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On Lyapunov Scaling Factors of Real Symmetric Matrices

DANIEL HERSHKOWITZ
Department of Mathematics, Technion – Israel Institute of Technology, Haifa 32000, Israel

and

HANS SCHNEIDER
Mathematics Department, University of Wisconsin – Madison, Madison, Wisconsin 53706, USA

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A necessary and sufficient condition for the identity matrix to be the unique Lyapunov scaling factor of a given real symmetric matrix A is given. This uniqueness is shown to be equivalent to the uniqueness of the identity matrix as a scaling D for which the kernels of A and AD are identical.

1. INTRODUCTION

A real square matrix A is said to be *Lyapunov diagonally semistable* if there exists a positive definite diagonal matrix D, called a *Lyapunov scaling factor of* A, such that the matrix $AD+DA^T$ is positive semidefinite.

Lyapunov diagonally semistable matrices play an important role in applications in several disciplines, and have been studied in many matrix theoretical papers, see for example [2] for some references.

In this paper we mainly discuss real Hermitian (symmetric) matrices. The problem of characterizing Lyapunov diagonally semistable symmetric matrices is easy. Clearly, every positive semidefinite

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symmetric matrix A is Lyapunov diagonally semistable, and the identity matrix is a Lyapunov scaling factor of A. Conversely, as is well known (e.g. [1]), a Lyapunov diagonally semistable matrix has nonnegative principal minors. Therefore, a symmetric matrix A is Lyapunov diagonally semistable if and only if A is positive semidefinite.

An interesting and natural question is the following: Given a positive semidefinite symmetric matrix A, is the identity matrix the unique (up to a nonzero scalar multiplication) Lyapunov scaling factor of A?

This problem is solved in this paper. The uniqueness of Lyapunov scaling factors for general matrices has recently been studied in [2], [4] and [5]. Here we employ and improve methods developed in [2] in order to give a necessary and sufficient condition for the above-mentioned uniqueness. We give two equivalent such conditions, a graph theoretic one as well as a rank condition. Also it is shown that the identity matrix is the unique Lyapunov scaling factor of a positive semidefinite symmetric matrix A if and only if the only nonsingular diagonal matrices D, for which the kernels of A and AD are identical, are the nonzero scalar matrices.

An important tool used in our study is the principal submatrix rank property. This property was introduced in [3], where it is also shown to be linked to Lyapunov diagonal semistability. Here we also define and use a somewhat weaker property. Another concept we define is the concept of A-minimal sets. We prove some properties of these sets, which are of interest by themselves.

2. NOTATION AND DEFINITIONS

2.1 *Notation* For a positive integer n we denote by:

$\langle n\rangle$ – the set $\{1, 2, \ldots, n\}$,

C^{nn} – the set of all $n \times n$ complex matrices,

C^n – the set of all n-dimensional complex (column) vectors.

2.2 *Notation* For a ser α we denote by $|\alpha|$ the cardinality of α.

2.3 *Notation* Let $A \in C^{nn}$, let $x \in C^n$ and let α and β be nonempty subsets of $\langle n\rangle$. We denote by:

$N(A)$ – the nullspace (kernel) of A,

$n(A)$ – the nullity of A, i.e., the dimension of $N(A)$,

$r(A)$ – the rank of A ($r(A) = n - n(A)$),

$A[\alpha|\beta]$ – the submatrix of A whose rows are indexed by α and whose columns are indexed by β in their natural orders.
$A[\alpha]$ $= A[\alpha|\alpha]$.
$x[\alpha]$ – the vector obtained from x by eliminating the components x_i such that $i \notin \alpha$,
$\text{supp}(x)$ $= \{i \in \langle n \rangle : x_i \neq 0\}$.

2.4 *Notation* Let V be a subset of C^{nn}. We denote

$$\text{supp}(V) = \bigcup_{x \in V} \text{supp}(x).$$

2.5 *Notation* Let V be an m-dimensional subspace of C^{nn}. We denote by $E(V)$ the $n \times m$ matrix which is in column reduced echelon form and such that the columns of $E(V)$ form a basis for V.

2.6 *Notation* Let G be a (nondirected) graph. An edge between i and j in G is denoted by $[i,j]$.

2.7 *Notation* Let G be a graph and let E be the edge set of G. The graph G is said to be full (or complete) if for every two vertices i and j in G we have $[i,j] \in E$.

2.8 *Notation* Let G be a graph and let E be the edge set of G. The graph G is said to be transitive if $[i,j]$, $[j,k] \in E$ implies that $[i,k] \in E$.

2.9 *Definition* Let G be a graph and let i and j be two vertices of G. We say that there exists a path between i and j in G if there exists a sequence $i_1, \ldots, i_k$ of vertices of G such that $i_1 = i$, $i_k = j$, and there is an edge between i_t and i_{t+1}, $t = 1, \ldots, k-1$, in G. The graph G is said to be connected if there exists a path between every two vertices in G.

2.10 *Definition* Let G_1 and G_2 be graphs. We say that G_1 is a subgraph of G_2 if the vertex sets of G_1 and G_2 are identical, and if the edge set of G_1 is contained in the edge set of G_2.

2.11 *Definition* Let A be an $m \times n$ matrix. The (nondirected) bipartite graph of A, denoted by $H(A)$, is the bipartite graph for which the two sets of vertices are $\{1, \ldots, m\}$ and $\{m+1, \ldots, m+n\}$, and where there is an edge between i and $j+m$ $(i \in \langle m \rangle, j \in \langle n \rangle)$ if and only if $a_{ij} \neq 0$.

The following Definitions 2.12, 2.13 and 2.14 were first given in [2].

2.12 *Definition* Let A be an $n \times n$ matrix and let α be a nonempty subset of $\langle n\rangle$. We define the set $s(A,\alpha)$ by

$$s(A,\alpha)=\alpha\cup\{j\in\langle n\rangle;\ A[\alpha|j]\notin \operatorname{Range}(A[\alpha]+A[\alpha]^T)\}.$$

2.13 *Definition* Let A be an $n\times n$ matrix and let α be a nonempty subset of $\langle n\rangle$. We define the set $s\hat{\ }(A,\alpha)$ by the following algorithm:

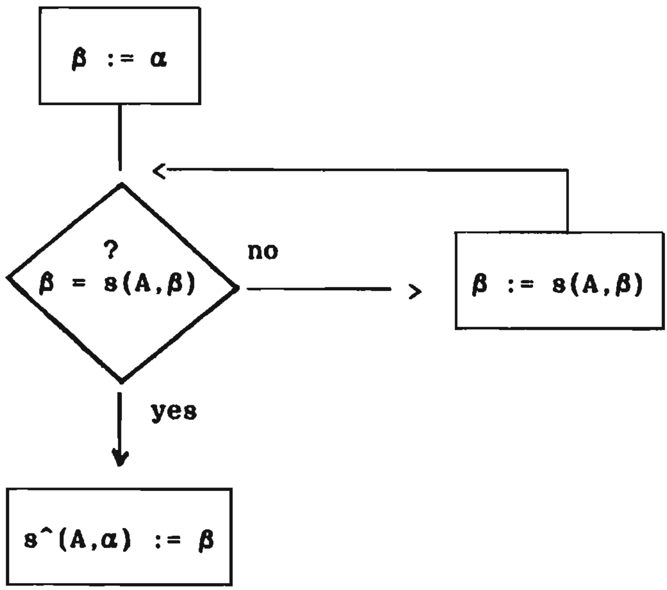

FIGURE 1.

2.14 *Definition* Let $A\in C^{nn}$. The graph $U(A)$ is defined as follows: The vertex set of $U(A)$ is $\langle n\rangle$, and there is an edge between the vertices i and j if i and j belong to a set $s\hat{\ }(A,\alpha)$ where $A[\alpha]$ is singular and $H(E(N([\alpha])))$ is connected.

2.15 *Definition* Let $A\in C^{nn}$. A set $\alpha\subseteq\langle n\rangle$ is said to be an A-minimal set if $A[\alpha]$ is singular but all proper submatrices of $A[\alpha]$ are nonsingular.

2.16 *Definition* Let $A\in C^{nn}$. The graph $U\tilde{\ }(A)$ is defined as follows: The vertex set of $U\tilde{\ }(A)$ is $\langle n\rangle$, and there is an edge between the vertices i and j if there is an A-minimal set α such that $i,j\in\alpha$.

2.17 *Definition* An $n\times n$ matrix A is said to have the *principal* *submatrix rank property* (PSRP) if it satisfies

$$r(A[\alpha] \mid \langle n\rangle) = r(A[\langle n\rangle \mid \alpha]) = r(A[\alpha])$$

for all nonempty sets $\alpha \subseteq \langle n\rangle$ (see also [3]).

The matrix A is said to have the *row principal submatrix rank property* (RPSRP) if it satisfies

$$r(A[\langle n\rangle \mid \alpha]) = r(A[\alpha])$$

for all nonempty sets $\alpha \subseteq \langle n\rangle$.

The matrix A is said to have the *column principal submatrix rank property* (CPSRP) if it satisfies

$$r(A[\alpha \mid \langle n\rangle]) = r(A[\alpha])$$

for all nonempty sets $\alpha \subseteq \langle n\rangle$.

2.18 *Remark* As is well known, positive semidefinite (Hermitian) matrices have the PSRP. This fact will be heavily used in the sequel.

3. MINIMAL SETS

3.1 LEMMA *Let A be a singular matrix and let $i \in \operatorname{supp}(N(A))$. If A has the* RPSRP *then there exists an A-minimal set which contains i.*

Proof Let $x^1, \ldots, x^p$ be the columns of the matrix $E(N(A))$, and let $i \in \operatorname{supp}(N(A))$. Obviously, there exists $j \in \langle p\rangle$ such that $i \in \operatorname{supp}(x^j)$. Since $E(N(A))$ is in column reduced form, the set $\operatorname{supp}(x^j)$ is an A-minimal set, and the result follows. ■

The requirement that A has the RPSRP cannot be omitted from Lemma 3.1 as demonstrated by the matrix

$$A = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}.$$

Since $N(A)$ is spanned by $[1\ -1]^T$, it follows that $\operatorname{supp}(N(A)) = \{1, 2\}$. However, there exists no A-minimal set that contains 2, since $\{1\}$ is the only A-minimal set.

3.2 PROPOSITION *Let $A \in C^{nn}$ have the* RPSRP *(or the* CPSRP*), and let α and β be A-minimal sets. If $\alpha \cap \beta \neq \varnothing$ then for every $k, l \in \alpha \cup \beta$ there exists an A-minimal set γ such that $k, l \in \gamma$.*

Proof Without loss of generality we may assume that A has the RPSRP. We prove our assertion by induction on $|\alpha\cup\beta|$. For $|\alpha\cup\beta|=1$ there is nothing to prove. Assume that our claim holds for $|\alpha\cup\beta|<m$, $m>1$, and let $|\alpha\cup\beta|=m$. If both k and l are in α or in β then there is nothing to prove. Therefore, without loss of generality we may assume that

$$k\in a\backslash\beta, \qquad l\in\beta\backslash\alpha. \tag{3.3}$$

Let y^1 and y^2 be nonzero vectors in $N(A[\alpha])$ and $N(A[\beta])$ respectively, and define $x^1, x^2\in C^n$ by

$$x_1=\begin{cases} y_1, & i\in\alpha\\ 0, & i\in\langle n\rangle\backslash\alpha,\end{cases}$$

$$x_2=\begin{cases} y_2, & i\in\beta\\ 0, & i\in\langle n\rangle\backslash\beta.\end{cases}$$

Since A has the RPSRP it follows that $x^1, x^2\in N(A)$. Observe that since α and β are A-minimal sets, we have $\operatorname{supp}(x^1)=\operatorname{supp}(y^1)=\alpha$ and $\operatorname{supp}(x^2)=\operatorname{supp}(y^2)=\beta$. Since $\alpha\cap\beta\neq\varnothing$ we can choose $t\in\alpha\cap\beta$. Define the vector $x=x_t^2x^1-x_t^1x^2$, and let $\delta=\operatorname{supp}(x)$. Observe that $t\notin\delta$ and that

$$k,l\in\delta. \tag{3.4}$$

Since $x\in N(A)$ it follows that the matrix $B=A[\delta]$ is singular. By Lemma 3.1 it follows from (3.4) that there exist B-minimal (and thus A-minimal) sets μ and ν such that

$$k\in\mu, \qquad l\in\nu. \tag{3.5}$$

It follows from the minimality of α and β that $\mu\not\subseteq\alpha$ and $\nu\not\subseteq\beta$. Thus,

$$\mu\cap(\beta\backslash\alpha)\neq\varnothing, \qquad \nu\cap(\alpha\backslash\beta)\neq\varnothing. \tag{3.6}$$

Distinguish between two cases:

(i) $\alpha\backslash\beta\subseteq\mu$. By (3.6) we have $\mu\cap\nu\neq\varnothing$. Since $\mu\cup\nu\subseteq\delta$ and since δ is a proper subset of $\alpha\cup\beta$, our assertion now follows from (3.5) by the inductive assumption:

(ii) $\alpha\backslash\beta\not\subseteq\mu$. In this case $\mu\cup\beta$ is a proper subset of $\alpha\cup\beta$. Since by (3.6) $\mu\cap\beta\neq\varnothing$, and since $k\in\mu$ and $l\in\beta$, our assertion now follows by the inductive assumption. ■

3.7 COROLLARY *Let $A \in C^{nn}$ have the* RPSRP (*or the* CPSRP). *Then the graph $U^{\sim}(A)$ is transitive.*

3.8 PROPOSITION *Let A be a positive semidefinite matrix. Then $U(A) = U^{\sim}(A)$.*

Proof Let α be an A-minimal set. Clearly, the bipartite graph $H(E(N(A[\alpha])))$ is connected. Thus, it follows from Definitions 2.14 and 2.16 that $U^{\sim}(A)$ is a subgraph of $U(A)$.

Conversely, since A is a symmetric matrix it follows that $A + A^T = 2A$. Furthermore, since A has the PSRP it follows that for every $\alpha \subseteq \langle n \rangle$ we have $s^{\wedge}(A, \alpha) = \alpha$. Now let $\alpha \subseteq \langle n \rangle$ be such that $H(E(N(A[\alpha])))$ is connected. Observe that for every column x of $E(N(A[\alpha]))$, the set supp(x) is A-minimal. It now follows from the connectedness of $H(E(N(A[\alpha])))$ and from Corollary 3.7 that for every $i, j \in \alpha$ there is an edge between i and j in $U^{\sim}(A)$. Therefore, the graph $U(A)$ is a subgraph of $U^{\sim}(A)$. ∎

3.9 *Remark* Proposition 3.8 does not hold in general if we weaken the requirement that A is positive semidefinite by requiring that A have the PRSR. This is demonstrated by the matrix

$$A = \begin{bmatrix} 2 & 3 & 4 & 2 \\ 1 & 2 & 3 & 1 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

The matrix A has the PSRP, and the only A-minimal set is $\{1, 2, 3\}$. Thus, the graph $U^{\sim}(A)$ is

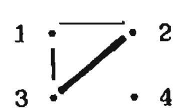

FIGURE 2.

However, since $s^{\wedge}(A, \alpha) = \{1, 2, 3, 4\}$, the graph $U(A)$ is

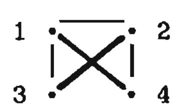

FIGURE 3.

4. UNIQUENESS OF LYAPUNOV SCALING FACTORS FOR HERMITIAN MATRICES

4.1 Lemma *Let an $n \times n$ matrix A have the* RPSRP *(or the* CPSRP*). Then for all nonempty sets $\alpha \subseteq \langle n \rangle$, $\alpha \neq \langle n \rangle$, we have*

$$r(A) \leqslant r(A[\alpha]) + r(A[\langle n \rangle \backslash \alpha]). \tag{4.2}$$

Furthermore, if for some nonempty $\alpha \subseteq \langle n \rangle$, $\alpha \neq \langle n \rangle$, equality holds in (4.2) *then for every nonempty $\beta \subseteq \alpha$ and every nonempty $\gamma \subseteq \langle n \rangle \backslash \alpha$ we have*

$$r(A[\beta \cup \gamma]) = r(A[\beta]) + r(A[\gamma]). \tag{4.3}$$

Proof Suppose that A has the RPSRP. It follows that for every nonempty $\alpha \subseteq \langle n \rangle$, $\alpha \neq \langle n \rangle$, we have

$$N(A[\alpha]) \oplus N(A[\langle n \rangle \backslash \alpha]) \subseteq N(A), \tag{4.4}$$

where $\oplus$ denotes a direct sum of vector spaces. Hence,

$$n(A) \geqslant n(A[\alpha]) + n(A[\langle n \rangle \backslash \alpha]),$$

which implies (4.2).

To prove the rest of the lemma let equality holds in (4.2) for some nonempty $\alpha \subseteq \langle n \rangle$, $\alpha \neq \langle n \rangle$. By (4.4) we now have

$$N(A[\alpha]) \oplus N(A[\langle n \rangle \backslash \alpha]) = N(A). \tag{4.5}$$

Let $\gamma \subseteq \langle n \rangle \backslash \alpha$, $\gamma \neq \varnothing$, and let

$$x \in N(A[\alpha \cup \gamma]). \tag{4.6}$$

Define the vector $y \in C^n$ by

$$y_i = \begin{cases} x_i, & i \in \alpha \cup \gamma \\ 0, & \text{otherwise}. \end{cases}$$

Since A has the RPSRP it follows that $y \in N(A)$. By (4.5) we now have

$$x[\alpha] = y[\alpha] \in N(A[\alpha]). \tag{4.7}$$

By the RPSRP it follows from (4.7) that $A[\gamma \mid \alpha]x[\alpha] = 0$, and hence

$$A[\gamma]x[\gamma] = A[\gamma \mid \alpha]x[\alpha] + A[\gamma]x[\gamma] = (A[\alpha \cup \gamma]x)[\gamma] = 0.$$

which means that

$$x[\gamma] \in N(A[\gamma]). \tag{4.8}$$

It now follows from (4.6), (4.7) and (4.8) that

$$N(A[\alpha\cup\gamma])\subseteq N(A[\alpha])\oplus N(A[\gamma]).$$

However, as observed in (4.4),

$$N(A[\alpha]\oplus N(A[\gamma])\subseteq N(A[\alpha\cup\gamma]),$$

and therefore

$$N(A[\alpha\cup\gamma])=N(A[\alpha])\oplus N(A[\gamma]). \tag{4.9}$$

We have proved that (4.5) implies (4.9). Choosing nonempty $\beta\subseteq\alpha$, it thus follows that (4.9) implies that

$$N(A[\beta\cup\gamma])=N(A[\beta])\oplus N(A[\gamma]),$$

which yields (4.3). ■

It is clear that the identity matrix is a Lyapunov scaling factor of every positive semidefinite symmetric matrix. We now characterize those positive semidefinite symmetric matrices for which the identity matrix is the unique Lyapunov scaling factor.

4.10 THEOREM *Let A be a $n\times n$ positive semidefinite symmetric matrix. Then the following are equivalent:*

(i) *The identity matrix is the unique Lyapunov scaling factor of A;*
(ii) *The graph $U^{\sim}(A)$ is connected;*
(iii) *The graph $U^{\sim}(A)$ is full;*
(iv) *For every nonempty $\alpha\subseteq\langle n\rangle$, $\alpha\neq\langle n\rangle$, we have*

$$r(A)<r(A[\alpha])+r(A[\langle n\rangle\backslash\alpha]).$$

(v) *The bipartite graph $H(E(N(A)))$ is connected;*
(vi) *For every nonsingular diagonal matrix D we have $N(AD)=N(A)$ if and only if D is a scalar matrix.*

Proof (i)$\Leftrightarrow$(ii) In view of Proposition 3.8, this equivalence follows from Theorems 6.18 and 6.20 of [2], observing that for every $\nu\subseteq\langle n\rangle$ we have $A[\nu]+A[\nu]^T=2A[\nu]$.

(ii)$\Leftrightarrow$(iii) By Corollary 3.7.

(i)$\Rightarrow$(iv) Let $\alpha\subseteq\langle n\rangle$, $\alpha\neq\emptyset,\langle n\rangle$. In view of (4.2) we have to prove that if

$$r(A)=r(A[\alpha])+r(A[\langle n\rangle\backslash\alpha]), \tag{4.11}$$

then A has a Lyapunov scaling factor which is not a scalar matrix. So, assume that (4.11) holds. Let D_a, $a>0$, be the $n\times n$ diagonal matrix whose diagonal entries are

$$d_{11}=\begin{cases}1, & i\in\alpha\\ a, & i\notin\alpha.\end{cases}$$

Clearly, if $a\neq 1$, then D_a is not a scalar matrix. Consider the matrix $C_a=AD_a+D_aA$. Observe that $C_1=2A$. Let $\beta\subseteq\langle n\rangle$, $\beta\neq\varnothing$. If $\beta\subseteq\alpha$, then $C_a[\beta]=2aA[\beta]$ and hence

$$\det C_a[\beta]\geqslant 0. \tag{4.12}$$

If $\beta\cap\alpha\neq\varnothing$ then $C_a[\beta]=2aA[\beta]$ and again (4.12) holds. The remaining case is where $\gamma=\alpha$, $\beta\neq\varnothing$ and $\delta=\beta\backslash\alpha\neq\varnothing$. Here we distinguish between two possibilities:

(a) At least one of the matrices $A[\gamma]$ and $A[\delta]$ is singular. Without loss of generality we assume that $A[\gamma]$ is singular. Since A has the PSRP it follows that for every $j\in\delta$ we have

$$C_a[\gamma\,|\,j]=(a+1)A[\gamma\,|\,j]\in\text{Range }A[\gamma],$$

and hence $C_a[\beta]$ is singular.

(b) Both $A[\gamma]$ and $A[\delta]$ are nonsingular. By Lemma 4.1 it follows from (4.11) that $A[\beta]$ is nonsingular and hence is positive definite. Therefore, the matrix $C_1[\beta]$ is positive definite. By continuity arguments it follows that (4.12) holds whenever $|a-1|$ is sufficiently small.

We have proved that in any case, if $|a-1|$ is sufficiently small then (4.12) holds. Since the number of sets β, $\beta\subseteq\langle n\rangle$ is finite, it follows that the matrix C_a is positive semidefinite for $|a-1|$ sufficiently small. Hence the nonscalar matrix D_a is a Lyapunov scaling factor of A.

(iv)$\Rightarrow$(ii) Suppose that (iv) holds. Assume that $U^{\sim}(a)$ is not connected, namely, that there exists a nonempty set $\alpha\subseteq\langle n\rangle$, $a\neq\langle n\rangle$, such that if β is an A-minimal set then either $\beta\subseteq\alpha$ or $\beta\cap\alpha=\varnothing$. Let r_1 and r_2 be the ranks of $A[\alpha]$ and $A[\langle n\rangle\backslash\alpha]$ respectively, and let $\gamma\subseteq\langle n\rangle$ be such that $A[\gamma\cap\alpha]$ and $A[\gamma\backslash\alpha]$ are $r_1\times r_1$ and $r_2\times r_2$ (respectively) positive definite matrices. If $A[\gamma]$ is singular then γ contains an A-minimal set which, by the choice of α, is contained in α or $\langle n\rangle\backslash\alpha$, contradicting the positive definiteness of $A[\gamma\cap\alpha]$ or $A[\gamma\backslash\alpha]$. Therefore, $A[\gamma]$ is nonsingular and hence

$$r(A[\gamma]) = r_1 + r_2. \tag{4.13}$$

It now follows from (4.13) that

$$r(A) \geq r_1 + r_2 = r(A[\alpha]) + r(A[\langle n\rangle \backslash \alpha]),$$

in contradiction to (iv). Thus, our assumption that $U^{\sim}(A)$ is not connected is false.

(ii)$\Leftrightarrow$(vi) Clearly, if D is a scalar matrix then $N(AD) = N(A)$. Suppose that $N(AD) = N(A)$ where D is a nonsingular diagonal matrix. Let $\alpha \subseteq \langle n\rangle$ be a nonempty set, and let $x \in C^n$ be such that $x_i = 0$ for all $i \notin \alpha$. Since A has the PSRP it follows that $x \in N(A)$ if and only if $x[\alpha] \in N(A[\alpha])$. Hence, we have $n(AD) = N(A)$ if and only if for every $\alpha \subseteq \langle n\rangle$, $\alpha \neq \emptyset$, we have $N(A[\alpha]D[\alpha]) = N(A[\alpha])$. By Corollary 5.5 of [2], if α is an A-minimal set then $D[\alpha]$ is a scalar matrix. Thus, if the graph $U^{\sim}(A)$ is connected then (vi) follows.

(vi)$\Rightarrow$(i) By Lemma 6.6 of [2].

(v)$\Leftrightarrow$(vi) By Theorem 5.2 of [2]. ∎

4.14 *Remark* The equivalence (v)$\Leftrightarrow$(vi) in Theorem 4.10 holds for general Hermitian matrices. Also it is easy to prove that the implication (v)$\Rightarrow$(ii) holds for general Hermitian matrices. However, the converse is not true in general, as demonstrated by the matrix

$$A = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 1 & 1 \\ 2 & 1 & 1 \end{bmatrix}.$$

The A-minimal sets are $\{1, 2\}$ and $\{2, 3\}$ and hence $U^{\sim}(A)$ is connected. Nevertheless, A is nonsingular and hence $H(E(N(A)))$ is not connected. This example also shows that the implication (ii)$\Rightarrow$(iii) does not hold for general Hermitian matrices.

4.15 *Remark* Our results raise the natural question whether Theorem 4.10 can be strengthened by replacing the phrase "Let A be an $n \times n$ positive semidefinite matrix" by "Let A be an $n \times n$ Lyapunov diagonally semistable matrix such that the identity matrix is a Lyapunov scaling factor of A and such that A has the PSRP". The answer to this question is negative as demonstrated by the matrix A defined in Remark 3.9. The matrix A is a Lyapunov diagonally semistable matrix such that the identity matrix is a Lyapunov scaling factor of A and such that A has the PSRP. Since $U(A)$ is connected, it follows from Theorem 6.18 in [2] that the identity matrix is the unique Lyapunov scaling factor of A. However, the graph $U^{\sim}(A)$ is not connected.

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