text{fix}(F')$. *Then:*

(i) *$E \prec\!\cdot E'$ in $\mathscr{E}$ implies $F \leqslant F'$;*

(ii) *$F \prec\!\cdot F'$ in $\mathscr{F}$ implies either $E \prec E'$ or else E and E′ are noncomparable with respect to $\prec$.*

Proof of (i). If $F = F'$, the result is clear. So suppose that $F \neq F'$. For convenience, denote the strictly positive element of $\mathscr{E}$ as E^+. If $E = E^+$, then $F = E^+$ by Lemma 6.2. Then by Theorem 3.7, $F \preccurlyeq \hat{F}$ for every $\hat{F}$ in $\mathscr{F}$. So suppose that $E \neq E^+$. Then $E \prec\!\cdot E'$ implies $E' \neq E^+$, and hence $F' \neq E^+$. Thus $\text{fix}(E^+)$ is not contained in $\text{fix}(F) \cup \text{fix}(F')$. By Lemma 6.1, $E \prec E'$. Extend this to a maximal chain in F and F', that is, to a maximal partially ordered chain of projectors from $\mathscr{E}$ each of whose invariant space is contained in the invariant space of either F or F'. Label the elements of this maximal chain as $E_1 \prec E_2 \prec \cdots \prec E_r$. Since E and E' are elements in this chain, $E = E_\sigma$ and $E' = E_\tau$ with $\sigma < \tau$. The sets $\{i : \text{fix}(E_i) \subseteq \text{fix}(F)\}$ and $\{i : \text{fix}(E_i) \subseteq \text{fix}(F')\}$ are a partition of the set $\{1, 2, \ldots, r\}$ into two nonempty sets. Label the set containing the element 1 as A, and label the other set as B. For convenience, label $\{F, F'\}$ as $\{F_A, F_B\}$ in the obvious manner. Since for each i, $E_i \neq E^+$, it follows from Theorem 3.7 that $\text{supp}(E_i) \supset \text{possupp}(E_i)$. In conjunction with Lemma 3.5, this implies

$$\text{supp}(E_1) \supset \text{possupp}(E_1) \supseteq \text{supp}(E_2) \supset \cdots \supset \text{supp}(E_r) \supseteq \text{possupp}(E_r).$$

Let t be the minimal element of B. Then

$$\text{possupp}(F_A) \supseteq \text{possupp}(E_{t-1}) \setminus \text{possupp}(E_t).$$

Since $\text{possup}(E_{t-1}) \supseteq \text{supp}(E_t)$, it follows that

$$\text{possupp}(E_{t-1}) \setminus \text{possupp}(E_t) \supseteq \text{negsupp}(E_t).$$

Note that $\text{negsupp}(E_t) \cap \text{supp}(E_j) = \varnothing$ for $j > t$. Thus

$$\text{negsupp}(F_B) \supseteq \text{negsupp}(E_t)$$

provided that it can be shown that there is no $\tilde{E}$ in $\mathscr{E}$ with $\text{fix}(\tilde{E}) \subseteq \text{fix}(F_B)$ such that $\text{possupp}(\tilde{E}) \cap \text{negsupp}(E_t)$ is nonempty. Assume that $\tilde{E}$ exists. Clearly, $\tilde{E} \prec E_t$. By the maximality of the chain, and by the choice of t, it follows that $\tilde{E}$ cannot be in the chain. Thus $E_{t-1} \prec \tilde{E}$ cannot occur. Observe that $\text{negsup}(E_t) \subset \text{possupp}(E_j)$ for $1 \leqslant j < t$. Hence for $1 \leqslant j < t$, $\tilde{E}$ and E_j have overlapping supports, and thus must be $\prec$-comparable by Theorem 3.2. This forces $\tilde{E} \prec E_{t-1}$. Then by maximality, $E_{t-2} \prec \tilde{E}$ cannot occur. Thus $\tilde{E} \prec E_{t-2}$. Iterating this process, $E_j \prec \tilde{E}$ cannot occur for any j with $1 \leqslant j < t$, so that $\tilde{E} \prec E_1$, contradicting the maximality of the chain.

Since $E_t \neq E^+$, $\text{negsupp}(E_t) \neq \varnothing$. Thus $\text{possupp}(F_A)$ and $\text{negsupp}(F_B)$ overlap. Thus $F_A \prec F_B$.

Assume that there is an $s \in A$ such that $s > t$. Choose the minimal such s. Note that $\text{negsupp}(E_s) \cap \text{supp}(E_j) = \varnothing$ for $j > s$, and that $\text{negsupp}(E_s) \subset \text{possupp}(E_j)$ for $j < s$. Since $s > t$, $\text{negsupp}(E_s) \subseteq \text{possupp}(F_B)$. It suffices to show that $\text{negsupp}(E_s) \cap \text{negsupp}(F_A) \neq \varnothing$ in order to imply that $F_B \prec F_A$—a contradiction, since $\mathscr{F}$ is inflation-generated and $F_A \prec F_B$.

Let (α, β) be in negsupp(E_s). Then

$$(F_A)_{\alpha\beta} = (E_1)_{\alpha\beta} + (E_2)_{\alpha\beta} + \cdots + (E_{t-1})_{\alpha\beta} + (E_s)_{\alpha\beta}.$$

Since $E_1 \prec E_2 \prec \cdots \prec E_s$, it follows by Theorem 3.2 that these projectors arise in this order for every inflation sequence for $\mathscr{E}$. Let $\{U_i\}_{i=1}^n$ be any inflation sequence for $\mathscr{E}$. Let U_{h_i} correspond to E_i for $1 \leqslant i \leqslant s$. Since $(E_j)_{\alpha\beta} > 0$ for $1 \leqslant i < s$, it follows by Lemma 2.4.5 that

$$\begin{aligned}
(E_1)_{\alpha\beta} &= (1 - u_{h_1})u_{h_1+1} \cdots u_{h_2} \cdots u_{h_{t-1}} \cdots u_{h_s} \cdots u_n,\\
(E_2)_{\alpha\beta} &= \qquad\qquad (1 - u_{h_2}) \cdots u_{h_{t-1}} \cdots u_{h_s} \cdots u_n,\\
&\ \vdots\\
(E_{t-1})_{\alpha\beta} &= \qquad\qquad\qquad (1 - u_{h_{t-1}}) \cdots u_{h_s} \cdots u_n,\\
(E_s)_{\alpha\beta} &= \qquad\qquad\qquad\qquad - u_{h_s} \cdots u_n.
\end{aligned}$$

Since each u_i satisfies $0 < u_i \leqslant 1$, it is easily verified that

$$(F_A)_{\alpha\beta} < -u_{h_1}u_{h_1+1} \cdots u_{h_2} \cdots u_{h_{t-1}} \cdots u_{h_s} \cdots u_n < 0.$$

Thus negsupp$(E_s) \subseteq$ negsupp(F_A). As indicated above, this yields a contradiction. Consequently, s does not exist. That is, $A = \{1, 2, \ldots, t-1\}$ and $B = \{t, t+1, \ldots, r\}$. It remains to show that $F_A = F$ and that $F_B = F'$. As noted above, $E = E_\sigma$ and $E' = E_\tau$ where $1 \leqslant \sigma < \tau \leqslant r$. Since E is a summand for F and since E' is not, it follows that $\sigma < t$, and hence $F_A = F$, and $F_B = F'$. Thus $E \prec E'$ implies $F \prec F'$. ■

Proof of (ii). Suppose that $F \prec\!\cdot\, F'$ in $\mathscr{F}$. Since $\prec\!\cdot$ is a partial order on $\mathscr{E}$, exactly one of the following must hold:

(1) $E \prec\!\cdot\, E'$ in $\mathscr{E}$,
(2) $E' \prec\!\cdot\, E$ in $\mathscr{E}$,
(3) E and E' are noncomparable with respect to $\prec\!\cdot$.

By Lemma 6.1, $\prec$ and $\prec\!\cdot$ are the same partial order on $\mathscr{E}$. Thus it suffices to show that $E' \prec E$ cannot hold. Assume the contray, that $E' \prec E$ does hold. Then by part (i) of this theorem, $F' \prec F$ in $\mathscr{F}$. This is a contradiction, since $\mathscr{F}$ is inflation-generated, and hence, $\prec\!\cdot$ is a partial order on $\mathscr{F}$. ■

The following result shows that the elements of $\mathscr{F}$ are "convex" sums of elements of $\mathscr{E}$.

COROLLARY 6.4. *Let $\mathscr{F}$ be a complete set of inflation-generated projectors. Let $\mathscr{E}$ be a complete set of inflation-generated projectors which is a rank-one refinement of $\mathscr{F}$. Let E and E' be distinct elements of $\mathscr{E}$. Suppose that there is an F in $\mathscr{F}$ such that* $\mathrm{fix}(E) \subset \mathrm{fix}(F)$ *and* $\mathrm{fix}(E') \subset \mathrm{fix}(F)$. *If there is an E'' in $\mathscr{E}$ such that $E \prec E''$ and $E'' \prec E'$, then* $\mathrm{fix}(E'') \subset \mathrm{fix}(F)$.

Proof. Suppose that $E \prec E''$ and $E'' \prec E'$. Let $\hat{F}$ be the unique element of $\mathscr{F}$ such that $\mathrm{fix}(E'') \subseteq \mathrm{fix}(\hat{F})$. By the preceding theorem, $F \preccurlyeq\cdot \hat{F}$ in $\mathscr{F}$ and $\hat{F} \preccurlyeq\cdot F$ in $\mathscr{F}$. Thus $F = \hat{F}$. ∎

Let $\mathscr{F}$ be a complete set of projectors. Suppose that $\mathscr{F}$ has an inflation-generated refinement $\mathscr{E}$. Then each element in $\mathscr{F}$ is a sum of elements of $\mathscr{E}$. The set $\mathscr{F}$ contains a *cyclic sum with respect to* $\mathscr{E}$ if there exist an integer $r \geqslant 2$, a sequence $\{F_i\}_{i=1}^r$ of distinct elements in $\mathscr{F}$, and two sequences $\{E_i\}_{i=1}^r$ and $\{E_i'\}_{i=1}^r$ of element sin $\mathscr{E}$ with the following properties:

(1) $\mathrm{fix}(E_i)$ and $\mathrm{fix}(E_i')$ are both in $\mathrm{fix}(F_i)$ for each i;
(2) $E_i \prec\cdot E_{i+1}'$ for $1 \leqslant i \leqslant r-1$;
(3) $E_r \prec\cdot E_1'$.

If the set $\mathscr{F}$ contains no cyclic sums with respect to $\mathscr{E}$, then $\mathscr{F}$ is *acyclic with respect to* $\mathscr{E}$.

THEOREM 6.5. *Let $\mathscr{E}$ be a complete set of rank-one inflation-generated projectors. Let E_1 be the unique strictly positive element of $\mathscr{E}$. Let $\mathscr{F}$ be a complete set of projectors such that $\mathscr{E}$ is a refinement of $\mathscr{F}$ and such that E_1 is an element of $\mathscr{F}$. Then the following are equivalent*:

(i) *the set $\mathscr{F}$ is inflation-generated;*
(ii) *the set $\mathscr{F}$ is acyclic with respect to $\mathscr{E}$.*

Proof of (i) $\Rightarrow$ (ii). Suppose that $\mathscr{F}$ is inflation-generated. Assume that $\mathscr{F}$ contains a cyclic sum. Then there exists an integer $r \geqslant 2$, and there exist sequences $\{F_i\}_{i=1}^r$, $\{E_i\}_{i=1}^r$, and $\{E_i'\}_{i=1}^r$ which satisfy the conditions in the definition of a cyclic sum. Due to conditions (1) and (2), Theorem 6.3 implies $F_i \prec F_{i+1}$ for $1 \leqslant i < r$. Because of conditions (1) and (3), Theorem 6.3 implies $F_r \prec F_1$. Then $F_1 \prec\cdot F_1$, a contradiction. ∎

Proof of (ii) ⇒ (i). Suppose that $|\mathscr{F}| = k$, and suppose that the matrices in $\mathscr{F}$ are $n \times n$. Let $F_1 = E_1$. If $k = 1$, the result is clear. So suppose $k \geqslant 2$. Let E_2 be the unique strictly nonzero element of $\mathscr{E} \setminus \{E_1\}$. (Note that by Theorem 3.7, E_2 is the unique minimal element of $\mathscr{E} \setminus \{E_1\}$.) Let F_2 be the unique element of $\mathscr{F}$ such that $\mathrm{fix}(E_2) \subseteq \mathrm{fix}(F_2)$. Label the summands from $\mathscr{E}$ so that $F_2 = E_2 + E_3 + \cdots + E_{r_2}$. Further, these labels can be assigned so that for $2 \leqslant i, j \leqslant r_2$, $E_1 \prec\cdot E_j$ in $\mathscr{E}$ implies $i < j$. Since $\mathscr{F}$ is acyclic, $E \in \mathscr{E} \setminus \{E_1, E_2, \dots, E_{r_2}\}$ implies $E \prec\cdot E_i$ in $\mathscr{E}$ must be false for each i with $1 \leqslant i \leqslant r_2$. Assume not. Then for some i, $E \prec\cdot E_i$ in $\mathscr{E}$. Clearly, $i > 2$. Let F be the element of $\mathscr{F}$ with $\mathrm{fix}(E) \subseteq \mathrm{fix}(F)$. Then $\mathscr{F}$ has a cycle with respect to $\mathscr{E}$: use the sequences $\{F_2, F\}$, $\{E_2, E\}$, and $\{E_i, E\}$. This contradicts the acyclicity of $\mathscr{F}$ with respect to $\mathscr{E}$.

Consider the minimal elements (with respect to $\prec\cdot$ in $\mathscr{E}$) of $\mathscr{E} \setminus \{E_1, E_2, \dots, E_{r_2}\}$. By acyclicity, there a minimal element E with corresponding element F of $\mathscr{F}$ such that for every E' in $\mathscr{E} \setminus \{E_1, E_2, \dots, E_{r_2}\}$ and for every E'' with $\mathrm{fix}(E'') \subseteq \mathrm{fix}(F)$, $E' \prec\cdot E''$ implies $\mathrm{fix}(E') \subseteq \mathrm{fix}(F)$. Label E as E_{r_2+1}, and label F as F_3. Then the summands in F_3 can be labeled so that $F_3 = E_{r_2+1} + E_{r_2+2} + \cdots + E_{r_3}$. Further, these labels can be assigned so that for $r_2 < i, j \leqslant r_3$, $E_i \prec\cdot E_j$ in $\mathscr{E}$ implies $i < j$. Then by the acyclicity of $\mathscr{F}$ and by the minimality of E_{r_2+1}, it follows that $E \in \mathscr{E} \setminus \{E_1, E_2, \dots, E_{r_2}, \dots, E_{r_3}\}$ implies $E \prec\cdot E_i$ must be false for each i with $1 \leqslant i \leqslant r_3$. Also note that for $1 \leqslant i, j \leqslant r_3$, $E_i \prec\cdot E_j$ in $\mathscr{E}$ implies $i < j$.

By iterating this process, at the hth step, $E_{r_{h-1}+1}$ is chosen as a minimal element of $\mathscr{E} \setminus \{E_1, E_2, \dots, E_{h-1}\}$ such that for every E' in $\mathscr{E} \setminus \{E_1, E_2, \dots, E_{h-1}\}$ and for every E'' with $\mathrm{fix}(E'') \subseteq \mathrm{fix}(F)$, $E' \prec\cdot E''$ implies $\mathrm{fix}(E') \subseteq \mathrm{fix}(F)$, where F is the element of $\mathscr{F}$ with $\mathrm{fix}(E_{r_{h-1}+1}) \subseteq \mathrm{fix}(F)$. The element F is labeled as F_h. Then the summands in F_h are labeled so that $F_h = E_{r_{h-1}+1} + E_{r_{h-1}+2} + \cdots + E_{r_h}$. Further, these labels are assigned so that for $r_{h-1} < i, j \leqslant r_h$, $E_i \prec\cdot E_j$ in $\mathscr{E}$ implies $i < j$. Then by the acyclicity of $\mathscr{F}$ and by the minimality of $E_{r_{h-1}+1}$, it follows that $E \in \mathscr{E} \setminus \{E_1, E_2, \dots, E_{r_h}\}$ implies that $E \prec\cdot E_i$ must be false for each i with $1 \leqslant i \leqslant r_h$. Also, note that for $1 \leqslant i, j \leqslant r_h$, $E_i \prec\cdot E_j$ in $\mathscr{E}$ implies $i < j$.

After k iterations, the elements of $\mathscr{E}$ have been labeled so that $F_1 = E_1$, and for $2 \leqslant i \leqslant k$,

$$F_i = \sum_{r_{i-1}}^{r_i} E_j.$$

And further, $E_i \prec\cdot E_j$ implies $i < j$. Thus the map sending i to the node corresponding to E_i in $\mathscr{L}(\mathscr{E})$ for each i with $1 \leqslant i \leqslant n$ is a linear extension on $\mathscr{L}(\mathscr{E})$. By the alternative inflation sequences theorem (Theorem 5.6), there is an inflation sequence $\{U_i\}_{i=1}^{n}$ for $\mathscr{E}$ such that E_i corresponds to U_i for each i. By repeated applications of the compression theorem (Theorem 2.5.4), the inflation sequence $\{W_i\}_{i=1}^{k}$ given by $W_1 = U_1$, and for $2 \leqslant i \leqslant k$ by

$$W_i = U_{r_{i-1}+1} \times\times \cdots \times\times U_{r_i},$$

is an inflation sequence for $\mathscr{F}$ such that W_i corresponds to F_i for each i. ∎

7. SLIDE-AROUND THEOREMS

In this section, we return to the subject of Z- and M-matrices in order to generalize the original "slide-around" theorem (Theorem 2.9.1). The essence of that theorem is that the numerical values in the spectrum of a ZME-matrix are not significant. Rather, what is significant is the manner in which the eigenvalues order the projectors. Thus it should not be surprising that the generalizations of this theorem will depend on the order relationship $\prec\cdot$.

THEOREM 7.1 (The strict-inequality slide-around theorem). *Let $\mathscr{F}$ be a complete set of inflation-generated projectors with $|\mathscr{F}| = k$. Label the elements of $\mathscr{F}$ as F_i for $1 \leqslant i \leqslant k$. Let $\{\alpha_1, \alpha_2, \dots, \alpha_k\}$ be a set of distinct real numbers satisfying $\alpha_1 < \alpha_2 < \cdots < \alpha_k$ and $|\alpha_1| \leqslant \alpha_2$. Let*

$$A = \sum_{i=1}^{k} \alpha_i F_i.$$

Then the following are equivalent:

(i) *$\alpha_r < \alpha_s$ whenever $F_r \prec\cdot F_s$ in $\mathscr{F}$,*
(ii) *A is a ZME-matrix.*

Further, if $\alpha_1 = -\alpha_2$, then (ii) *is equivalent to the assertion that A is a ZMO-matrix; if $\alpha_1 > -\alpha_2$, then* (ii) *is equivalent to the assertion that A is a ZMA-matrix; and if $\alpha_1 \geqslant 0$, then* (ii) *is equivalent to the assertion that A is an MMA-matrix.*

Proof of (i) ⇒ (ii). Label the nodes of $\mathscr{L}(\mathscr{F})$ so that node v_i corresponds to F_i for each i. Define a map f sending the set $\{1, 2, \dots, k\}$ to the node set of $\mathscr{L}(\mathscr{F})$ by $f(i) = v_i$ for each i. Suppose that there is a path from v_r to v_s in $\mathscr{L}(\mathscr{E})$; then $F_r \prec\cdot F_s$ in $\mathscr{F}$, and hence $\alpha_r < \alpha_s$. Since the α_i are increasing with i, it follows that $r < s$. Thus $f^{-1}(v_r) < f^{-1}(v_s)$. Thus f is a linear extension on $\mathscr{L}(\mathscr{F})$. By the alternative-inflation-sequences theorem (Theorem 5.6), there is a normalized inflation sequence $\{U_i\}_{i=1}^{k}$ for $\mathscr{E}$ such that F_i corresponds to U_i for each i. By the inflation theorem (Theorem 2.6.3), A is a *ZME*-matrix. ∎

Proof of (ii) ⇒ (i). By the inflation theorem, there is a normalized inflation sequence $\{U_i\}_{i=1}^{k}$ such that F_i corresponds to U_i for each i. Suppose that $F_i \prec\!\cdot\, F_j$ in $\mathscr{F}$. Then by Lemma 3.4, F_i precedes F_j; hence $i<j$. Thus $\alpha_i < \alpha_j$. ∎

In the preceding theorem, the real numbers α_i were required to be distinct. This requirement can be weakened.

THEOREM 7.2 (The weak-inequality slide-around theorem). *Let $\mathscr{F}$ be a complete set of inflation-generated projectors with $|\mathscr{F}|=k$. Label the elements of $\mathscr{F}$ as F_i for $1 \leqslant i \leqslant k$. Let $\alpha_1, \alpha_2, \ldots, \alpha_k$ be real numbers which are not necessarily distinct, but which satisfy*

$$-\alpha_2 \leqslant \alpha_1 < \alpha_2 \leqslant \alpha_3 \leqslant \cdots \leqslant \alpha_k .$$

Let

$$A=\sum_{i=1}^{k} \alpha_i F_i .$$

Then the following are equivalent:

(i) $\alpha_r \leqslant \alpha_s$ *whenever* $F_r \prec\!\cdot\, F_s$ *in* $\mathscr{F}$,
(ii) *A is a ZME-matrix.*

Further, A is a ZMO-, ZMA-, or MMA-matrix depending on the relationship of α_1 and the next smallest eigenvalue of A as given in Theorem 2.6.1.

Proof of (i) ⇒ (ii). Without loss of generality, it may be assumed that $\alpha_r = \alpha_s$ and $F_r \prec\!\cdot\, F_s$ in $\mathscr{F}$ together imply $r<s$. Then $F_r \prec\!\cdot\, F_s$ in $\mathscr{F}$ implies $r<s$. For $\epsilon>0$ and for $1<i \leqslant k$, define $\alpha_i(\epsilon)=\alpha_i + i\epsilon$. Define $\alpha_1(\epsilon)=\alpha_1$ for $\epsilon>0$. Note that for small ϵ (that is, for ϵ such that $k\epsilon$ is less than the minimum separation of distinct α_i), the $\alpha_i(\epsilon)$ are distinct real numbers satisfying:

$$-\alpha_2(\epsilon)<\alpha_1(\epsilon)<\alpha_2(\epsilon)<\alpha_3(\epsilon)<\cdots \leqslant \alpha_k(\epsilon).$$

For $\epsilon > 0$, define $A(\epsilon)$ by

$$A(\epsilon) = \sum_{i=1}^{k} \alpha_i(\epsilon) F_i.$$

Since $F_r \prec\cdot F_s$ in $\mathscr{F}$ implies $r < s$, it follows for small $\epsilon > 0$ that $\alpha_r(\epsilon) < \alpha_s(\epsilon)$ whenever $F_r \prec\cdot F_s$. By Theorem 7.1, $A(\epsilon)$ is a *ZME*-matrix for all small $\epsilon > 0$. By Lemma 2.6.5, A is a *ZME*-matrix. Then using Theorem 2.6.1, it is easy to check that the relationship between the restrictions on α_1 and A being a *ZMO*-, *ZMA*-, or *MMA*-matrix hold. ∎

Proof of (ii) ⇒ (i). By Theorem 2.8.2, $\mathscr{F}$ has a rank-one, inflation-generated refinement. Let $\mathscr{E}$ be such a complete set of projectors. Suppose that $|\mathscr{E}| = n$. Label the elements of $\mathscr{E}$ as E_i for $1 \leqslant i \leqslant n$. There exists a k-partition Ω of n given by the sets $C_1, C_2, \ldots, C_k$, such that for each i,

$$F = \sum_{j \in C_i} E_j.$$

There also exist n real numbers $\beta_1, \beta_2, \ldots, \beta_n$ such that for each i and every j in C_i, $\beta_j = \alpha_i$. Thus,

$$A = \sum_{j=1}^{n} \beta_j E_j.$$

Let $\hat{\alpha}_1, \hat{\alpha}_2, \ldots, \hat{\alpha}_p$ be the distinct elements from among the α (hence from among the β_j). Without loss, it may be assumed that the $\hat{\alpha}_h$ are labeled so that

$$\hat{\alpha}_1 < \hat{\alpha}_2 < \cdots < \hat{\alpha}_p.$$

For each h, let

$$H_h = \sum_{j:\, \alpha_j = \hat{\alpha}_h} E_j.$$

Let $\mathscr{H}$ be the set $\mathscr{H} = \{H_1, H_2, \ldots, H_p\}$. Then $\mathscr{H}$ is a complete set of projectors which has $\mathscr{E}$ as a refinement. Note that

$$A = \sum_{h=1}^{p} \hat{\alpha}_h H_h,$$

where the $\hat{\alpha}_h$ are distinct. Since A is a *ZME*-matrix, it follows by the inflation theorem (Theorem 2.6.3) that $\mathscr{H}$ is inflation-generated. And further, since $\hat{\alpha}_i < \hat{\alpha}_j$ implies $i < j$, there is an inflation sequence $\{V_i\}_{i=1}^{p}$ for $\mathscr{H}$ such that V_i corresponds to H_i for each i. Then by Theorem 3.2, $H_i \preccurlyeq\cdot H_j$ in $\mathscr{H}$ implies $i \leqslant j$, and hence, $\hat{\alpha}_i \leqslant \hat{\alpha}_j$.

Suppose that $F_r \prec\cdot F_s$ in $\mathscr{F}$. Then by the definition of $\prec\cdot$, there is a sequence $F_{i_1} = F_r, F_{i_2}, \ldots, F_{i_g} = F_s$ in $\mathscr{F}$ such that

$$F_{i_1} \prec F_{i_2} \prec \cdots \prec F_{i_g}.$$

By Theorem 3.6, $\mathrm{supp}(F_{i_h}) \cap \mathrm{supp}(F_{i_{h+1}}) \neq \emptyset$ for $1 \leqslant h < g$. Then for each pair in the sequence $\{F_{i_h}\}_{h=1}^{g}$, there are two sequences $\{E_{j_h}\}_{h=1}^{g-1}$ and $\{E'_{j_h}\}_{h=2}^{g}$ of elements of ε such that:

(1) $\mathrm{fix}(E_{j_h}) \subseteq \mathrm{fix}(F_{i_h})$ for $1 \leqslant h < g$;
(2) $\mathrm{fix}(E'_{j_h}) \subseteq \mathrm{fix}(F_{i_h})$ for $2 \leqslant h \leqslant g$;
(3) $\mathrm{supp}(E_{j_h}) \subseteq \mathrm{supp}(E'_{j_{h+1}}) \neq \phi$ for $1 \leqslant h < g$.

For each h, E_{j_h} and $E'_{j_{h+1}}$ are $\prec$-comparable, by Theorem 3.2. Then by Theorem 6.3, $E_{j_h} \prec E'_{j_{h+1}}$ for $1 \leqslant h < g$.
For each h, let β'_{j_h} be the β corresponding to E'_{j_h}. Observe that $\beta_{j_h} = \beta'_{j_h} = \alpha_{i_h}$ for each h. Let $\{H_{k_h}\}_{h=1}^{g}$ and $\{H'_{k_h}\}_{h=1}^{g}$ be the sequences of elements of $\mathscr{H}$ such that

(4) $\mathrm{fix}(E_{j_h}) \subseteq \mathrm{fix}(H_{k_h})$ for $1 \leqslant i < g$;
(5) $\mathrm{fix}(E'_{j_h}) \subseteq \mathrm{fix}(H'_{k_h})$ for $2 \leqslant i \leqslant g$.

Let $\hat{\alpha}'_{k_h}$ be the α corresponding to H_{k_h}. Then $\beta_{j_h} = \hat{\alpha}_{k_h}$ and $\beta'_{j_h} = \hat{\alpha}'_{k_h}$ for each h, and hence $\hat{\alpha}_{k_h} = \hat{\alpha}'_{k_h}$ for each h. Since (3) holds for each h with $1 \leqslant h < g$, it follows by applying Theorem 6.3 that $H_{k_h} \preccurlyeq\cdot H'_{k_{h+1}}$ for each h. Then as noted above, $\hat{\alpha}_{k_h} \leqslant \hat{\alpha}'_{k_{h+1}}$ for $1 \leqslant h < g$. Thus

$$\alpha_r = \alpha_{i_1} = \beta_{j_1} = \hat{\alpha}_{k_1} \leqslant \alpha'_{k_g} = \beta'_{j_g} = \alpha'_{i_g} = \alpha_s.$$

That is, $F_r \prec\cdot F_s$ in $\mathscr{F}$ implies $\alpha_r \leqslant \alpha_s$. ∎

REMARK. These theorems have a nice pictorial interpretation. Given $\mathscr{L}(\mathscr{F})$, assign the α_i's to the nodes so that along any directed path the α_i's form a nondecreasing sequence. Then the resultant matrix A is a *ZME*-matrix.

8. PRODUCTS OF COMMUTING *ZME*-MATRICES

In this section, we determine conditions for the product of commuting *ZME*-matrices to be a *ZME*-matrix.

It is well known that if two diagonalizable matrices commute, then they have a common complete set of spectral projectors. That is, if A and B are diagonalizable matrices in $\mathscr{M}_n(\mathbb{C})$ and if $AB = BA$, then there exists a complete set of projectors $\mathscr{E} = \{E_i : 1 \leqslant i \leqslant n\}$, there exist complex numbers α_i (not necessarily distinct) for $1 \leqslant i \leqslant n$, and there exist complex numbers β_i (not necessarily distinct) for $1 \leqslant i \leqslant n$ such that

$$A = \sum_{i=1}^{n} \alpha_i E_i \quad \text{and} \quad B = \sum_{i=1}^{n} \beta_i E_i.$$

Even if each of the matrices has an inflation-generated complete set of projectors, it is not immediate that the common complete set is also inflation-generated. The question of when a decomposition into pairs of commuting, inflation-generated projectors exists appears to be a difficult one, and only certain partial results are known. (See [4, Chapter 8].)

The following notation is adopted for the remainder of this section:

NOTATION. Let A and B be commuting ZME-matrices which are in $\mathscr{M}_n(\mathbb{R})$ for some $n \geqslant 2$. Since A and B commute, they have a common set $\mathscr{E}$ of rank-one projectors with $n = |\mathscr{E}|$. Let $-\alpha_2 \leqslant \alpha_1 < \alpha_2 \leqslant \alpha_3 \leqslant \cdots \leqslant \alpha_n$ be the spectrum of A. (Unlike the notation previously used, in this section the α_i *need not be pairwise distinct.*) Label the elements of $\mathscr{E}$ so that

$$A = \sum_{i=1}^{n} \alpha_i E_i.$$

Label the spectrum of B as $\beta_1, \beta_2, \ldots, \beta_n$ so that

$$B = \sum_{i=1}^{n} \beta_i E_i.$$

Then there exists a permutation σ of the set $\{1, 2, \ldots, n\}$ such that $-\beta_{\sigma(2)} \leqslant \beta_{\sigma(1)} < \beta_{\sigma(2)} \leqslant \beta_{\sigma(3)} \leqslant \cdots \leqslant \beta_{\sigma(n)}$. Since α_1 and $\beta_{\sigma(1)}$ are the simple, minimum eigenvalues of A and B, respectively, it follows that E_1 is the unique, strictly positive element of $\mathscr{E}$, and hence, $\sigma(1) = 1$. The product of A and B is

$$AB = \sum_{i=1}^{n} \alpha_i \beta_i E_i.$$

LEMMA 8.1. *Let A and B be commuting ZME-matrices. Then AB has a real spectrum $\{\alpha_1\beta_1, \alpha_2\beta_2, \dots, \alpha_n\beta_n\}$, and $|\alpha_1\beta_1| \leqslant \alpha_i\beta_i$ for each $i \geqslant 2$. Further, if at least one of A and B is a ZMA-matrix and if λ is the minimal eigenvalue of AB, then λ is simple.*

Proof. Clearly the spectrum of AB is real. Since A is a *ZME*-matrix, it follows from the notational convention that $|\alpha_1| \leqslant \alpha_i$ for each $i \geqslant 2$. Since B is a *ZME*-matrix, both $|\beta_1| \leqslant \beta_i$ and $0 < \beta_i$ hold for each i with $i \geqslant 2$. Then $\alpha_1\beta_1 \leqslant |\alpha_1||\beta_1| \leqslant \alpha_i\beta_i$ for each $i \geqslant 2$.

Now suppose that one of A and B is a *ZMA*-matrix—without loss of generality, A. Then $|\alpha_1| < \alpha_i$ for each $i \geqslant 2$. Then $\alpha_1\beta_1 \leqslant |\alpha_1||\beta_1| < \alpha_i\beta_i$ for each $i \geqslant 2$. The minimum eigenvalue of AB is $\lambda = \alpha_1\beta_1$, which is simple. ■

REMARK. Suppose that A and B are *ZMO*-matrices; then $\alpha_1 = -\alpha_2$ and $\beta_1 = -\beta_{\sigma(2)}$. Let r be the largest integer such that $\alpha_r = \alpha_2$. Let s be the largest integer such that $\beta_{\sigma(s)} = \beta_{\sigma(2)}$. Then $\alpha_1\beta_1$ is a simple eigenvalue for AB if and only if $\{2,3,\dots,r\} \cap \{\sigma(2), \sigma(3), \dots, \sigma(s)\} = \varnothing$. To see this, suppose that j is in the intersection. Then $\alpha_j\beta_j = \alpha_2\beta_{\sigma(2)} = (-\alpha_1)(-\beta_1)$. Since $\alpha_1\beta_1 \leqslant \alpha_i\beta_i$ for $i \geqslant 2$, the existence of such a j implies that the minimal eigenvalue of AB has multiplicity greater than one. Hence AB cannot be a *ZME*-matrix if such a j exists. As a particular example, note that if A is a *ZMO*-matrix, then A^2 is not irreducible; hence A^2 is not a *ZME*-matrix. (This case corresponds to σ being the identity map and $j = 2$ being in the intersection.) Additionally, note that if $(r-1)+(s-1) > (n-1)$, then the intersection must be nonempty, and hence AB cannot be a *ZME*-matrix.

THEOREM 8.2. *Let A and B be ZMO-matrices. Suppose that A and B have a common complete set $\mathscr{F}$ of projectors which is inflation-generated. Label the elements of $\mathscr{F}$ as F_i for $1 \leqslant i \leqslant k$. For $1 \leqslant i \leqslant k$, let $\tilde{\alpha}_i$ and $\tilde{\beta}_i$ be the (not necessarily distinct) real numbers such that A and B can be represented as*

$$A = \sum_{i=1}^{k} \tilde{\alpha}_i F_i \quad \text{and} \quad B = \sum_{i=1}^{k} \tilde{\beta}_i F_i.$$

Then AB is not a ZME-matrix.

Proof. Without loss of generality, suppose that the elements of $\mathscr{F}$ are labeled so that $A = \sum_{i=1}^{k}\tilde{\alpha}_i F_i$ with $\tilde{\alpha}_1 < \tilde{\alpha}_2 \leqslant \cdots \leqslant \tilde{\alpha}_k$. Clearly $\tilde{\alpha}_1 = \alpha_1$ and $\tilde{\alpha}_2 = -\tilde{\alpha}_1$. Since A is a *ZME*-matrix, Theorem 3.7 applies to A. Thus F_1 is the unique strictly positive element of $\mathscr{F}$, and F_2 is the unique element of $\mathscr{F}$ with mixed signs and no zero entries. Since B is a ZMO-matrix, Theorem 3.7 applies to B. Consequently, $\tilde{\beta}_1 = \beta_1$ and $\tilde{\beta}_2 = -\beta_1$. Then $\tilde{\alpha}_1\tilde{\beta}_1 = \tilde{\alpha}_2\tilde{\beta}_2$. Since $\tilde{\alpha}_2\tilde{\beta}_2$ is in the spectrum of AB, it follows that the minimum eigenvalue of AB is not simple; hence AB cannot be a ZME-matrix. ∎

COROLLARY 8.3. *Let A and B be $n \times n$ commuting ZMO-matrices. Suppose that at least one of A and B has n simple eigenvalues. Then AB is not a ZME-matrix.*

Proof. Without loss, suppose that A has n distinct eigenvalues. Then $\mathscr{E}$ must be inflation-generated, by the inflation theorem (Theorem 2.6.3). Since A and B commute, the set $\mathscr{E}$ is the set $\mathscr{F}$ of the preceding theorem. ∎

Theorem 8.2 and the remarks which precede it suggest the following conjecture:

CONJECTURE 8.4. The product of two commuting ZMO-matrices is not a ZME-matrix.

We turn now to the case where at least one of A and B is a ZMA-matrix.

THEOREM 8.5. *Let A and B be ZME-matrices such that at least one of A and B is a ZMA-matrix. Suppose that A and B have a common complete set $\mathscr{F}$ of projectors which is inflation-generated. Label the elements of $\mathscr{F}$ as F_i for $1 \leqslant i \leqslant k$. For $1 \leqslant i \leqslant k$, let $\tilde{\alpha}_i$ and $\tilde{\beta}_i$ be the (not necessarily distinct) real numbers such that A and B can be represented as*

$$A = \sum_{i=1}^{k} \tilde{\alpha}_i F_i \quad \textit{and} \quad B = \sum_{i=1}^{k} \tilde{\beta}_i F_i .$$

Then AB is a ZMA-matrix.

Proof. Without loss of generality, suppose that the elements of $\mathscr{F}$ are labeled so that $A = \sum_{i=1}^{k} \tilde{\alpha}_i F_i$ with $\tilde{\alpha}_1 < \tilde{\alpha}_2 < \cdots < \tilde{\alpha}_k$. Clearly $\tilde{\alpha}_1 = \alpha_1$. Since A is a ZME-matrix, F_1 is the unique strictly positive element of $\mathscr{F}$. Consequently, $\tilde{\beta}_1 = \beta_1$, since B is a ZME-matrix. By Lemma 8.1, AB has a real spectrum with simple minimum eigenvalue $\tilde{\alpha}_1\tilde{\beta}_1$. Since A is a ZME-matrix, and since $\mathscr{F}$ is inflation-generated, Theorem 7.2 can be applied to A

TABLE 8.1
PRODUCTS OF *ZME*-MATRICES WHICH HAVE
A COMMON SET OF INFLATION-GENERATED PROJECTORS

A	B	AB
ZMO	*ZMO*	Not *ZME*
ZMA	*ZME*	*ZMA*
ZME	*ZMA*	*ZMA*

to yield $\tilde{\alpha}_i \leqslant \tilde{\alpha}_j$ whenever $F_i \prec\cdot F_j$ in $\mathscr{F}$. Similarly Theorem 7.2 can be applied to B to yield $\tilde{\beta}_i \leqslant \tilde{\beta}_j$ whenever $F_i \prec\cdot F_j$ in $\mathscr{F}$. By Lemma 8.1, $|\tilde{\alpha}_1\tilde{\beta}_1| < \tilde{\alpha}_j\tilde{\beta}_j$ for $j \geqslant 2$. Thus $F_1 \prec\cdot F_j$ in $\mathscr{F}$ implies $\tilde{\alpha}_1\tilde{\beta}_1 < \tilde{\alpha}_j\tilde{\beta}_j$ for $j \geqslant 2$. Suppose that $i \geqslant 2$. Then $F_i \prec\cdot F_j$ in $\mathscr{F}$ implies both $0 < \tilde{\alpha}_i \leqslant \tilde{\alpha}_j$ and $0 < \tilde{\beta}_i \leqslant \tilde{\beta}_j$, and thus $\tilde{\alpha}_i\tilde{\beta}_i \leqslant \alpha_j\tilde{\beta}_j$. Then by Theorem 7.2, AB is a *ZME*-matrix, and hence a *ZMA*-matrix. ∎

Table 1 summarizes the results of Theorems 8.2 and 8.4.

COROLLARY 8.6. *Let* A *and* B *be* $n \times n$ *commuting ZME-matrices. Suppose that at least one of* A *and* B *is a ZMA-matrix. Suppose that at least one of* A *and* B *has* n *simple eigenvalues. Then* AB *is a ZMA-matrix.*

Proof. Without loss, suppose that A has n distinct eigenvalues. Then $\mathscr{E}$ must be inflation-generated, by the inflation theorem (Theorem 2.6.3). Since A and B commute, the set $\mathscr{E}$ is the set $\mathscr{F}$ of the preceding theorem. ∎

EXAMPLE. The product of commuting *ZMA*-matrices all of those eigenvalues are simple need not have all of its eigenvalues simple. Let $\mathscr{E} = \{E_i : 1 \leqslant i \leqslant 4\}$ be the complete set of inflation-generated projectors discussed in Section 2.7. Let A and B be the *MMA*-matrices

$$A = 1E_1 + 2E_2 + 3E_3 + 4E_4,$$

$$B = 1E_1 + 2E_2 + 4E_3 + 3E_4.$$

Then AB is the *MMA*-matrix

$$AB = 1E_1 + 4E_2 + 12(E_3 + E_4).$$

We end this section with a final open question:

QUESTION 8.7. Suppose that A and B are commuting ZME-matrices. Do A and B have a common complete set of projectors which is inflation-generated?

It should be noted that if the answer is affirmative, then it follows that the product of commuting ZMO-matrices is never a ZME-matrix, and that the product of commuting ZME-matrices is always a ZMA-matrix if one of the matrices is a ZMA-matrix.

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