SIAM J. MATRIX ANAL. APPL.
Vol. 9, No. 1, January 1988

001

EQUALITY CLASSES OF MATRICES*

DANIEL HERSHKOWITZ† AND HANS SCHNEIDER†‡

Abstract. Recent results of Neumaier for irreducible matrices on the equality case of a classical matrix inequality due to Ostrowski are generalized to general matrices. Several graph and number theoretic concepts are employed in the proof of various further results.

Key words. irreducible matrix, Frobenius normal form, diagonal similiarity, equality class, twist, cycle product, access cover

AMS(MOS) subject classifications. 15, 05

1. Introduction. Let A be a complex $n \times n$ matrix and define the absolute value matrix $B = |A|$ of A by $b_{ij} = |a_{ij}|$, $i, j = 1, \cdots, n$. Let $\rho(A)$ be the spectral radius of A.

Let $\mathscr{U}$ be the set of all complex matrices A such that $\rho(|A|) < 1$. In [7] Ostrowski proves the now very well known result that, for $A \in \mathscr{U}$,

$$|(I-A)^{-1}| \leqq (I-|A|)^{-1}, \tag{1.1}$$

where the inequality is entrywise.

In [6] Neumaier shows that for $A \in \mathscr{I}$, the set of $n \times n$ irreducible matrices $A \in \mathscr{U}$,

$$|(I-A)^{-1}| = (I-|A|)^{-1}, \tag{1.2}$$

if and only if

(1.3) all circuit products of A are positive.

It is well known ([2], [3]) that for irreducible A, (1.3) is equivalent to

(1.4) A is diagonally similar to $|A|$, i.e., there exists a diagonal matrix X such that $A = X|A|X^{-1}$.

Neumaier also shows in [6] that the condition

$$|(I-A^{-1})|_{ij} = (I-|A|^{-1})_{ij}, \quad \text{for some } i,j, \quad 1 \leqq i,j \leqq n, \tag{1.5}$$

which is apparently weaker than (1.2), is in fact equivalent to (1.2)–(1.4) for $A \in \mathscr{I}$. (We have stated special cases of the results of Ostrowski and Neumaier, from which, however, the general theorems may easily be derived.)

In this paper we generalize Neumaier's results in various directions. We consider the equality (1.2) for general $A \in \mathscr{U}$, omitting the requirement of irreducibility. We use the concept of two-twisted chain of the graph $G(A)$ of A, which was defined in [5] (see also § 2 of this paper). Intuitively, a chain in a directed graph is obtained by putting a pointer at a vertex and moving it either in the direction or against the direction of a connected sequence of arcs to another vertex. Each change in direction is a twist. A two-twisted chain (e.g., cycle) is a chain with at most two twists. Thus, a circuit (directed

* Received by the editors May 19, 1986; accepted for publication October 27, 1987. This work was supported by the United States–Israel Binational Science Foundation under joint grant 85-00153.

† Mathematics Department, University of Wisconsin, Madison, Wisconsin 53706 and Mathematics Department, Technion-Israel Institute of Technology, Haifa 32000, Israel. The first author is permanently located at the Technion and the second at the University of Wisconsin.

‡ The work of this author was supported in part by National Science Foundation grants DMS-8320189 and DMS-8521521, Office of Naval Research grant N00014-85-K-1613, and the Lady Davis Foundation of Israel.

cycle) is a special case of a two-twisted cycle. We show that, for $A \in \mathscr{U}$, condition (1.2) is equivalent to

(1.6) all cycle products of A corresponding to two-twisted cycles are positive

(and other conditions). This generalizes (1.3).

If C is an $s \times s$ matrix and A is an $n \times n$ matrix, where $s \leqq n$, we generalize both the Kronecker and Hadamard products in [4] by defining the $n \times n$ matrix $C \times\times A$, see also § 3. Thus, if A is partitioned into s^2 matrices A_{ij}, $i, j = 1, \cdots, s$, then $C \times\times A$ is the matrix whose blocks are $c_{ij}A_{ij}$, $i, j = 1, \cdots, s$. Here we show that if $A \in \mathscr{U}$ is in Frobenius normal form then A satisfies (1.2) if and only if

(1.7) A is diagonally similar to $C \times\times |A|$, where C is an upper triangular $s \times s$ matrix ($s \leqq n$) such that $|c_{ij}|$ is 1 or 0, c_{ii} is 1 or 0, $i, j = 1, \cdots, s$, and zC satisfies (1.2) for $0 < z < 1$.

This generalizes (1.4).

We also generalize (1.5) by defining the concept of a $G(A)$-access cover, see also § 2. A subset Γ of $\langle n \rangle \times \langle n \rangle$, where $\langle n \rangle = \{1, \cdots, n\}$, is a $G(A)$-access cover if for each $(i, j) \in \langle n \rangle \times \langle n \rangle$ there is an $(h, k) \in \Gamma$ such that h has access to i in $G(A)$ and j has access to k in $G(A)$. We observe that $\{(i, j)\}$ is a $G(A)$-access cover for all $(i, j) \in \langle n \rangle \times \langle n \rangle$ if and only if A is irreducible (or equivalently, $G(A)$ is strongly connected). Thus, if Γ is a $G(A)$-access cover and $A \in \mathscr{U}$, then (1.2) is equivalent to

$$|(I - A)^{-1}|_{ij} = (I - |A|)^{-1}_{ij} \quad \text{for } (i, j) \in \Gamma. \tag{1.8}$$

The results above may be found as part of Theorem 5.14.

It is easily seen that (1.2) is equivalent to

$$\left| \sum_{s \in N} A^s \right| = \sum_{s \in N} |A|^s \tag{1.9}$$

for $A \in \mathscr{U}$, where N is the set of natural numbers. Since, for all subsets S of N,

$$\left| \sum_{s \in S} A^s \right| \leqq \sum_{s \in S} |A|^s, \tag{1.10}$$

it is natural to define Equ $(\mathscr{A}, \Gamma, S)$ to be the set of all $A \in \mathscr{A}$ such that

$$\left| \sum_{s \in S} A^s \right| = \sum_{s \in S} |A|^s \quad \text{for } (i, j) \in \Gamma, \tag{1.11}$$

where $\mathscr{A} \subseteq \mathscr{U}$, $\Gamma \subseteq \langle n \rangle \times \langle n \rangle$ and $S \subseteq N$.

The equivalences stated above, and others, are stated in terms of Equ $(\mathscr{U}, \Gamma, N)$. It is clear that Equ $(\mathscr{A}, \Gamma, S) \supseteq$ Equ $(\mathscr{A}, \Gamma, N)$ for $S \subseteq N$. We therefore call a subset S of N $(\mathscr{A}, \Gamma)$-sufficient if Equ $(\mathscr{A}, \Gamma, S) =$ Equ $(\mathscr{A}, \Gamma, N)$.

We give conditions equivalent to $(\mathscr{J}, \langle n \rangle \times \langle n \rangle)$-sufficiency and $(\mathscr{U}, \langle n \rangle \times \langle n \rangle)$-sufficiency. The general problem of characterizing $(\mathscr{A}, \Gamma)$-sufficient sets and minimal $(\mathscr{A}, \Gamma)$-sufficient sets, for $\mathscr{A} \subseteq \mathscr{U}$ and $\Gamma \subseteq \langle n \rangle \times \langle n \rangle$, is open.

Section 2 contains graph theoretic preliminaries. Section 3 contains preliminaries from combinatorial matrix theory. The basic definitions and results on Equ $(\mathscr{A}, \Gamma, S)$ are collected in § 4. Sections 5 and 6 contain our principal results on Equ $(\mathscr{A}, \Gamma, N)$ and $(\mathscr{J}, \Gamma)$-sufficient and $(\mathscr{U}, \Gamma)$-sufficient sets.

2. Graph theoretic definitions and preliminaries.

DEFINITION 2.1. A (simple, directed) *graph* $G = (V, E)$ is a pair of finite sets with $E \subseteq V \times V$. An element of V is called a *vertex* of G, and an element of E is called an *arc* of G. We call $G = (V', E')$ a *subgraph* of G if $V' \subseteq V$ and $E' \subseteq E$.

DEFINITION 2.2. Let G be a graph. A *chain* in G of *length* s from a vertex i_0 to a vertex i_s of G is a sequence

$$\gamma = (i_0, e_1, i_1, e_2, i_2, \cdots, i_{s-1}, e_s, i_s) \tag{2.3}$$

where either $e_p = 1$ and (i_{p-1}, i_p) is an arc of G or $e_p = -1$ and (i_p, i_{p-1}) is an arc of G, $p = 1, \cdots, s$. The arc (i_{p-1}, i_p), $[(i_p, i_{p-1})]$, $1 \leqq p \leqq s$, is said to *lie on* γ if $e_p = 1$ $[e_p = -1]$. The length of a chain γ is denoted by $|\gamma|$. The chain γ is *simple* if the vertices $i_0, \cdots, i_s$ are distinct. The chain γ is *closed* if $i_0 = i_s$, and γ is called a *cycle* if it is closed and the vertices $i_1, \cdots, i_s$ are distinct. A chain given by (2.3) such that $e_1 = \cdots e_s = 1$ is called a *path*. A path that is a cycle is called a *circuit*. A closed chain of form

$$\gamma = (i_0, e_1, i_1, \cdots, i_s, -e_s, i_{s-1}, \cdots, -e_1, i_0)$$

will be called *trivial*. The empty chain $\varnothing$ will be considered a chain of length 0 from any vertex to itself and is defined to be simple. The set $\{i_0, \cdots, i_m\}$ is called the *vertex set* of the chain g given by (2.3).

Thus the empty chain is the only simple circuit.

Intuitively, the chain (i, e, j) is a step from vertex i to vertex j along the arc (i, j) if $e = 1$ and a step from i to j along the arc (j, i) if $e = -1$. We normally write $i \to j$ or $i \leftarrow j$ in place of (i, e, j) accordingly as $e = 1$ or $e = -1$. For example, $1 \to 2 \to 3 \to 1$ is a circuit and $1 \to 2 \to 3 \leftarrow 1$ is a cycle. Note also that as a consequence of the above definition certain chains are cycles that normally are not considered as such, e.g., $1 \to 2 \leftarrow 1$. It would make no difference to our results to eliminate such cycles from consideration.

DEFINITION 2.4. A vertex i *has access to* a vertex j in a graph G if there is a path from i to j in G and we write $i >\!\!- j$ or $j -\!\!< i$. If U, W are subsets of the vertex set V of G, then the notation $U >\!\!- W$ indicates that every vertex of U has access to every vertex of W.

Observe that a vertex i has access to itself since $\varnothing$ is a path from i to i.

DEFINITION 2.5. A graph G is *strongly connected* if every vertex of G has access to every vertex of G. A subgraph H of G is called a *component* of G if H is a maximal strongly connected subgraph of V, viz. H is strongly connected but no subgraph properly containing H is connected.

DEFINITION 2.6. Let $G = (V, E)$ be a graph and let (i, j), $(h, k) \in V \times V$. Then (i, j) is a *G-access cover* for (h, k) (or (i, j) *G-access covers* (h, k)) if $i >\!\!- h$ and $k >\!\!- j$. Let Γ be a subset of $V \times V$. Then the set of all (h, k) that are G-access covered by elements of Γ will be denoted by $A_G(\Gamma)$. If $\Lambda \subseteq A_G(\Gamma)$, we shall say that Γ is a *G-access cover* for Λ (or that Γ G-access covers Λ). If α is a chain in G [G' is a subgraph of G] with vertex set V', then Γ will be called a *G-access cover* for α [G'] if Γ access covers $V' \times V'$. A G-access cover for $V \times V$ will be called a *G-access cover.*

It is easy to show that A_G considered as an operator from the set of subsets of $V \times V$ into itself is a closure operator in the sense of [1, p. 42].

The following lemma is clear:

LEMMA 2.7. *Let $G = (V, E)$ be a graph. Then the following conditions are equivalent*:

(i) *G is strongly connected.*

(ii) *Every nonempty subset of* $V \times V$ *is a G-access cover.*

(iii) *Every pair* $(i, j) \in V \times V$ *is a G-access cover.*

Remark 2.8. Let G be a graph and let $H_1, \cdots, H_s$ be the components of G with vertex sets $V_1, \cdots, V_s$, respectively. It is possible to order the components of G so that

$$V_p >\!- V_q \Rightarrow p < q \quad \text{for } p, q = 1, \cdots, s.$$

DEFINITION 2.9. (i) Let β and γ be the chains $(i_0, e_1, \cdots, e_s, i_s)$ and $(j_0, f_1, \cdots, f_t, j_t)$, respectively. If $i_s = j_0$ we define the *concatenated chain* $\beta\gamma$ by $(i_0, e_1, \cdots, e_j, i_s, f_1, \cdots, f_t, j_t)$. (If $i_s \neq j_0$ then $\beta\gamma$ is not defined.)

(ii) Let α and β be chains. We call α an *extension* (chain) of β (and β a *subchain* of α) if $\beta = \beta_1\beta_2$ and $\alpha = \beta_1\alpha'\beta_2$ where β_1, β_2, and α' are chains (which may be empty). Also, an extension of an extension of β is defined to be an extension of β.

It is easy to see that if α is an extension of β then α and β may be written in the forms $\beta = \beta_1\beta_2 \cdots \beta_p$ and $\alpha = \alpha_0\beta_1\alpha_1 \cdots \beta_p\alpha_p$, where the α_i, $i = 0, \cdots, p$, β_i, $i = 1, \cdots, p$ are chains and α_i, $i = 1, \cdots, p-1$ is closed.

DEFINITION 2.10.

(i) Let γ be the chain given by (2.3). Then the *reverse chain* of γ is defined to be $(i_s, -e_s, i_{s-1}, \cdots, -e_1, i_0)$, and is denoted by γ^*.

(ii) We call e_1 $[e_s]$ the *initial* [*final*] *sign* of γ.

DEFINITION 2.11. Let γ be a chain given by (2.3).

(i) If $e_p \neq e_{p+1}$, $1 \leqq p < s$, then we say that γ *has a twist at* p (or that p is a *twist* of γ). If γ is a closed chain then we allow $p = 0$ and we let $e_0 = e_s$.

(ii) If γ has exactly k twists then γ is said to be *exactly k-twisted* and we put $t(\gamma) = k$.

(iii) If $t(\gamma) \leqq m$ for an integer m then γ is said to be *m-twisted.*

Note that if γ is not closed then $t(\gamma)$ is equal to the number of sign changes in the sequence $e_1, \cdots, e_s$. If γ is closed then $t(\gamma)$ is equal to the number of sign changes in the sequence $e_1, \cdots, e_s, e_1$. Also note that a closed chain in form (2.3) may have a twist at $0, \cdots, s-1$ but not at s.

Observe that a chain [cycle] is 0-twisted if and only if it is a path [circuit] or a reversed path [reversed circuit], and that a closed chain has an even number of twists.

LEMMA 2.12. *Let G be a graph.*

(i) *If* α *is a chain in G and* γ *is a subchain of* α *then*

$$t(\gamma) \leqq t(\alpha) + 1. \tag{2.13}$$

(ii) *If, further,* α *and* γ *are closed then*

$$t(\gamma) \leqq t(\alpha). \tag{2.14}$$

Proof. (i) Let $\gamma = \gamma_1 \cdots \gamma_p$ and $\alpha = \alpha_0\gamma_1\alpha_1 \cdots \alpha_{p-1}\gamma_p\alpha_p$. We shall establish a $1-1$ mapping of the set of twists of γ (excluding a possible twist at 0) into the set of twists of α. Suppose that $|\alpha_i| = s_i$, $i = 0, \cdots, p$ and that $|\gamma_i| = t_i$, $i = 1, \cdots, p$. Let $1 \leqq r \leqq t_1 + \cdots + t_p$ and suppose that γ has a twist at r. Then

$$r = t_1 + \cdots + t_i + q$$

where $0 \leqq i < p$ and $1 \leqq q \leqq t_{i+1}$. If $q < t_{i+1}$ then α has a twist at $r + s_0 + \cdots + s_i$. If $q = t_{i+1}$ then $i < p - 1$ (since γ does not have a twist at $t_1 + \cdots + t_p$) and, since the final sign of γ_{i+1} and the initial sign of γ_{i+2} are inequal, it follows that α must have a twist at $r + s_1 + \cdots + s_i + q'$ for some q' satisfying $0 \leqq q_i \leqq s_{i+1}$. This proves the existence of the claimed injection and (i) follows.

(ii) If α and γ are closed, then $t(\alpha)$ and $t(\gamma)$ are both even and (ii) follows from (i). □

3. Definitions and preliminaries in combinatorial matrix theory.

DEFINITION 3.1. Let c be a complex number. The *sign* of c is defined by

$$\operatorname{sgn}(c) = \begin{cases} c/|c|, & \text{if } c \neq 0. \\ 0, & \text{if } c = 0. \end{cases}$$

We call a complex number c a *sign* if $|c|$ is either 0 or 1. If $A \in \mathbb{C}^{nn}$, then we call A a *sign matrix* if a_{ij} is a sign for $i, j = 1, \cdots, n$.

DEFINITION 3.2. Let $A \in \mathbb{C}^{nn}$.

(i) Then $C = |A| \in \mathbb{C}^{nn}$ is defined by $c_{ij} = |a_{ij}|$ for $i, j = 1, \cdots, n$.

(ii) The matrix A is called *nonnegative* ($A \geqq 0$) if $a_{ij} \geqq 0$, $i, j = 1, \cdots, n$.

DEFINITION 3.3. If $A \in \mathbb{C}^{nn}$ (the set of $n \times n$ complex matrices) then the *graph* $G(A)$ of A is defined to be $(\langle n \rangle, E)$ where $\langle n \rangle = \{1, \cdots, n\}$ and $(i, j) \in E$ whenever $a_{ij} \neq 0$.

DEFINITION 3.4. Let $A \in \mathbb{C}^{nn}$ and let $\alpha = (i_0, e_1, i_1, \cdots, e_q, i_q)$ be a chain in $G(A)$. Then we define the *chain product* $\prod_\alpha(A)$ by

$$\prod_\alpha(A) = \prod_{p=1}^{q} a_{i_{p-1} i_p}^{e_p}.$$

We put $\prod_\varnothing(A) = 1$. If α is a cycle (path, circuit) we call the $\prod_\alpha(A)$ a cycle (path, circuit) product.

Note that if $\alpha = (i_0, e_1, i_1, \cdots, e_q, i_q)$ is a closed path and

$$\beta = (i_k, e_{k+1}, \cdots, e_q, i_0, e_1, \cdots, i_k), \qquad 0 \leqq k < q,$$

then $\prod_\alpha(A) = \prod_\beta(A)$.

DEFINITION 3.5. Let $A, B \in \mathbb{C}^{nn}$. We say that A and B are *diagonally similar* if there exists a nonsingular diagonal matrix X such that $B = X^{-1}AX$, and we say that A and B are *sign similar* if there exists a nonsingular diagonal sign matrix X such that $B = X^{-1}AX$. We say that A and B are *permutation similar* if there exists a permutation matrix P such that $B = P^{-1}AP$. We say that A and B are *diagonally equivalent* if there exist nonsingular diagonal matrices X and Y such that $B = YAX$.

DEFINITION 3.6. Let $A, B \in \mathbb{C}^{nn}$. We say that A and B are *c-equivalent* if $G(A) = G(B)$ and for all circuits α in $G(A)$ we have $\prod_\alpha(A) = \prod_\alpha(B)$.

Definition 3.6 and some implications may be found in [2]. In particular, it is well known that for irreducible matrices A and B, the matrices A and B are diagonally similar if and only if they are c-equivalent (see [2, Thm. 4.1]).

DEFINITION 3.7. If $V, W \subseteq \langle n \rangle$ and $A \in \mathbb{C}^{nn}$, then $A[V, W]$ is the submatrix of A whose rows are indexed by V and whose columns are indexed by W (in their natural orders).

DEFINITION 3.8. Let $A \in \mathbb{C}^{nn}$.

(i) The matrix A is called *irreducible* if $G(A)$ is strongly connected.

(ii) The matrix A is said to be in *Frobenius normal form* if A may be written in the block form

$$A = \begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1s} \\ 0 & A_{22} & & \vdots \\ \vdots & & \ddots & \\ 0 & \cdots & 0 & A_{ss} \end{bmatrix}, \tag{3.9}$$

where A_{ii} is an irreducible square matrix, $i = 1, \cdots, s$.

(iii) Let $B \in \mathbb{C}^{nn}$. The matrix B is said to be a *Frobenius normal form* of A if B is in Frobenius normal form and if A and B are permutation similar.

Remark 3.10. Let $A \in \mathbb{C}^{nn}$. We may obtain a Frobenius normal form of A by reordering the vertices of $G(A)$ so that V_p consists of consecutive integers, $p = 1, \cdots, s$, and so that (2.8) holds. It follows from Definition 3.8 that a Frobenius normal form of A is unique up to permutation similarity. The diagonal blocks of a Frobenius normal form of A will be called the *components* of A.

In [4, § 4] we introduced the inflation product $C \times\times A$ of two matrices where $C \in \mathbb{C}^{ss}$, $A \in \mathbb{C}^{nn}$, and A is partitioned into s blocks. In this paper we use the notation $C \times\times A$ only in the special case when A is in Frobenius normal form and C satisfies (3.12) below.

DEFINITION 3.11. Let $A \in \mathbb{C}^{nn}$ be in Frobenius normal form (3.9) and suppose that $C \in \mathbb{C}^{ss}$ satisfies

$$C \text{ is a sign matrix,} \tag{3.12}$$

$$c_{pp} \text{ is equal to 0 or 1}, \quad p \in \langle s \rangle, \tag{3.13}$$

$$c_{pq} = 0 \Leftrightarrow A_{pq} = 0, \quad p, q \in \langle s \rangle. \tag{3.14}$$

Then the matrix $B = C \times\times A \in \mathbb{C}^{nn}$ is defined to be the matrix with blocks $B_{pq} = c_{pq}A_{pq}$, $p, q \in \langle s \rangle$.

4. Preliminaries on equality classes and sufficient sets.

Notation 4.1. We use the following notation:

N = the set $\{0, 1, 2, \cdots\}$

Δ = the set $\{(i, i) : i \in \langle n \rangle\}$.

Notation 4.2. Let $A \in \mathbb{C}^{nn}$.

$\rho(A)$ = the spectral radius of A.

$\mathcal{U}_n$ = the set $\{A \in \mathbb{C}^{nn} : \rho(|A|) < 1\}$.

We normally write $\mathcal{U}$ in place of $\mathcal{U}_n$.

$\mathcal{J}$ = the set of irreducible matrices contained in $\mathcal{U}$.

If $G = (\langle n \rangle, E)$ is a graph, then $\mathcal{U}(G)$ is the set $\{A \in \mathcal{U} : G(A) = G\}$.

Note that for every $A \in \mathbb{C}^{nn}$ we have $cA \in \mathcal{U}$ for all complex numbers c whose absolute value is sufficiently small. Let $A \in \mathcal{U}$ and let $S \subseteq N$. Observe that

$$\left| \sum_{s \in S} A^s \right| \leqq \sum_{s \in S} |A|^s \leqq \sum_{s \in N} |A|^s = (I - |A|)^{-1}. \tag{4.3}$$

Hence the series in (4.3) converge. In order to discuss the cases when the equalities hold in (4.3) we shall make several definitions. The first of these allows us to discuss the case of equality in the first inequality in (4.3).

DEFINITION 4.4. Let $\Gamma \subseteq \langle n \rangle \times \langle n \rangle$, let $S \subseteq N$, and let $\mathcal{A} \subseteq \mathcal{U}$. Then the $(\mathcal{A}, \Gamma, S)$-*equality class* is defined to consist of all $A \in \mathcal{A}$ such that

$$\left(\left| \sum_{s \in S} A^s \right| \right)_{ij} = \left(\sum_{s \in S} |A|^s \right)_{ij}, \tag{4.5}$$

for all $(i, j) \in \Gamma$, and it is denoted by Equ $(\mathcal{A}, \Gamma, S)$.

The first two parameters in Equ $(\mathcal{A}, \Gamma, S)$ are optional and default to $\mathcal{U}$ and $\langle n \rangle \times \langle n \rangle$, respectively. Thus (by convention)

Equ (Γ, S) = Equ $(\mathcal{U}, \Gamma, S)$,

Equ $(\mathcal{A}, S)$ = Equ $(\mathcal{A}, \langle n \rangle \times \langle n \rangle, S)$,

Equ (S) = Equ $(\mathcal{U}, \langle n \rangle \times \langle n \rangle, S)$.

We have the following easy but fundamental lemma.

LEMMA 4.6. *Let $i, j \in \langle n \rangle$ and let $S \subseteq N$. Then the following conditions are equivalent*:

(i) $A \in \mathrm{Equ}\,((i,j), S)$.

(ii) $\mathrm{sgn}\,(\prod_\alpha(A)) = \mathrm{sgn}\,(\prod_\beta(A))$, *for all paths* α, β *from* i *to* j *in* $G(A)$ *such that* $|\alpha|, |\beta| \in S$.

Proof. Note that (i) is equivalent to (4.5) by definition of $\mathrm{Equ}\,((i,j), S)$. The equivalence of (i) and (ii) follows from the conditions for equality in the triangle inequality and the result that for $i, j \in \langle n\rangle$ and $s \in N$ we have

$$A^s_{ij} = \sum_{\sigma \in P(i,j;s)} \prod\nolimits_\sigma(A) \tag{4.7}$$

where $P(i, j; s)$ is the set of all paths from i to j of length s in $G(A)$. □

The proof of the following lemma is easy and will be omitted.

LEMMA 4.8. *Let* $\mathscr{A} \subseteq \mathscr{B} \subseteq \mathscr{U}$ *and let* Γ, Γ', Λ *be subsets of* $\langle n\rangle \times \langle n\rangle$ *such that* $\Gamma \subseteq \Lambda$. *Let* $S \subseteq T \subseteq N$. *Then*

$$\mathrm{Equ}\,(\mathscr{A}, \Gamma, S) = \mathrm{Equ}\,(\mathscr{B}, \Gamma, S) \cap \mathscr{A}, \tag{4.9}$$

$$\mathrm{Equ}\,(\mathscr{A}, \Gamma \cup \Gamma', S) = \mathrm{Equ}\,(\mathscr{A}, \Gamma, S) \cap \mathrm{Equ}\,(\mathscr{A}, \Gamma', S), \tag{4.10}$$

$$\mathrm{Equ}\,(\mathscr{A}, \Lambda, T) \subseteq \mathrm{Equ}\,(\mathscr{B}, \Gamma, S). \tag{4.11}$$

Let $S \subseteq N$. Then by Lemma 4.8 it follows that $\mathrm{Equ}\,(\mathscr{A}, \Gamma, N) \subseteq \mathrm{Equ}\,(\mathscr{A}, \Gamma, S)$ for all $\mathscr{A} \subseteq \mathscr{U}$ and $\Gamma \subseteq \langle n\rangle \times \langle n\rangle$. This remark motivates the following definition which allows us to discuss the case of equality in the second inequality in (4.3).

DEFINITION 4.12. We say that the subset S of N is $(\mathscr{A}, \Gamma)$*-sufficient* if $\mathrm{Equ}\,(\mathscr{A}, \Gamma, S) = \mathrm{Equ}\,(\mathscr{A}, \Gamma, N)$. We say that S is *minimal* $(\mathscr{A}, \Gamma)$*-sufficient* if S is $(\mathscr{A}, \Gamma)$-sufficient but no proper subset of S is $(\mathscr{A}, \Gamma)$-sufficient. We say that S is *optimal* $(\mathscr{A}, \Gamma)$*-sufficient* if S is an $(\mathscr{A}, \Gamma)$-sufficient of minimal cardinality, viz. there exists no $(\mathscr{A}, \Gamma)$-sufficient set of lower cardinality. The two parameters in the term (minimal, optimal) $(\mathscr{A}, \Gamma)$-sufficient are optional and default to $\mathscr{U}$ and $\langle n\rangle \times \langle n\rangle$, respectively. Thus S is Γ-sufficient means that S is $(\mathscr{U}, \Gamma)$-sufficient, S is $\mathscr{A}$-sufficient means that S is $(\mathscr{A}, \langle n\rangle \times \langle n\rangle)$-sufficient, S is sufficient means that S is $(\mathscr{U}, \langle n\rangle \times \langle n\rangle)$-sufficient.

Of course, an optimal $(\mathscr{A}, \Gamma)$-sufficient set is minimal $(\mathscr{A}, \Gamma)$-sufficient.

LEMMA 4.13. *Let* $\mathscr{A} \subseteq \mathscr{B} \subseteq \mathscr{U}$, *let* $\Gamma \subseteq \langle n\rangle \times \langle n\rangle$, *and let* $S \subseteq T \subseteq N$. *If* S *is* $(\mathscr{B}, \Gamma)$*-sufficient then* T *is* $(\mathscr{A}, \Gamma)$*-sufficient.*

Proof. By Lemma 4.8 we have

$$\mathrm{Equ}\,(\mathscr{B}, \Gamma, N) \subseteq \mathrm{Equ}\,(\mathscr{B}, \Gamma, T) \subseteq \mathrm{Equ}\,(\mathscr{B}, \Gamma, S).$$

But by our hypothesis $\mathrm{Equ}\,(\mathscr{B}, \Gamma, S) = \mathrm{Equ}\,(\mathscr{B}, \Gamma, N)$ and it follows that

$$\mathrm{Equ}\,(\mathscr{B}, \Gamma, T) = \mathrm{Equ}\,(\mathscr{B}, \Gamma, N).$$

Therefore, by (4.9), it follows that

$$\mathrm{Equ}\,(\mathscr{A}, \Gamma, T) = \mathrm{Equ}\,(\mathscr{B}, \Gamma, T) \cap \mathscr{A} = \mathrm{Equ}\,(\mathscr{B}, \Gamma, N) \cap \mathscr{A} = \mathrm{Equ}\,(\mathscr{A}, \Gamma, N). \qquad \square$$

5. The equality class of N. In this section we prove necessary and sufficient conditions for $A \in \mathrm{Equ}\,(\Gamma, N)$ for irreducible and general $A \in \mathscr{U}$. In view of Definition 4.4, $A \in \mathrm{Equ}\,(\Gamma, N)$ is equivalent to

$$|(I-A)^{-1}|_{ij} = (I-|A|)^{-1}_{ij} \quad \text{for } (i,j) \in \Gamma. \tag{5.1}$$

THEOREM 5.2. *Let* $i, j \in \langle n\rangle$, *and let* Λ *be a subset of* $\langle n\rangle \times \langle n\rangle$ *such that* $(i,j) \in \Lambda$ *and* (i,j) *access covers* Λ. *Let* $A \in \mathscr{U}$. *Then the following conditions are equivalent.*

(i) $A \in \mathrm{Equ}\,((i,j), N)$.

(ii) $\mathrm{sgn}\,(\prod_\alpha(A)) = \mathrm{sgn}\,(\prod_\beta(A))$, *for all paths* α, β *from* i *to* j *in* $G(A)$.

(iii) $\operatorname{sgn}(\prod_\alpha(A)) = \operatorname{sgn}(\prod_\beta(A))$, *for all paths* α, β *from* h *to* k *in* $G(A)$, *where* $(h, k) \in \Lambda$.

(iv) $A \in \operatorname{Equ}(\Lambda, N)$.

(v) *Both*

(a) $\operatorname{sgn}(\prod_\beta(A)) = \operatorname{sgn}(\prod_\gamma(A))$, *for all simple paths* β, γ *from* i *to* j *in* $G(A)$ *and*

(b) *If* α *is a circuit of* $G(A)$ *which is* $G(A)$*-access covered by* (i, j) *then* $\prod_\alpha(A) > 0$.

(vi) *All chain products of two-twisted closed chains of* $G(A)$ *which are* $G(A)$*-access covered by* (i, j) *are positive.*

(vii) *All cycle products of two-twisted cycles of* $G(A)$ *which are* $G(A)$*-access covered by* (i, j) *are positive.*

Proof. We shall show that (i) $\Leftrightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv) $\Rightarrow$ (i), (ii) $\Leftrightarrow$ (v), and (ii) $\Leftrightarrow$ (vi) $\Leftrightarrow$ (vii).

(i) $\Leftrightarrow$ (ii). This is given by Lemma 4.6.

(ii) $\Rightarrow$ (iii). Suppose (ii) holds. Let $(h, k) \in \Lambda$ and let α and β be paths from h to k in $G(A)$. Since (i, j) is a $G(A)$-access cover for (h, k) there exist paths γ and δ in $G(A)$ from i to h and k to j, respectively. Since

$$\prod\nolimits_{\gamma\alpha\delta}(A) = \prod\nolimits_{\gamma\beta\delta}(A)$$

by (ii), and since

$$\prod\nolimits_{\gamma\alpha\delta}(A) = \prod\nolimits_{\gamma}(A)\prod\nolimits_{\alpha}(A)\prod\nolimits_{\delta}(A),$$

$$\prod\nolimits_{\gamma\beta\delta}(A) = \prod\nolimits_{\gamma}(A)\prod\nolimits_{\beta}(A)\prod\nolimits_{\delta}(A),$$

we obtain (iii).

(iii) $\Rightarrow$ (iv). By (4.10) and Lemma 4.6.

(iv) $\Rightarrow$ (i). By (4.11), since $(i, j) \in \Lambda$.

(ii) $\Rightarrow$ (v). Suppose (ii) holds. Then obviously we have (a). To prove (b), let $\alpha = (i_0, \cdots, i_s)$ be a circuit of $G(A)$ that is $G(A)$-access covered by (i, j). Then there is a vertex k of α for which there exist paths δ from i to k and ψ from k to j. Without loss of generality we may assume that $k = i_0$. By (ii) the path products corresponding to the paths $\delta\psi$ and $\delta\alpha\psi$ have the same (nonzero) sign. It follows that $\prod_\alpha(A) > 0$ and (v) is proved.

(v) $\Rightarrow$ (ii). Suppose that (a) and (b) hold. Let δ be a path in $G(A)$ from i to j. Then $\prod_\delta(A)$ is a product of $\prod_\beta(A)$ and factors of type $\prod_\alpha(A)$, where β is a simple path from i to j and α is a circuit of $G(A)$ for which (i, j) is a $G(A)$-access cover. By (b), $\operatorname{sgn}(\prod_\delta(A)) = \operatorname{sgn}(\prod_\beta(A))$. Hence it follows from (a) that products corresponding to every pair of paths from i to j have the same sign.

(ii) $\Rightarrow$ (vi). Let $\alpha = (i_0, e_1, \cdots, i_m)$ with $i_0 = i_m$ be a two-twisted closed chain which is $G(A)$-access covered by (i, j). If $t(\alpha) = 0$ then the positivity of $\prod_\alpha(A)$ follows as in the proof of (ii) implies (v) with "circuit" replaced by "closed path." Suppose $t(\alpha) = 2$. Let α have twists at p and q, respectively. Without loss of generality we may assume that $p = 0$ and $e_1 = 1$. Observe that $e_{q+1} = -1$. Let $\alpha_1 = (i_0, \cdots, i_q)$ and let $\alpha_2 = (i_q, \cdots, i_s)^*$. Observe that both α_1 and a_2 are paths from i_0 to i_q. Since (i, j) is a $G(A)$-access cover for α, there exists paths δ from i to i_0 and ψ from i_q to j. By (ii), the nonzero path products corresponding to $\delta\alpha_1\psi$ and $\delta\alpha_2\psi$ have the same sign. Thus the path products corresponding to α_1 and α_2 have the same sign. Since $\alpha = \alpha_1\alpha_2^*$ our claim follows.

(vi) ⇒ (ii). Let α and β be two paths from i to j in $G(A)$. Then $\alpha\beta^*$ is a two-twisted closed chain (possibly trivial). Since

$$\prod_{\alpha\beta^*}(A) = \prod_\alpha(A)/\prod_\beta(A)$$

clearly (vi) implies (ii).

(vi) ⇒ (vii). This is trivial.

(vii) ⇒ (vi). Assume that (vii) holds and let $\alpha = (i_0, \cdots, i_s)$, $i_0 = i_s$, be a two-twisted closed chain which is $G(A)$-access covered by (i, j). The proof is by induction on the length s. If $s = 1$, then α is a cycle and the result holds. So let $s > 1$ and assume that $\prod_\gamma(A) > 0$ for every two-twisted closed chain γ that is $G(A)$-access covered by (i, j) and such that $|\gamma| < s$. If α is a cycle the result holds. Otherwise, there exist p and q, $0 \leqq p < q < s$, such that $\delta = (i_p, \cdots, i_q)$ is a cycle. Further, $\beta = (i_0, \cdots, i_p, i_{q+1}, \cdots, i_s)$ is a closed chain of length less than s which is $G(A)$-access covered by (i, j). By Lemma 2.12, δ and β are two-twisted and hence by the inductive assumption the corresponding chain products are positive. But $\prod_\alpha(A) = \prod_\beta(A)\prod_\delta(A)$, and hence $\prod_\alpha(A) > 0$. We now deduce (vi). □

It is easy to construct an example to show that the assumption $(i, j) \in \Lambda$ cannot be omitted from the hypothesis of Theorem 5.2. However, we have the following corollary.

COROLLARY 5.3. *Let $A \in \mathbb{C}^{nn}$ and let $i, j, h, k \in \langle n\rangle$. Let (i, j) be a $G(A)$-access cover for (h, k). Then* Equ $((i, j), N) \subseteq$ Equ $((h, k), N)$.

Proof. Let $A \in$ Equ $((i, j), N)$. Let α and β be paths in $G(A)$ from h to k. Since (i, j) $G(A)$-access covers (h, k), there exist paths γ from i to h and δ from k to j in $G(A)$. By Theorem 5.2,

$$\prod_{\gamma\alpha\delta}(A) = \prod_{\gamma\beta\delta}(A)$$

and it follows that

$$\prod_\alpha(A) = \prod_\beta(A).$$

Hence, by Theorem 5.2, $A \in$ Equ $((h, k), N)$. □

COROLLARY 5.4. *Let $G = (\langle n\rangle, E)$ be a graph and let $\Gamma \subseteq \Lambda \subseteq A_G(\Gamma) \subseteq \langle n\rangle \times \langle n\rangle$. Then*

$$\text{Equ}(\mathcal{U}(G), \Lambda, N) = \text{Equ}(\mathcal{U}(G), \Gamma, N).$$

Proof. Since $\Gamma \subseteq \Lambda$, it follows from (4.11) that

$$\text{Equ}(\mathcal{U}(G), \Lambda, N) \subseteq \text{Equ}(\mathcal{U}(G), \Gamma, N).$$

Hence we need only prove that

$$\text{Equ}(\mathcal{U}(G), \Gamma, N) \subseteq \text{Equ}(\mathcal{U}(G), \Lambda, N). \tag{5.5}$$

By (4.10) we have

$$\text{Equ}(\mathcal{U}(G), \Gamma, N) = \bigcap\{\text{Equ}(\mathcal{U}(G), (i,j), N) : (i,j) \in \Gamma\}. \tag{5.6}$$

and similarly

$$\text{Equ}(\mathcal{U}(G), \Lambda, N) = \bigcap\{\text{Equ}(\mathcal{U}(G), (h,k), N) : (h,k) \in \Lambda\}. \tag{5.7}$$

It follows from the definition of $A_G(\Gamma)$ that for each $(h, k) \in \Lambda$ there exists $(i, j) \in \Gamma$ such that (i, j) G-access covers (h, k). Hence (5.5) now follows from (5.6), (5.7), and Corollary 5.3. □

As a special case of Theorem 5.2 we obtain the following corollary, which is essentially known.

COROLLARY 5.8. *Let $A \in \mathcal{U}$. Then the following are equivalent:*

(i) $A \in \operatorname{Equ}(\Delta, N)$.

(ii) *Every circuit product for $G(A)$ is positive.*

Proof. (i) $\Rightarrow$ (ii). Let $A \in \operatorname{Equ}(\Delta, N)$. Since Δ is a $G(A)$-access cover for every circuit it follows by Theorem 5.2, Part (v) that every circuit product is positive.

(ii) $\Rightarrow$ (i). This follows from Theorem 5.2, Part (v), since the only simple paths from i to i, $i \in \langle n \rangle$, are circuits. □

For irreducible matrices there is the following stronger result which is essentially due to Neumaier [6] and which motivated our investigations.

COROLLARY 5.9. *Let Γ be a nonempty subset of $\langle n \rangle \times \langle n \rangle$ and let $A \in \mathcal{J}$. Then the following are equivalent:*

(i) $A \in \operatorname{Equ}(N)$.

(ii) $A \in \operatorname{Equ}(\Gamma, N)$.

(iii) *All circuit products of $G(A)$ are positive.*

(iv) *All closed path products of $G(A)$ are positive.*

(v) *A is sign similar to $|A|$.*

Proof. (i) $\Rightarrow$ (ii). This implication follows from Lemma 4.8.

(ii) $\Rightarrow$ (iii). Suppose that (ii) holds. Since $G(A)$ is strongly connected, it follows from Lemma 2.7 that Γ is a $G(A)$-access cover for $\langle n \rangle$ and (iii) follows immediately from Theorem 5.2.

(iii) $\Rightarrow$ (iv). Every closed path product is a product of circuit products.

(iv) $\Rightarrow$ (v). Suppose (iv) holds. Then corresponding circuit products of A and $|A|$ are equal. Thus, since A is irreducible, as is well known (e.g., [2, Thm. 4.1]), there exists a diagonal matrix X such that $X^{-1}AX = |A|$. Let $D = |X^{-1}|X$. Then D is a diagonal sign matrix satisfying $D^{-1}AD = |A|$.

(v) $\Rightarrow$ (i). Let D be a diagonal sign matrix such that $D^{-1}AD = |A|$. Since $|A|^k = D^{-1}A^kD$ and $\rho(A) < 1$, it follows that

$$D^{-1}(I-A)^{-1}D = (I-|A|)^{-1}.$$

Hence, since D is a diagonal sign matrix, (5.1) holds for $\Gamma = \langle n \rangle \times \langle n \rangle$ and (i) is proved. □

LEMMA 5.10. *Let $A \geqq 0$ be an $n \times n$ matrix in Frobenius normal form and let C be an (upper triangular) $s \times s$ matrix satisfying (3.12)–(3.14). Let $B = C \times\times A$. Let $i, j \in \langle n \rangle$ and suppose that a_{ij} is an element of A_{pq}, where $1 \leqq p, q \leqq s$. Then for every path β in $G(B)$ from i to j there is a path γ in $G(C)$ from p to q such that*

$$\operatorname{sgn}\left(\textstyle\prod_\beta(B)\right) = \textstyle\prod_\gamma(C). \tag{5.11}$$

Conversely, for every path γ in $G(C)$ from p to q there is a path β in $G(B)$ from i to j such that (5.11) *is satisfied.*

Proof. Suppose the rows and columns of the component A_{rr} of A are indexed by the subset V_r of $\langle n \rangle$, $r = 1, \cdots, s$. Since A and B are in Frobenius normal form, there exist p_t, $t = 0, \cdots, k$, $1 \leqq p_t \leqq s$ with $p_0 = p$ and $p_k = q$ and i_t, j_t in V_{p_t}, $t = 0, \cdots, k$, with $i_0 = i$ and $j_k = j$, such that

$$\beta = \beta_0\delta_1\beta_1 \cdots \beta_k, \tag{5.12}$$

where β_t is a path from i_t to j_t in $G(B_{p_tp_t})$, $t = 0, \cdots, k$ and $\delta_t = j_{t-1} \rightarrow i_t$, $t = 1, \cdots, k$. Since $A \geqq 0$ and $c_{p_tp_t} = 1$ or 0, $t = 1, \cdots, k$ and $c_{p_tp_t} = 0$ if and only if $B_{p_tp_t}$ is a zero 1×1 block in which case β_t is empty, we have $\prod_{\beta_t}(B) > 0$ and $\prod_{\delta_t}(B) = c_{p_tp_t}$. Hence if we define

$$\gamma = p_0 \rightarrow \cdots \rightarrow p_k, \tag{5.13}$$

then γ is a path in $G(C)$ from p to q such that (5.11) holds.

Conversely, let $B = C \times\times |A|$ and let γ given by (5.13) be a path in $G(C)$ from p to q. We may choose $i_t, j_t \in V_{p_t}$, $t = 0, \cdots, k$ and paths β_t from i_t to j_t in $G(B_{p_t})$, $t = 0, \cdots, k$. If δ_t is again defined to be $j_{t-1} \to i_t$, $t = 1, \cdots, k$ and β is defined by (5.12) then (5.11) holds, since $c_{pp} \geqq 0$, $p = 1, \cdots, s$. □

We now apply Theorem 5.2 to obtain the final result in this section.

THEOREM 5.14. *Let $A \in \mathcal{U}$ and let Γ be a $G(A)$-access cover. Then the following are equivalent.*

(i) $A \in \mathrm{Equ}\,(\Gamma, N)$.

(ii) $\mathrm{sgn}\,(\prod_\alpha(A)) = \mathrm{sgn}\,(\prod_\beta(A))$, *for all paths α, β in $G(A)$ from i to j, where $(i, j) \in \Gamma$.*

(iii) $\mathrm{sgn}\,(\prod_\alpha(A)) = \mathrm{sgn}\,(\prod_\beta(A))$, *for all paths α, β in $G(A)$ from i to j, where $(i, j) \in \langle n\rangle \times \langle n\rangle$.*

(iv) $A \in \mathrm{Equ}\,(N)$.

(v) *Both*

(a) $\mathrm{sgn}\,(\prod_\beta(A)) = \mathrm{sgn}\,(\prod_\gamma(A))$, *for all simple paths β, γ in $G(A)$ from i to j, where $(i, j) \in \Gamma$,*

and

(b) $\prod_\alpha(A) > 0$ *for all circuits α of $G(A)$.*

(v′) *Both*

(a′) $\mathrm{sgn}\,(\prod_\beta(A)) = \mathrm{sgn}\,(\prod_\gamma(A))$, *for all simple paths β, γ in $G(A)$ from i to j, where $(i, j) \in \langle n\rangle \times \langle n\rangle$,*

and

(b′) $\prod_\alpha(A) > 0$ *for all circuits α of $G(A)$.*

(vi) *All chain products of two-twisted closed chains of $G(A)$ are positive.*

(vii) *All cycle products of two-twisted cycles are positive.*

(viii) *If A is in Frobenius normal form* (3.9) *then there exists an $s \times s$ sign matrix C such that $zC \in \mathrm{Equ}\,(\mathcal{U}_s, N)$, for $0 < z < 1$ and A is sign similar to $C \times\times |A|$.*

Proof. The equivalence of conditions (i)–(vii) follows immediately from the equivalence of the correspondingly numbered conditions in Theorem 5.2 and the fact that

$$\mathrm{Equ}\,(\Gamma, N) = \cap\,\{\mathrm{Equ}\,((i,j), N) : (i,j) \in \Gamma\}$$

by (4.10). The equivalence of conditions (v) and (v′) is easily derived by means of Conditions (iv) and (v) of Theorem 5.2. So it suffices to prove the equivalence of Conditions (iv) and (viii).

(iv) ⇒ (viii). Suppose that (iv) holds. Since A_{pp} is irreducible, $p = 1, \cdots, s$, by Corollary 5.9 there exist diagonal sign matrices X_p that satisfy $X_p^{-1}AX_p = |A_{pp}|$, $p = 1, \cdots, s$. Let $X = X_1 \oplus \cdots \oplus X_s$ and let $B = X^{-1}AX$. Then $|B| = |A|$ and $B_{pp} \geqq 0$, $p = 1, \cdots, s$. We shall show that $B = C \times\times |A|$, where C is a suitably chosen sign matrix satisfying conditions (3.12)–(3.14).

Let $1 \leqq i, j, h, k \leqq n$ and suppose that both b_{ij} and b_{hk} are nonzero elements of B_{pq}, where $1 \leqq p, q \leqq s$. Since B_{pp} and B_{qq} are irreducible, there exist chains α and γ in $G(B_{pp})$ from i to h and in $G(B_{qq})$ from k to j, respectively. Since $B_{pp} \geqq 0$ and $B_{qq} \geqq 0$, the products $\prod_\alpha(B)$ and $\prod_\gamma(B)$ are positive. Let β, δ be chains $h \to k$ and $i \to j$ of length 1, respectively. Then $\alpha\beta\gamma$ and δ are paths from i to j in $G(B)$. Since $A \in \mathrm{Equ}\,(N)$ we also have $B \in \mathrm{Equ}\,(N)$ and it follows from (ii) of Theorem 5.2 that

$$\mathrm{sgn}\,(\textstyle\prod_{\alpha\beta\gamma}(B)) = \mathrm{sgn}\,(\prod_\delta(B)).$$

We deduce that $\mathrm{sgn}\,(b_{hk}) = \mathrm{sgn}\,(b_{ij})$.

Thus we may define

$$c_{pq} = \begin{cases} 0 & \text{if } B_{pq} = 0 \\ \operatorname{sgn}(b_{ij}) & \text{if } B_{pq} \neq 0, \text{ where } b_{ij} \text{ is any nonzero entry of } B_{pq}. \end{cases}$$

Then c_{pp} is equal to 0 or 1 since $B_{pp} \geqq 0$, $p = 1, \cdots, s$. Thus $C \in \mathbb{C}^{ss}$ is an (upper triangular) matrix that satisfies conditions (3.12)–(3.14). Further, $B = C \times\times |A|$.

We must still show that $zC \in \operatorname{Equ}(\mathcal{U}, N)$ for $0 < z < 1$. Let $p, q \in \langle s \rangle$ and let γ be a chain from p to q in $G(C)$. Let i and j be elements of the sets V_p and V_q (which index the corresponding components), respectively. Then by Lemma 5.10 there exists a chain from i to j in $G(B)$ such that (5.11) holds. It follows that path products corresponding to any two paths from p to q in $G(B)$ have the same sign. Let $0 < z < 1$. Since $\rho(zC) < 1$, we now obtain $zC \in \operatorname{Equ}(N)$ by Theorem 5.2.

(viii) $\Rightarrow$ (iv). Suppose that (viii) holds and put $B = C \times\times |A|$. Let $i, j \in \langle n \rangle$ and let α, β be paths from i to j in $G(B)$. It follows from Lemma 5.10 that there exists a path γ in $G(C)$ such that

$$\operatorname{sgn}(\textstyle\prod_\alpha(B)) = \operatorname{sgn}(\prod_\gamma(C)) = \operatorname{sgn}(\prod_\beta(B)).$$

Hence $B \in \operatorname{Equ}(N)$ by Theorem 5.2. Since A is sign similar to B we obtain (iv). □

For the terminology and definitions employed in the following remark see [5].

Remark 5.15. (i) Our proof of (vii) $\Rightarrow$ (vi) of Theorem 5.2 shows that every algebraic two-twisted chain in a graph G is an integral linear combination of algebraic two-twisted cycles.

(ii) Suppose that $A \in \mathcal{U}$ and let W be the subspace of the flow space of $G(A)$ which is spanned by the algebraic two-twisted closed chains of $G(A)$. Let X be an integral spanning set for W. If the chain products corresponding to the closed chains in X are positive, then all chain products corresponding to chains in W are positive. Hence (vi) of Theorem 5.14 holds, and it follows that $A \in \operatorname{Equ}(N)$. However, this conclusion does not follow for arbitrary (nonintegral) spanning sets as one may see from Example 5.2 in [8]. A similar remark may be made concerning (vi) of Theorem 5.2.

6. Sufficient sets. We begin this section with some applications of Corollary 5.4.

COROLLARY 6.1. *Let $G = (\langle n \rangle, E)$ be a graph. Let $\Gamma \subseteq \Lambda \subseteq \langle n \rangle \times \langle n \rangle$ and suppose that Γ is a G-access cover for Λ. Let $S \subseteq N$. If S is $(\mathcal{U}(G), \Gamma)$-sufficient then S is $(\mathcal{U}(G), \Lambda)$-sufficient.*

Proof. By (4.11) we have

$$\operatorname{Equ}(\mathcal{U}(G), \Lambda, S) \subseteq \operatorname{Equ}(\mathcal{U}(G), \Gamma, S). \tag{6.2}$$

By assumption,

$$\operatorname{Equ}(\mathcal{U}(G), \Gamma, S) = \operatorname{Equ}(\mathcal{U}(G), \Gamma, N), \tag{6.3}$$

and by Corollary 5.4,

$$\operatorname{Equ}(\mathcal{U}(G), \Gamma, N) = \operatorname{Equ}(\mathcal{U}(G), \Lambda, N). \tag{6.4}$$

It follows from (6.2)–(6.4) that

$$\operatorname{Equ}(\mathcal{U}(G), \Lambda, S) \subseteq \operatorname{Equ}(\mathcal{U}(\Gamma), \Lambda, N).$$

But hence by (4.11) we obtain

$$\operatorname{Equ}(\mathcal{U}(G), \Lambda, S) = \operatorname{Equ}(\mathcal{U}(\Gamma), \Lambda, N)$$

which proves the corollary. □

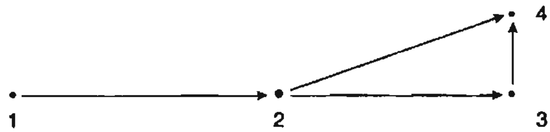

FIG. 1

Example 6.5. Let $\langle n \rangle = 4$ and let G be given by Fig. 1. Let $S_1 = \{2, 3\}$. Then S_1 is $(\mathscr{U}(G), (1, 4))$-sufficient but not $(\mathscr{U}(G), (2, 4))$-sufficient. Since $(1, 4)$ is a G-access cover for $(2, 4)$, this shows that the condition $\Gamma \subseteq \Lambda$ cannot be omitted in Corollary 6.1.

Next let $S_2 = \{0\}$ and let $\Gamma = \{(3, 4)\}$. Then S_2 is $(\mathscr{U}(G), \Gamma)$-sufficient but not Λ-sufficient if $(2, 4) \in \Lambda \subseteq \langle n \rangle \times \langle n \rangle$. By choosing $\{(2, 4), (3, 4)\} \subseteq \Lambda$, we obtain an example with S_2 is $(\mathscr{U}(G), \Gamma)$-sufficient but not $(\mathscr{U}(G), \Lambda)$-sufficient even though $\Gamma \subseteq \Lambda$, and thus the condition that Γ is a G-access cover for Λ cannot be omitted in Corollary 6.1. Finally, observe that S_1 is $(\mathscr{U}(G), \Lambda)$-sufficient for any set Λ such that $(1, 4) \in \Lambda \subseteq \langle n \rangle \times \langle n \rangle$. Choosing $\{(1, 4), (2, 4)\} \subseteq \Lambda$ and putting $\Gamma = \{(2, 4)\}$ it follows from our previous remarks that S_1 is $(\mathscr{U}(G), \Lambda)$-sufficient, but not $(\mathscr{U}(G), \Gamma)$-sufficient. Note that $\Gamma \subseteq \Lambda$. Thus there appears to be no simple relation in general (without the condition that Γ is a G-access cover for Λ) between $(\mathscr{U}(G), \Gamma)$-sufficiency and $(\mathscr{U}(G), \Lambda)$-sufficiency when $\Gamma \subseteq \Lambda$.

We shall give two proofs of our next corollary. The first is an application of Corollary 6.1 and the second is based directly on Lemma 4.6.

COROLLARY 6.6. *Let* $\Gamma \subseteq \Lambda \subseteq \langle n \rangle \times \langle n \rangle$ *where* $\Gamma \neq \varnothing$. *Let* $S \subseteq N$. *If* S *is* $(\mathscr{J}, \Gamma)$-*sufficient then* S *is* $(\mathscr{J}, \Lambda)$-*sufficient.*

First proof. Let $A \in \operatorname{Equ}(\mathscr{J}, \Lambda, S)$. Then $A \in \operatorname{Equ}(\mathscr{U}(G(A)), \Lambda, S)$. Since

$$\mathscr{U}(G(A)) \subseteq \mathscr{J},$$

it follows from Lemma 4.13 that S is $(\mathscr{U}(G(A)), \Gamma)$-sufficient. But since $G(A)$ is strongly connected it follows from Lemma 2.7 that Γ is a $G(A)$-access cover for Λ. Hence, by Corollary 6.1, S is $(\mathscr{U}(G(A)), \Lambda)$-sufficient. It follows that

$$A \in \operatorname{Equ}(\mathscr{U}(G(A)), \Lambda, N) \subseteq \operatorname{Equ}(\mathscr{J}, \Lambda, N).$$

The result follows.

Second proof. Let $A \in \operatorname{Equ}(\mathscr{J}, \Lambda, S)$. Then, by Lemma 4.6, for all $(i, j) \in \Gamma$, $\operatorname{sgn} \prod_\alpha(A) = \operatorname{sgn} \prod_\beta(A)$, for all paths α, β from i to j in $G(A)$ such that $|\alpha|, |\beta| \in S$. Hence, since S is $(\mathscr{J}, \Gamma)$-sufficient, it follows that $A \in \operatorname{Equ}(\mathscr{J}, \Gamma, N)$ and consequently $\operatorname{sgn} \prod_\alpha(A) = \operatorname{sgn} \prod_\beta(A)$ for all paths α, β from i to j in $G(A)$, where $(i, j) \in \Gamma$, without restriction on the lengths of α and β. Hence, also, $\operatorname{sgn} \prod_\gamma(A) = \operatorname{sgn} \prod_\delta(A)$ for all paths γ, δ from h to k in $G(A)$, where $(h, k) \in \Lambda$, since, by Lemma 2.7, these paths can be extended to paths α, β, respectively, from i to j with $(i, j) \in \Gamma$. But this proves that S is $(\mathscr{J}, \Gamma)$-sufficient. $\square$

Of course, the most interesting case of Corollary 6.6 arises when

$$\{(i, j)\} = \Gamma \subseteq \Lambda = \langle n \rangle \times \langle n \rangle.$$

DEFINITION 6.7. Let S be a nonempty subset of N. Then we define

$D(S) = \{s - t : s, t \in S \text{ and } s > t\}$,
$\gcd(S) =$ the greatest common divisor of the elements of S,
$C(S) = \{\gcd(T) : T \subseteq S, T \neq \varnothing\}$,

$CD(S) = C(D(S))$,
$D(\varnothing) = C(\varnothing) = \varnothing$.

Observe that $S \subseteq C(S)$. For example, if $S = \{3, 9, 13, 18\}$ then

$$D(S) = \{4, 5, 6, 9, 10, 15\}$$

and $CD(S) = \{1, 2, 3, 4, 5, 6, 9, 10, 15\}$. Note also that $C(C(S)) = C(S)$. Since $C(CD(S)) = CD(S)$, it follows that every element of $CD(S)$ is a multiple of the minimal element of $CD(S)$.

LEMMA 6.8. *Let $S \subseteq N$ and let $A \in$ Equ (S). Then for a closed path α in $G(A)$ with length $s \in CD(S)$ we have $\prod_\alpha(A) > 0$.*

Proof. Let

$$\alpha = i_0 \to \cdots \to i_{s-1} \to i_0.$$

We first show $\prod_\alpha(A) > 0$ for $s \in D(S)$. Then $s = v - u$, where $u, v \in S$. Write $u = as + t$, where a and t are nonnegative integers and $t < s$. Then $v = (a + 1)s + t$. We take $\beta[\gamma]$ to be the path from i_0 to i_t of length $u[v]$ obtained by repeating $a[a + 1]$ times the path α and adjoining $i_0 \to \cdots \to i_t$. Since $A \in$ Equ (S), it follows from Lemma 4.6 that the nonzero path products $\prod_\beta(A)$ and $\prod_\gamma(A) = \prod_\beta(A)\prod_\alpha(A)$ have the same sign. Hence $\prod_\alpha(A) > 0$.

We now consider the general case of $s \in CD(S)$. Then there exist $s_1, s_2, \cdots, s_k$ in $D(S)$ whose gcd is s. As is well known, there exist integers a_i, $i = 1, \cdots, k$, such that

$$s = \sum_{i=1}^{k} a_i s_i. \tag{6.9}$$

Without loss of generality, assume that $a_i \leqq 0$ if and only if $1 \leqq i \leqq q$. Let ω_i be the closed path from i_0 to i_0 obtained by repeating s_i/s times the path α. By the first part of the proof, ω_i has a positive path product. Let $\mu[\nu]$ be the closed path from i_0 to i_0 obtained by repeating $|a_i|$ times the path ω_i, $i = 1, \cdots, t$ $[i = t + 1, \cdots, k]$. By (6.9), ν is obtained by adjoining α to μ. Since μ and ν have positive path products it follows that $\prod_\alpha(A) > 0$. □

COROLLARY 6.10. *If $A \in$ Equ (S) then $A \in$ Equ $(\Delta, CD(S))$.*

Proof. Immediate by Lemmas 6.8 and 4.6. □

The converses of Lemma 6.8 and Corollary 6.10 are false if $n > 1$ even for irreducible matrices. In fact, we shall give an example of an irreducible matrix A and a set S, for which every closed path of length $s \in CD(S)$ has positive path product, yet the matrix A is not even in Equ $((i, i), S)$, for any $i \in \langle n \rangle$.

Example 6.11. Let A be the $n \times n$ matrix with all entries on and above the diagonal equal to 1 and all entries below the diagonal equal to -1. Let $S = \{1, 2\}$ and let $i, j \in \langle n \rangle$, $i \neq j$. Observe that the circuit product corresponding to $i \to i$ is positive while the circuit product corresponding to $i \to j \to i$ is negative. Hence, by Lemma 4.6, $A \notin$ Equ $((i, i), S)$. However $CD(S) = \{1\}$ and all circuit products of length 1 are positive. Hence $A \in$ Equ $(CD(S))$, by Lemma 4.6.

THEOREM 6.12. *Let S be a subset of N. Then the following are equivalent.*

(i) *S is Δ-sufficient.*
(ii) *S is $(\mathscr{J}, \Delta)$-sufficient.*
(iii) *S is $\mathscr{J}$-sufficient.*
(iv) *$CD(S)$ contains $\langle n \rangle$.*
(v) *For all $A \in$ Equ (S), all circuit products of A are positive.*
(vi) *For all $A \in$ Equ (S), A is diagonally similar to a matrix B such that all irreducible diagonal blocks in the Frobenius normal form of B are nonnegative.*

Proof. (i) $\Rightarrow$ (ii) is obvious.

(ii) $\Rightarrow$ (iii). By Corollary 6.6.

(iii) $\Rightarrow$ (iv). Let $k \in \langle n \rangle$ and let λ be a nonzero complex number. Suppose that $k \notin CD(S)$. We shall prove the claimed implication by constructing an irreducible $n \times n$ matrix $A(k, \lambda)$ such that, for suitable λ, $A(k, \lambda) \in \mathrm{Equ}\,(\mathscr{J}, S) \setminus \mathrm{Equ}\,(\mathscr{J}, N)$. If $k = 1$, we let $A(k, \lambda)$ be the $n \times n$ matrix all of whose entries are λ. If $k \in \{2, \cdots, n\}$ we define $A(k, \lambda) = A$ by

$$\begin{aligned} a_{i,i+1} &= 1, \quad i = 1, \cdots, k-2, \\ a_{k-1,j} &= \lambda, \quad j = k, \cdots, n, \\ a_{j,1} &= 1, \quad j = k, \cdots, n, \\ a_{ij} &= 0, \quad \text{otherwise}, \quad i, j \in \langle n \rangle, \end{aligned}$$

e.g., for $n = 5$ and $k = 4$

$$A(k, \lambda) = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & \lambda & \lambda \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Note that for all $k \in \langle n \rangle$ the matrix $A(k, \lambda)$ is irreducible and the length of every closed path of $A(k, \lambda)$ is a multiple of k and, provided that $k \geqq 2$, every circuit product of $A(k, \lambda)$ equals λ. For all $k \in \langle n \rangle$, it follows that for every closed path δ of $G(A(k, \lambda))$ we have

$$\prod\nolimits_\delta (A(k, \lambda)) = \lambda^h, \quad \text{where} \quad |\delta| = hk. \tag{6.13}$$

We now choose λ depending on two cases.

(a) No multiple of k lies in $CD(S)$. Then let $\lambda = -1$.

(b) Some positive multiple of k is in $CD(S)$. Then let pk be the smallest such multiple and λ be a primitive pth root of unity. Since $k \notin CD(S)$ we have $p > 1$.

Let $i, j \in \langle n \rangle$ and let α and β be paths from i to j in $G(A)$. Suppose that $|\alpha|$ and $|\beta|$ belong to S and assume without loss of generality that $|\alpha| \geqq |\beta|$. Let $d = |\alpha| - |\beta|$. Let γ be a path from j to i in $G(A(k, \lambda))$, which exists since $A(k, \lambda)$ is irreducible. Observe that $\alpha\gamma$ and $\beta\gamma$ are closed paths and hence $d = |\alpha\gamma| - |\beta\gamma|$ is divisible by k.

Suppose first that $d = 0$. Then $\alpha\gamma$ and $\beta\gamma$ are closed paths of the same length. It follows from (6.13) that the closed path products corresponding to $\alpha\gamma$ and $\beta\gamma$ are equal. Suppose now that $d > 0$. Then $d \in D(S) \subseteq CD(S)$. Hence (b) above holds. We recall that $C(CD(S)) = CD(S)$. Hence, since pk is the minimal multiple of k in $CD(S)$, it follows that d must be a multiple of pk. But (6.13) then again implies that the closed path products corresponding to $\alpha\gamma$ and $\beta\gamma$ are equal. Hence, in either case, $\prod_\alpha(A) = \prod_\beta(A)$. Since i, j are arbitrary in $\langle n \rangle$, it follows from Lemma 4.6 that $A \in \mathrm{Equ}\,(S)$.

On the other hand, since $A(k, \lambda)$ has a circuit α of length k and $\prod_\alpha(A(k, \lambda))$ is not positive, we have by Theorem 5.2 that $A(k, \lambda) \notin \mathrm{Equ}\,(N)$. The implication (iii) $\Rightarrow$ (iv) is proved.

(iv) $\Rightarrow$ (v). Immediate by Lemma 6.8.

(v) $\Rightarrow$ (vi). By Fiedler and Ptak [3] or Engel and Schneider [2] an irreducible matrix that satisfies (v) is diagonally similar to a nonnegative matrix. By applying this result to the Frobenius normal form of A we obtain (vi) from (v).

(vi) $\Rightarrow$ (i). Let $A \in \mathrm{Equ}\,(\Delta, S)$ and let B be a matrix diagonally similar to A and such that B has nonnegative diagonal blocks in a (and therefore every) Frobenius normal form. Since the diagonal blocks of B are clearly in Equ (Δ, N) it follows that $B \in$ Equ (Δ, N). Hence $A \in$ Equ (Δ, N) and (i) follows from (vi). $\square$

THEOREM 6.14. *Let $S \subseteq N$.*

I. *If $n \leqq 2$, then the following are equivalent.*

(i) *S is sufficient,*

(ii) $\langle n \rangle \subseteq CD(S)$.

II. *If $n \geqq 3$, then* (i) *is equivalent to*

(iii) (a) $\{n-1, n\} \subseteq CD(S)$.

and

(b) $\langle n-1 \rangle \subseteq S$.

Proof. I: Let $n \leqq 2$.

(i) $\Rightarrow$ (ii). Since S is sufficient, it is also Δ-sufficient and the result follows from Theorem 6.12.

(ii) $\Rightarrow$ (i). Let $\langle n \rangle \subseteq CD(S)$. Suppose $A \in$ Equ (S). Then, by Lemma 6.8 all circuit products of A are positive. Since, for $i, j \in \langle n \rangle$ there is at most one nonempty simple path from i to j in $G(A)$, the conditions of Theorem 5.14, Part (v) are satisfied for all $i, j \in \langle n \rangle$. Hence, by Theorem 5.14, $A \in$ Equ (N) and the implication (ii) $\Rightarrow$ (i) follows.

II: (i) $\Rightarrow$ (iii), Part (a). With the same proof as in Part I, we have $\langle n \rangle \subseteq CD(S)$.

(i) $\Rightarrow$ (iii), Part (b). Let $2 \leqq k \leqq n-1$. To prove this implication it is enough to construct a matrix $B(k) \in$ Equ $(S)\backslash$Equ (N) if either $1 \notin S$ or $k \notin S$. We let the arc set of $G(B(k))$ consist of $1 \rightarrow k+1$, and $i \rightarrow i+1$, $i = 1, \cdots, k$. We define the $(1, k+1)$-element of $B(k)$ to be -1 and all other nonzero elements to be 1. For example, if $k = 2$ and $n = 4$, then

$$B(k) = \begin{bmatrix} 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Let $i, j \in \langle n \rangle$. If either $1 \notin S$ or $k \notin S$ then there is at most one path from i to j in $G(B(k))$ whose length lies in S. Hence, by Lemma 4.6, we have $A \in$ Equ (S). But there are two paths from 1 to k in $G(B(k))$ whose corresponding products have different signs. Hence, again by Lemma 4.6, $A \notin$ Equ (N).

(iii) $\Rightarrow$ (i). Suppose that (iii) holds. Let $A \in$ Equ (S). Let $i, j \in \langle n \rangle$. Let α and β be simple paths in $G(A)$. Since $|\alpha| < n$, and $|\beta| < n$, we have by Lemma 4.6 that $\prod_\alpha(A) = \prod_\beta(A)$. Since $\langle n \rangle \subseteq CD(S)$, it follows from Lemma 6.8 that all circuit products of A are positive. Hence the conditions of Theorem 5.2, Part (v) are satisfied. By Theorem 5.2 we now obtain $A \in$ Equ (N) and (iii) $\Rightarrow$ (i) is proved. $\square$

We note that, for $n \geqq 3$, neither of the conditions (iii)(a) or (iii)(b) of Theorem 6.14 alone implies that S is sufficient, or even that S is Δ-sufficient. This is clear from Theorem 6.12 since neither condition implies that $\langle n \rangle \subseteq CD(S)$.

COROLLARY 6.15. *Let $n \geqq 3$. Let $S \subseteq N$.*

I. *If S is sufficient then $|S| \geqq n$.*

II. *The following conditions are equivalent:*

(i) *S is sufficient and $|S| = n$.*

(ii) $S = \{1, \cdots, n-1, m\}$ *where* $n+1 \leqq m \leqq 2n-2$.

(iii) *S is optimal sufficient.*

Proof.

I. This is obvious by Theorem 6.14.

II. (i) $\Rightarrow$ (ii). By Theorem 6.14 we have $S = \{1, \cdots, n-1, m\}$. If $m = 0$ or

$m = n$ then $n \notin CD(S)$ and S is not sufficient by Theorem 6.14. Hence $m > n$. Suppose that $m > 2n - 2$. Then it follows that

$$D(S) = \{1, \cdots, n-2, m-n+1, \cdots, m-1\}. \tag{6.16}$$

Let $p, q \in D(S)$ where $p < q$. Then, by (6.16), either $p < n - 1$ or $q - p < n - 1$. Hence, gcd $(p, q) < n - 1$. It follows that gcd $(T) < n - 1$ for any subset T of $D(S)$ with $|T| > 1$. Since $n - 1 \notin D(S)$ and just one positive multiple of $n - 1$ belongs to $D(S)$ we also have $n - 1 \notin CD(S)$, which contradicts Theorem 6.14. The implication is now proved.

(ii) $\Rightarrow$ (iii). By Theorem 6.14, S is sufficient. The optimal sufficiency of S follows from Part I.

(iii) $\Rightarrow$ (i). Let S be an optimal sufficient set. Then clearly S is sufficient. By Theorem 6.14 the set $T = \{1, \cdots, n - 1, n + 1)$ is sufficient with $|T| = n$. Hence, by Part I we have $|S| = n$. □

We now use Corollary 6.15 to show that a minimal sufficient set is not necessarily an optimal sufficient set.

Example 6.17. Let $n \geqq 3$ and let $S = \{1, \cdots, n - 1, 2n - 1, 3n - 2\}$. Then $\langle n \rangle \subseteq CD(S)$ and so, by Theorem 6.14, S is sufficient. Let S' be a subset of S of cardinality n. Observe that S' cannot satisfy condition (ii) of Corollary 6.15. Hence, by Corollary 6.15, S' is not sufficient. Thus, S is a minimal sufficient set, but, again by Corollary 6.15, S is not an optimal sufficient set.

It is clear that our definitions and results raise a number of interesting questions. Some are purely number theoretic, others involve a mixture of matrix and number theory. A general problem is to characterize the $(\mathscr{A}, \Gamma)$-sufficient [minimal $(\mathscr{A}, \Gamma)$-sufficient, optimal $(\mathscr{A}, \Gamma)$-sufficient] sets for given $\mathscr{A} \subseteq \mathscr{U}$ and $\Gamma \subseteq \langle n \rangle \times \langle n \rangle$.

In view of Theorem 6.12 the following open questions are of interest.

Open Questions 6.18.

(i) Characterize subsets S of N such that $CD(S) \supseteq \langle n \rangle$.

(ii) Characterize subsets S of N which are minimal with respect to the property $CD(S) \supseteq \langle n \rangle$.

Remark 6.19. In Definition 4.4 the restriction to $A \in \mathscr{U}$ (viz. $A \in \mathbb{C}^{nn}$ such that $\rho(|A|) < 1$) and the use of power series with all coefficients equal to 1 are technicalities. Alternatively, we could have considered throughout arbitrary $A \in \mathbb{C}^{nn}$ and nonnegative sequences

$$\langle C \rangle = (c_1, c_2, \cdots\cdots\cdots)$$

such that $\sum_{s \in N} c_s |A|^s$ converges. In this approach one then defines the equality class Equ $(\mathscr{A}, \Gamma\ S)$ to consist of all $A \in \mathscr{A}$ such that for some nonnegative sequences $\langle C \rangle$ with $c_s \neq 0$ if and only if $s \in S$, $\sum_{s \in N} c_s |A|^s$ converges and

$$\left| \sum_{s \in N} (c_s A^s)_\Gamma \right| = \sum_{s \in N} (c_s |A|^s)_\Gamma .$$

Since the proof of our fundamental lemma, Lemma 4.6, is unchanged, our results go through to this more general situation and reduce to the previous results for $A \in \mathscr{U}$. The concept of sufficiency remains unchanged. We illustrate by means of an example.

Example 6.20. Let $n \leqq 10$. If $S = \{3, 9, 10, 13, 18\}$ then $CD(S) = \langle 10 \rangle \cup \{15\}$ and hence, by Theorem 6.12, S is $(\mathscr{J}, \langle n \rangle)$-sufficient. In other words, let A be an irreducible

$n \times n$ matrix, $n \leqq 10$, and let c_s be positive, $s = 3, 9, 10, 13, 18$. Then the equality

$$|c_3A^3 + c_9A^9 + c_{10}A^{10} + c_{13}A^{13} + c_{18}A^{18}|$$
$$= c_3|A|^3 + c_9|A|^9 + c_{10}|A|^{10} + c_{13}|A|^{13} + c_{18}|A|^{18}$$

implies that for all nonnegative d_s, $s \in N$, we have

$$\left|\sum_{s \in N} d_sA^s\right| = \sum_{s \in N} d_s|A|^s,$$

provided that the second series converges. In particular, if $\rho(|A|) < 1$, then

$$|(I - A)^{-1}| = (I - |A|)^{-1}.$$

REFERENCES

[1] P. M. COHN, *Universal Algebra*, Harper and Row, New York, 1965.

[2] G. M. ENGEL AND H. SCHNEIDER, *Cyclic and diagonal products on a matrix*, Linear Algebra Appl., 7 (1973), pp. 301–335.

[3] M. FIEDLER AND V. PTAK, *Cyclic products and an inequality for determinants*, Czechoslovak Math. J., 19 (1969), pp. 428–450.

[4] S. FRIEDLAND, D. HERSHKOWITZ, AND H. SCHNEIDER, *Matrices whose powers are M-matrices or Z-matrices*, Trans. Amer. Math. Soc., 300 (1987), pp. 343–366.

[5] D. HERSHKOWITZ AND H. SCHNEIDER, *On 2k-twisted graphs*, European J. Combin. (to appear).

[6] A. NEUMAIER, *The extremal case of some matrix inequalities*, Arch. Math., 43 (1984), pp. 137–141.

[7] A. OSTROWSKI, *Über die Determinanten mit überwiegender Hauptdiagonale*, Comment. Math. Helv., 10 (1937), pp. 69–96.

[8] B. D. SAUNDERS AND H. SCHNEIDER, *Flows on graphs applied to diagonal similarity and diagonal equivalence for matrices*, Discrete Math., 24 (1981), pp. 205–220.