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SEQUENCES, WEDGES AND ASSOCIATED SETS OF COMPLEX NUMBERS

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1. INTRODUCTION

Let $R = (r_1, r_2, \ldots)$ be a finite or infinite (strictly increasing) sequence of positive integers and let $(W_1, W_2, \ldots)$ be a sequence of wedges in the complex plane. Consider the following problem: Characterize those complex numbers c for which

$$c^{r_k} \in W_k\,, \tag{1.1}$$

for $k = 1, 2, \ldots$.

It is shown in [1] and [4] that, under certain assumptions on the wedges and on the density of the sequence R, the set of all complex numbers satisfying (1.1) for $k = 1, 2, \ldots$ is finite. The set itself is not identified there.

In this paper we assume that

$$W = W_1 = W_2 = \ldots,$$

where W is the open wedge $W(\alpha)$ (the closed wedge $W[\alpha]$) of width 2α symmetrically located around the nonnegative real axis. We then discuss the set $S(R, \alpha, n)$ ($S[R, \alpha, n]$) of nonzero complex numbers c which satisfy (1.1) for $k = 1, \ldots, n$, where n is either a positive integer or ∞.

Section 2 is devoted to the case of finite n. Let $W = W(\alpha)$. Obviously, $W(\alpha/r_n) \subseteq S(R, \alpha, n)$. We prove a necessary condition (Proposition 2.14) and a sufficient condition (Proposition 2.22) for

$$W(\alpha/r_n) = S(R, \alpha, n)\,. \tag{1.2}$$

These results are then combined to obtain a characterization of the case (1.2) (Theorem 2.29). In the case that (1.2) does not hold, we give a necessary condition (Proposition 2.39), a sufficient condition (Proposition 2.53) and a characterization (Theorem 2.63) for

$$S(R, \alpha, n) \subseteq W(\alpha/r_{n-1})\,. \tag{1.3}$$

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Our conditions involve α and two or three consecutive terms of R. Similar results hold for closed wedges.

Let T be a subset of the set $\mathbb{C}$ of complex numbers. In Section 3 we introduce the concept of (T, α)-*forcing* ((T, α)-*semiforcing*) *sequence*. A sequence R with cardinality $|R|$ is said to be such a sequence if $W = W[\alpha]$ ($W = W(\alpha)$) and if $S \cap T$ is contained in the positive real axis, where $S = S(R, \alpha, n)$ ($S = S[R, \alpha, n]$). We prove several sufficient conditions for R to be (T, α)-(semi)forcing (Theorems 3.4, 3.10, 3.11, 3.14; Corollaries 3.15, 3.16, 3.28; Proposition 3.27). Section 3 is concluded with a discussion of an interesting relation between (semi)-forcing sequences and continued fractions.

Examples of $(\mathbb{C}, \pi/2)$-forcing sequences are $(k, k+1, k+2, \ldots)$, where k is a positive integer, $(1, 2, 3, 8, \ldots)$ and $(1, 3, 10, 31, 94, \ldots)$. Examples of $(\mathbb{C}, \pi/2)$-semiforcing (but not forcing) sequences are $(1, 3, 9, 27, \ldots)$ and $(1, 3, 4, 8, 16, \ldots)$. Additional examples are given in Section 4, which also contains examples pertaining to the results in Section 2.

Applications of forcing and semiforcing sequences to linear algebra are contained in [3]. These applications motivated the present investigations. In view of the results presented here and their applications it would be of interest to characterize forcing and semiforcing sequences.

2. WEDGES

In this paper we shall use the notation $(r_1, r_2, \ldots)$ for an infinite sequence of integers and the notation $(r_1, r_2, \ldots, r_t)$ for a sequence of integers which is finite if t is a positive integer and is the infinite sequence $(r_1, r_2, \ldots)$ if $t = \infty$. Further, "sequence of positive integers" will always mean "strictly increasing sequence of positive integers".

The cardinality of a set (or sequence) R is denoted by $|R|$. The set of all complex numbers is denoted by $\mathbb{C}$. We assume that the argument of a nonzero complex number is chosen in the half open interval $(-\pi, \pi]$.

Definition 2.1. Let $-\pi \leqq \alpha, \beta \leqq \pi$. If $\alpha \leqq \beta$ then we define the *closed wedge* (excluding 0) $W[\alpha, \beta]$ *of width* $\beta - \alpha$ to be the set

$$\{c \in \mathbb{C}: c \neq 0,\ \alpha \leqq \arg(c) \leqq \beta\}.$$

If $\beta < \alpha$ then we define the *closed wedge* $W[\alpha, \beta]$ *of width* $2\pi + \beta - \alpha$ to be the set

$$\{c \in \mathbb{C}: c \neq 0,\ \alpha \leqq \arg(c) \leqq \pi \text{ or } -\pi \leqq \arg(c) \leqq \beta\}.$$

Remark 2.2. Consider the rays l_α nad l_β which form angles α and β respectively with the nonnegative real axis. Observe that $W[\alpha, \beta]$ is the wedge covered when we move from l_α to l_β counter-clockwise.

Definition 2.3. Let $-\pi \leqq \alpha, \beta \leqq \pi$. We define the *open wedge* $W(\alpha, \beta)$ to be the interior of the closed wedge $W[\alpha, \beta]$ in the Euclidean topology. The *width* of $W(\alpha, \beta)$ is defined to equal the width of $W[\alpha, \beta]$.

Remark 2.4. (i) By Definition 2.1 we have $W[\alpha, \beta] = \mathbb{C} \setminus \{0\}$ if and only if $\alpha = -\pi$ and $\beta = \pi$.

(ii) By Definition 2.3, for every α and β, $-\pi \leqq \alpha, \beta \leqq \pi$, we have $W(\alpha, \beta) \subsetneqq \mathbb{C} \setminus \{0\}$.

Notation 2.5. For $0 \leqq \alpha \leqq \pi$ we denote:

$$W[\alpha] = W[\alpha, \alpha],$$
$$W(\alpha) = W(\alpha, \alpha).$$

Notation 2.6. Let $R = (r_1, r_2, \ldots, r_t)$ be a sequence of positive integers. Let n be either a positive integer or ∞, where $n \leqq t$, and let $0 \leqq \alpha \leqq \pi$. We denote:

$$S[R, \alpha, n] = \{c \in \mathbb{C}\colon c^{r_k} \in W[\alpha],\ k = 1, \ldots, n\},$$
$$S(R, \alpha, n) = \{c \in \mathbb{C}\colon c^{r_k} \in W(\alpha),\ k = 1, \ldots, n\}.$$

Notation 2.7. Let $0 \leqq \alpha \leqq \pi$ and let n be a positive integer. We denote:

$$Q_n[\alpha] = \{c \in \mathbb{C}\colon c^n \in W[\alpha]\},$$
$$Q_n(\alpha) = \{c \in \mathbb{C}\colon c^n \in W(\alpha)\}.$$

Let n be a positive integer and let m be the largest nonnegative integer such that $m \leqq (n-1)/2$. Let $0 \leqq \alpha \leqq \pi$. It is easy to verify that

$$Q_n[\alpha] = \left(\bigcup_{k=-m}^{m} W[(2\pi k - \alpha)/n,\ (2\pi k + \alpha)/n]\right) \cup W^*, \tag{2.8}$$

where

$$W^* = \begin{cases} \emptyset, & n \text{ odd}, \\ W[\pi - \alpha/n,\ -\pi + \alpha/n], & n \text{ even}. \end{cases} \tag{2.9}$$

Also, $Q_n(\alpha)$ is the interior of $Q_n[\alpha]$.

For example, the sets $Q_4[\pi/3]$ and $Q_5[\pi/4]$ are the darkened areas in the figures 1 and 2 on page 141.

The following observation is immediate.

Observation 2.10. *Let $R = (r_1, r_2, \ldots)$ be a sequence of positive integers, let n be a positive integer and let $0 \leqq \alpha \leqq \pi$. Then*

$$S[R, \alpha, n] = \bigcap_{k=1}^{n} Q_{r_k}[\alpha]$$

and

$$S(R, \alpha, n) = \bigcap_{k=1}^{n} Q_{r_k}(\alpha).$$

Since R is an increasing sequence we now obtain from (2.8) and Observation 2.10

that for $0 \leqq \alpha \leqq \pi$,

(2.11) $$W[\alpha/r_n] \subseteq S[R, \alpha, n]$$

and similarly

(2.12) $$W(\alpha/r_n) \subseteq S(R, \alpha, n)\,.$$

We now find a necessary condition and a sufficient condition for equality of the sets in (2.11) and (2.12). We prove our assertions for open wedges. The results in the case of closed wedges are similar, and will be stated later.

The case $n = 1$ is easy.

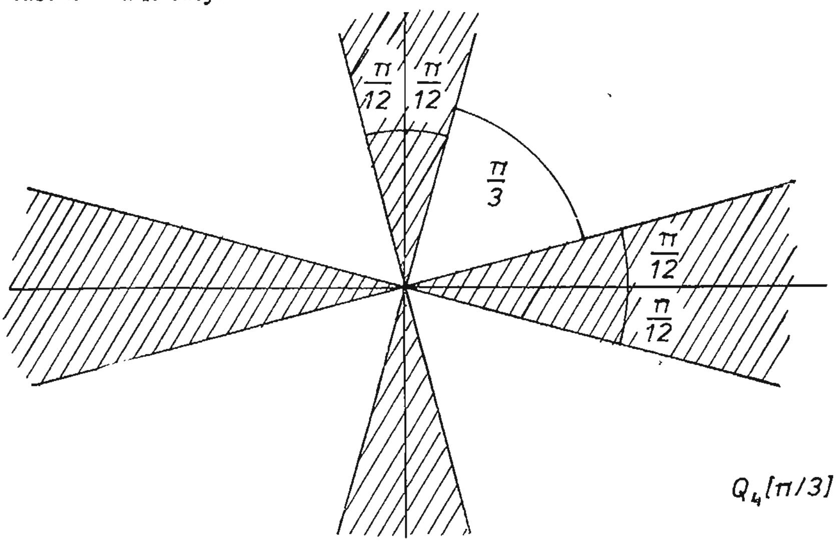

Figure 1

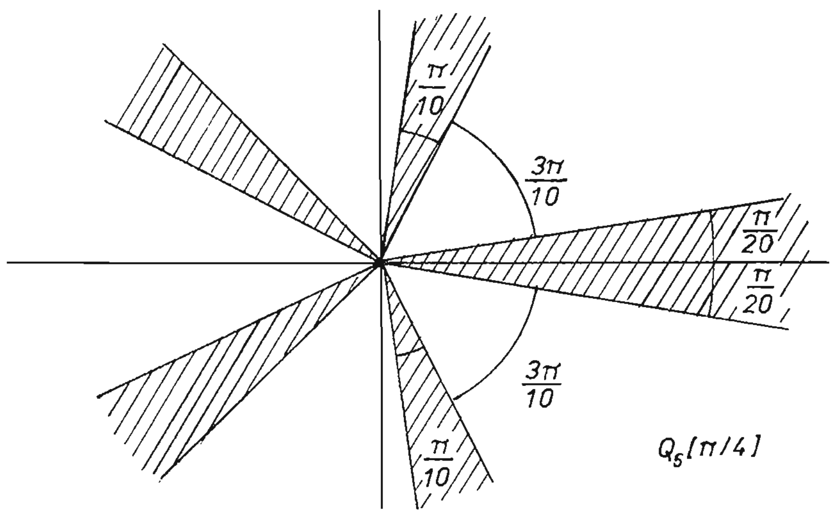

Figure 2

Theorem 2.13. *Let $R = (r_1, r_2, \dots)$ be a sequence of positive integers and let $0 < \alpha \leqq \pi$. Then*

$$W(\alpha/r_1) = S(R, \alpha, 1)$$

if and only if $r_1 = 1$.

Proof. In view of (2.8) and Observation 2.10, the set $S(R, \alpha, 1)$ consists of the single wedge $W(\alpha/r_1)$ if and only $r_1 = 1$. □

Proposition 2.14. *Let $R = (r_1, r_2, \dots)$ be a sequence of positive integers, let n be a positive integer, $n > 1$, and let $0 < \alpha \leqq \pi$. If*

$$W(\alpha/r_n) = S(R, \alpha, n) \tag{2.15}$$

then

$$\alpha r_n \leqq (2\pi - \alpha)\, r_{n-1}\,. \tag{2.16}$$

Proof. Observe that

$$S(R, \alpha, n) = S(R, \alpha, n-1) \cap Q_{r_n}(\alpha)\,. \tag{2.17}$$

Assume that

$$\alpha r_n > (2\pi - \alpha)\, r_{n-1}\,, \tag{2.18}$$

or equivalently,

$$(2\pi - \alpha)/r_n < \alpha/r_{n-1}\,.$$

Let $\beta = \min\{(2\pi + \alpha)/r_n, \alpha/r_{n-1}\}$, and let $W = W((2\pi - \alpha)/r_n, \beta)$. By (2.8) we have $W \subseteq Q_{r_n}(\alpha)$. Also, by (2.12),

$$W \subseteq W(\alpha/r_{n-1}) \subseteq S(R, \alpha, n-1)\,.$$

Therefore, it follows from (2.17) that

$$W \subseteq S(R, \alpha, n)\,.$$

Since $\beta > \alpha/r_n$ it follows that $W \nsubseteq W(\alpha/r_n)$ and we now have a contradiction to (2.15). Hence, our assumption (2.18) is false and we have proved (2.16). □

Remark 2.19. Inequality (2.16) is equivalent to

$$\alpha \leqq 2\pi r_{n-1}/(r_{n-1} + r_n)\,. \tag{2.20}$$

Therefore, in view of Proposition 2.14, the equality (2.15) yields an upper bound on α. In particular, since $r_n \geqq r_{n-1} + 1$, it follows from (2.20) that

$$\alpha \leqq 2\pi r_{n-1}/(2r_{n-1} + 1)\,.$$

The converse of Proposition 2.14 is in general false, as demonstrated by Example 4.1.

In order to prove a sufficient condition for (2.15) in terms of (2.16) we introduce the following notation.

Notation 2.21. Let $T \subseteq \mathbb{C}$, $T \setminus \{0\} \neq \emptyset$.

We denote:

$$\mu(T) = \sup\{|\arg(c)| : c \in T,\ c \neq 0\}\,.$$

Proposition 2.22. *Let* $R = (r_1, r_2, \ldots)$ *be a sequence of positive integers, let* n *be a positive integer,* $n > 1$, *and let* $0 < \alpha \leqq \pi$. *If*

$$\mu(S(R, \alpha, n)) \leqq \alpha / r_{n-1}, \tag{2.23}$$

and if

$$\alpha r_n \leqq (2\pi - \alpha) r_{n-1}, \tag{2.24}$$

then

$$W(\alpha / r_n) = S(R, \alpha, n).$$

Proof. It is easy to verify, by (2.8), that if (2.24) holds then

$$W[\alpha / r_{n-1}] \cap Q_{r_n}(\alpha) = W(\alpha / r_n). \tag{2.25}$$

Our assertion now follows from (2.23) and (2.25). □

The following corollary clearly follows from Proposition 2.22.

Corollary 2.26. *Let* $R = (r_1, r_2, \ldots)$ *be a sequence of positive integers, let* n *be a positive integer,* $n > 1$, *and let* $0 < \alpha \leqq \pi$. *If*

$$S(R, \alpha, n - 1) = W(\alpha / r_{n-1}), \tag{2.27}$$

and if

$$\alpha r_n \leqq (2\pi - \alpha) r_{n-1},$$

then

$$S(R, \alpha, n) = W(\alpha / r_n). \tag{2.28}$$

The converse of Corollary 2.26 is not true in general. Example 4.2 demonstrates that (2.28) does not imply (2.27). Example 4.2 also shows an application of Corollary 2.26.

Propositions 2.14 and 2.22 yield the following theorem.

Theorem 2.29. *Let* $R = (r_1, r_2, \ldots)$ *be a sequence of positive integers, let* n *be a positive integer,* $n > 1$, *and let* $0 < \alpha \leqq \pi$. *Then*

$$S(R, \alpha, n) = W(\alpha / r_n) \tag{2.30}$$

if and only if

$$\mu(S(R, \alpha, n)) \leqq \alpha / r_{n-1} \tag{2.31}$$

and

$$\alpha r_n \leqq (2\pi - \alpha) r_{n-1} \tag{2.32}$$

Proof. Obviously, (2.30) implies (2.31). The implication (2.30) ⇒ (2.32) is proved in Proposition 2.14. Inequalities (2.31) and (2.32) imply (2.30) by Proposition 2.22. □

For a sequence $R = (r_1, r_2, \ldots)$ of positive integers and for $0 < \alpha \leqq \pi$, (2.12) implies that

$$\mu(S(R, \alpha, n) \geqq \alpha / r_n, \tag{2.33}$$

Furthermore, by (2.8) and Observation (2.10) equality in (2.33) holds if and only if (2.15) holds.

We now assume that strict inequality holds in (2.33) and we seek an upper bound for the left hand side of (2.33) under additional conditions.

We start with the case $n = 2$.

Theorem 2.34. *Let $R = (r_1, r_2, \ldots)$ be a sequence of positive integers, and let $0 < \alpha \leqq \pi$. Then*

$$\alpha/r_2 < \mu(S(R, \alpha, 2)) \leqq \alpha/r_1 \tag{2.35}$$

if and only if

$$r_1 = 1$$

and

$$\alpha r_2 > 2\pi - \alpha .$$

Proof. If $r_1 = 1$ then by Theorem 2.13

$$\mu(S(R, \alpha, 2)) \leqq \mu(S(R, \alpha, 1)) = \alpha/r_1 .$$

Furthermore, if $\alpha r_2 > 2\pi - \alpha$, it follows from Proposition 2.14 that

$$\mu(S(R, \alpha, 2)) > \alpha/r_2 . \tag{2.36}$$

Conversely, assume that (2.35) holds. Since we now have (2.36), in view of the right hand inequality in (2.35) it follows from Proposition 2.22 that

$$\alpha r_2 > (2\pi - \alpha)\, r_1 . \tag{2.37}$$

By (2.8), every open wedge of width greater than $(2\pi - 2\alpha)/r_2$ has a nonempty intersection with $Q_{r_2}(\alpha)$. Observe that $Q_{r_1}(\alpha)$ is a union of r_1 wedges, each of width $2\alpha/r_1$. By (2.37) we have $2\alpha/r_1 > (2\pi - 2\alpha)/r_2$, and hence, each of the r_1 wedges contained in $Q_{r_1}(\alpha)$ has a nonempty intersection with $Q_{r_2}(\alpha)$. By the right hand inequality of (2.35), only the wedge $W(\alpha/r_1)$ has such a nonempty intersection, and hence we necessarily have $r_1 = 1$. □

Notation 2.38. Let $T \subseteq \mathbb{C}$ and let m be a positive integer. We denote by T^m the set $\{c^m : c \in T\}$.

Proposition 2.39. *Let $R = (r_1, r_2, \ldots)$ be a sequence of positive integers, let n be a positive integer, $n > 2$, and let $0 < \alpha \leqq \pi$. If*

$$\alpha/r_n < \mu(S(R, \alpha, n)) \leqq \alpha/r_{n-1} , \tag{2.40}$$

then either

$$\alpha r_{n-1} \leqq (2\pi - \alpha)\, r_{n-2} \tag{2.41}$$

or

$$\frac{2\pi k + \alpha}{2\pi - \alpha}\, r_{n-1} \leqq r_n \leqq \frac{2\pi(k+1) - \alpha}{\alpha}\, r_{n-2} , \tag{2.42}$$

for some positive integer k.

Proof. Let (2.40) hold. By Proposition 2.22, the two inequalities in (2.40) yield that

$$\alpha r_n > (2\pi - \alpha)\, r_{n-1}\,. \tag{2.43}$$

We assume that (2.41) is not satisfied, namely

$$(2\pi - \alpha)/r_{n-1} < \alpha/r_{n-2}\,, \tag{2.44}$$

and we shall prove (2.42). Assume first that

$$(2\pi + \alpha)\, r_{n-1} \leqq \alpha/r_{n-2}\,. \tag{2.45}$$

By (2.8) and Observation 2.10 we now obtain

$$\begin{aligned} W = W((2\pi - \alpha)/r_{n-1}, (2\pi + \alpha)/r_{n-1}) \subseteq \\ \subseteq S(R, \alpha, n-2) \cap Q_{r_{n-1}}(\alpha) = S(R, \alpha, n-1)\,. \end{aligned} \tag{2.46}$$

It now follows from the right hand inequality in (2.40) and from (2.46) that

$$W^{r_n} \cap W(\alpha) = \emptyset\,. \tag{2.47}$$

Observe that W^{r_n} is an open wedge of width $d = 2\alpha r_n/r_{n-1}$ if $d \leqq 2\pi$ and $W^{r_n} = C$ if $d > 2\pi$. Hence it follows from (2.47) that $d \leqq 2\pi - 2\alpha$ and so $(\pi - \alpha)\, r_{n-1} \geqq \alpha r_n$, which contradicts (2.43). Therefore, our assumption (2.45) is false and consequently we deduce that $(2\pi + \alpha)/r_{n-1} > \alpha/r_{n-2}$. In view of (2.44) we now have

$$V = W((2\pi - \alpha)/r_{n-1}, \alpha/r_{n-2}) \subseteq S(R, \alpha, n-2) \cap Q_{r_{n-1}}(\alpha) = S(R, \alpha, n-1)\,.$$

As before

$$V^{r_n} \cap W(\alpha) = \emptyset\,. \tag{2.48}$$

Let k be the integer such that

$$2\pi k - \alpha < \alpha r_n/r_{n-2} \leqq 2\pi(k+1) - \alpha\,. \tag{2.49}$$

Observe that since $r_n > r_{n-1}$ it follows from (2.44) that $k \geqq 1$. By (2.48) and the left hand inequaity of (2.49) we have

$$(2\pi - \alpha)\, r_n/r_{n-1} \geqq 2\pi k + \alpha\,. \tag{2.50}$$

Inequalities (2.42) now follow from (2.49) and (2.50). □

We remark that (2.40) implies neither (2.41) nor (2.42), as demonstated by Examples 4.3 and 4.4.

Remark 2.51. As in Remark 2.19, we note that (2.40) yields some upper bound on α. The precise computation of such a bound in general might be tedious. However, to demonstrate our assertion we now show that if (2.40) holds with $r_1 = 1$ and $n = 3$ then $\alpha < 3\pi/4$. By Proposition 2.39, (2.40) implies either (2.41) or (2.42) for some positive integer k. If (2.41) holds then, since $r_2 \geqq 2$, we have $2\alpha \leqq (2\pi - \alpha)$, or, equivalently, $\alpha \leqq 2\pi/3$. If (2.42) holds and if we assume that

$$\alpha \geqq 3\pi/4 \tag{2.52}$$

then we obtain

$$\frac{16k+6}{5} \leqq 2\,\frac{2\pi k+\alpha}{2\pi-\alpha} \leqq \frac{2\pi(k+1)-\alpha}{\alpha} \leqq \frac{8k+5}{3},$$

which implies that $k \leqq 7/8$, in contradiction to the fact that k is a positive integer. Thus, our assumption (2.52) is false and we have $\alpha < 3\pi/4$.

The converse of Proposition 2.39 is false in general as demonstrated by Example 4.1 in which (2.41) is satisfied for n sufficiently large. Another example which involves (2.42) is Example 4.7.

If we add an additional hypothesis then we obtain the following converse to Proposition 2.39.

Proposition 2.53. *Let $R = (r_1, r_2, \ldots)$ be a sequence of positive integers, let n be a positive integer, $n > 2$, and let $0 < \alpha \leqq \pi$. If*

$$\mu(S(R, \alpha, n)) \leqq \alpha/r_{n-2}\,, \tag{2.54}$$

and if either

$$\alpha r_{n-1} \leqq (2\pi - \alpha)\, r_{n-2}\,, \tag{2.55}$$

or

$$\frac{2\pi k+\alpha}{2\pi-\alpha}\, r_{n-1} \leqq r_n \leqq \frac{2\pi(k+1)-\alpha}{\alpha}\, r_{n-2} \tag{2.56}$$

for some positive integer k, then

$$\mu(S(R, \alpha, n)) \leqq \alpha/r_{n-1}\,.$$

Proof. Let

$$c \in S(R, \alpha, n)\,, \tag{2.57}$$

and let $\beta = \arg(c)$. We have to prove that

$$|\beta| \leqq \alpha/r_{n-1}\,. \tag{2.58}$$

By (2.54) we have

$$|\beta| \leqq \alpha/r_{n-2}\,. \tag{2.59}$$

If (2.55) holds then by (2.59) we obtain

$$|\beta r_{n-1}| \leqq (2\pi - \alpha)\, |\beta r_{n-2}|/\alpha \leqq 2\pi - \alpha\,. \tag{2.60}$$

Since $\beta r_{n-1} = \arg(c)$ (up to addition or subtraction of 2π), it now follows from (2.57) and (2.60) that $|\beta r_{n-1}| < \alpha$ and (2.58) follows.

Suppose that (2.56) holds. By (2.57) we deduce that either

$$|\beta r_{n-1}| < \alpha\,, \tag{2.61}$$

or

$$|\beta r_{n-1}| > 2\pi - \alpha\,. \tag{2.62}$$

If (2.61) holds then (2.58) follows. To complete the proof we show that (2.62) leads to a contradiction. Observe that (2.56), (2.59) and (2.62) yield

$$2\pi k + \alpha \leqq (2\pi - \alpha)\, r_n/r_{n-1} < |\beta r_n| \leqq \frac{2\pi(k+1) - \alpha}{\alpha}\, |\beta r_{n-2}| \leqq 2\pi(k+1) - \alpha\,,$$

which means that $c^{r_n} \notin W(\alpha)$, in contradiction to (2.57). □

Propositions 2.39 and 2.53 yield the following theorem

Theorem 2.63. *Let $R = (r_1, r_2, \ldots)$ be a sequence of positive integers, let n be a positive integer, $n > 2$, and let $0 < \alpha \leqq \pi$. Then*

$$\alpha/r_n < \mu(S(R, \alpha, n)) \leqq \alpha/r_{n-1}\,, \tag{2.64}$$

if and only if

$$\mu(S(R, \alpha, n)) \leqq \alpha/r_{n-2}\,, \tag{2.65}$$

$$\alpha r_n > (2\pi - \alpha)\, r_{n-1}\,, \tag{2.66}$$

and either

$$\alpha r_{n-1} \leqq (2\pi - \alpha)\, r_{n-2} \tag{2.67}$$

or

$$\frac{2\pi k + \alpha}{2\pi - \alpha}\, r_{n-1} \leqq r_n \leqq \frac{2\pi(k+1) - \alpha}{\alpha}\, r_{n-2}\,, \tag{2.68}$$

for some positive integer k.

Proof. Obviously, (2.64) implies (2.65). The implication (2.64) ⇒ (2.67) or (2.68) is proved in Proposition 2.39. In view of Proposition 2.22, the two inequalities in (2.64) imply (2.66). Conversely, by Proposition 2.53, (2.65) and (2.67) or (2.68) imply the right hand inequality in (2.64). The left hand inequality in (2.64) follows from (2.66) by Proposition 2.14. □

In the case of closed wedges we have the following similar theorems. The proofs are essentially the same as the proofs of the corresponding theorems for open wedges.

Theorem 2.69. *Let $R = (r_1, r_2, \ldots)$ be a sequence of positive integers and let $0 \leqq \alpha \leqq \pi$. Then*

$$W[\alpha/r_1] = S[R, \alpha, 1]$$

if and only if $r_1 = 1$.

Theorem 2.70. *Let $R = (r_1, r_2, \ldots)$ be a sequence of positive integers, let n be a positive integer, $n > 1$, and let $0 \leqq \alpha \leqq \pi$. Then*

$$S[R, \alpha, n] = W[\alpha/r_n]$$

if and only if

$$\mu(S[R, \alpha, n]) \leqq \alpha/r_{n-1}$$

and

$$\alpha r_n < (2\pi - \alpha)\, r_{n-1}\,.$$

Theorem 2.71. *Let $R = (r_1, r_2, \ldots)$ be a sequence of positive integers, let n be a positive integer, $n > 2$, and let $0 < \alpha \leqq \pi$. Then*

$$\alpha/r_n < \mu(S[R, \alpha, n]) \leqq \alpha/r_{n-1},$$

if and only if

$$\mu(S[R, \alpha, n]) \leqq \alpha/r_{n-2},$$
$$\alpha r_n \geqq (2\pi - \alpha)\, r_{n-1},$$

and either

$$\alpha r_{n-1} < (2\pi - \alpha)\, r_{n-2}$$

or

$$\frac{2\pi k + \alpha}{2\pi - \alpha}\, r_{n-1} < r_n < \frac{2\pi(k+1) - \alpha}{\alpha}\, r_{n-2}$$

for some positive integer k.

For $\alpha = 0$ we also have the following immediate theorem.

Theorem 2.72. *Let $R = (r_1, r_2, \ldots)$ be a sequence of positive integers and let n be a positive integer. Then*

$$S[R, 0, n] = W[0]$$

if and only if the greatest common divisor of $r_1, r_2, \ldots, r_n$ is 1.

Proof. Notice that, for a positive integer m, the set $Q_m[0]$ consists of all positive multiples of the m-th roots of unity. Our assertion now follows from Observation 2.10. □

3. FORCING AND SEMIFORCING SEQUENCES

Let $\mathbb{R}_+[\mathbb{R}_+^0]$ be the set of all positive [nonnegative] numbers.

Definition 3.1. Let $T \subseteq \mathbb{C}$, let R be a (finite or infinite) sequence of positive integers, and let $0 \leqq \alpha \leqq \pi$.

(i) The sequence R is called a (T, α)-*forcing sequence* if

$$T \cap S[R, \alpha, |R|] \subseteq \mathbb{R}_+ .$$

(ii) The sequence R is called a (T, α)-*semiforcing sequence* if

$$T \cap S(R, \alpha, |R|) \subseteq \mathbb{R}_+ .$$

(iii) The parameters T and α in definitions (i) and (ii) are optional, where the defaults are $\mathbb{C}$ and $\pi/2$ respectively. Thus for example:

$$\alpha\text{-forcing sequence} = (\mathbb{C}, \alpha)\text{-forcing sequence},$$
$$T\text{-forcing sequence} = (T, \pi/2)\text{-forcing sequence},$$
$$\text{forcing sequence} = (\mathbb{C}, \pi/2)\text{-forcing sequence},$$

and similarly for semiforcing sequences.

Remark 3.2. (i) Clearly, every sequence is (T, α)-(semi)forcing if $T \subseteq \mathbb{R}_+^0$.

(ii) If $T \nsubseteq \mathbb{R}_+^0$ then there exists no (T, π)-forcing sequence.

(iii) Let n be an odd positive integer. If T contains a number c which is a nonreal n-th root of some positive number, and if $\alpha \geqq \pi(n-1)/n$ $[\alpha > \pi(n-1)/n]$ then there exists no (T, α)-forcing $[(T, \alpha)$-semiforcing$]$ sequence, since all the powers of c are in $W[\alpha]$ $[W(\alpha)]$. In particular there is no α-forcing [α-semiforcing] for $\alpha \geqq 2\pi/3 [\alpha > 2\pi/3]$. However, it is easy to verify that the sequence $(1, 2, \ldots)$ is α-forcing [α-semiforcing] whenever $\alpha < 2\pi/3$ $[\alpha \leqq 2\pi/3]$.

(iv) Let $T \nsubseteq \mathbb{R}_+^0$. If T contains no nonreal odd root of a positive number then there exist (T, α)-(semi)forcing sequences whenever $0 \leqq \alpha < \pi$. An example for such a sequence is $(1, 2, \ldots)$: Let $c \in T$. If $\arg(c)$ is an irational multiple of π then the observation follows from Kronecker's theorem, e.g. [2, p. 375, Theorem 4.38]. Otherwise, c is a root fo a negative number and so some power of c is negative.

(v) Let $T \subseteq \mathbb{C}$ and let $0 \leqq \beta < \alpha \leqq \pi$. Observe that every (T, α)-semiforcing sequence is (T, β)-forcing.

Notation 3.3. Let $T \subseteq \mathbb{C}$, $T \nsubseteq \mathbb{R}_+^0$. We denote:

$$\nu(T) = \inf\{|\arg(c)| : c \in T \setminus \mathbb{R}_+^0\} .$$

The following assertion follows from the results of the previous section.

Theorem 3.4. *Let $0 < \alpha \leqq \pi$ and let $R = (r_1, r_2, \ldots)$ be a sequence of positive integers with*

$$r_1 = 1 , \tag{3.5}$$

and such that for every m, $m = 3, 4, \ldots$, either

$$\alpha r_{m-1} \leqq (2\pi - \alpha)\, r_{m-2} , \tag{3.6}$$

or

$$\frac{2\pi k + \alpha}{2\pi - \alpha} r_{m-1} \leqq r_m \leqq \frac{2\pi(k+1) - \alpha}{\alpha} r_{m-2} \tag{3.7}$$

for some positive integer k (which depends on m). Then R is an α-semiforcing sequence.

Furthermore, let $T \subseteq \mathbb{C}, T \nsubseteq \mathbb{R}_+^0$, be such that $\nu(T) > 0$, and suppose that t is the (smallest) positive integer such that

$$r_t\, \nu(T) \geqq \alpha . \tag{3.8}$$

Then the sequence $(r_1, r_2, \ldots, r_{t+1})$ is (T, α)-semiforcing.

Proof. By Theorem 2.13 and Proposition 2.53, it follows from (3.5) and (3.6) or (3.7) that

$$\mu(S(R, \alpha, m) \leqq \alpha/r_{m-1} , \quad m = 2, 3, \ldots . \tag{3.9}$$

Clearly, it follows from (3.9) that $S(R, \alpha, \infty) \subseteq \mathbb{R}_+$, which proves that R is α-semiforcing.

Now let $T \subseteq \mathbb{C}$, $T \nsubseteq \mathbb{R}_+^0$, be such that $v(T) > 0$. By (3.8) and (3.9) we have $S(R, \alpha, t+1) \subseteq W(v(T))$, and by the definition of $v(T)$ we obtain $T \cap S(R, \alpha, t+1) \subseteq \subseteq \mathbb{R}_+$, which proves the second part of our theorem. □

We remark that for the secpnd part of Theorem 3.4 it is enough to assume that (3.6) or (3.7) are satisfied for $m = 3, 4, \ldots, t+1$. A similar remark holds for Theorem 3.11 below.

An application of Theorem 3.4 is demonstrated in Example 4.9.

We remark that condition (3.5) cannot be omitted from Theorem 3.4, as demonstrated by Examples 4.1 and 4.7. Another example will be given later in this section.

If in Theorem 3.4 we have (3.6) for every m, $m = 3, 4, \ldots$, then we can strengthen the second part of the theorem.

Theorem 3.10. *Let $T \subseteq \mathbb{C}$, $T \nsubseteq \mathbb{R}_+^0$, be such that $v(T) > 0$, let $0 < \alpha \leqq \pi$ and let $R = (r_1, r_2, \ldots, r_t)$ be a finite sequence of positive integers such that $r_t\, v(T) \geqq \alpha$. Suppose that $r_1 = 1$ and that*

$$\alpha r_{m-1} \leqq (2\pi - \alpha)\, r_{m-2}\,, \quad m = 3, 4, \ldots, t+1\,.$$

Then R is a (T, α)-semiforcing sequence.

Proof. The proof is essentially the same as for the previous theorem using Corollary 2.26 instead of Proposition 2.53. Here we obtain the equality

$$S(R, \alpha, m) = W(\alpha / r_m)\,, \quad m = 1, 2, \ldots,$$

which is stronger than (3.9) and which yields the better result. □

Example 4.10 shows applications of Theorem 3.10.

For forcing sequences one can similarly prove the following.

Theorem 3.11. *Let $0 \leqq \alpha \leqq \pi$ and let $R = (r_1, r_2, \ldots)$ be a sequence of positive integers with $r_1 = 1$ and such that for every m, $m = 3, 4, \ldots$, either*

$$\alpha r_{m-1} < (2\pi - \alpha)\, r_{m-2}\,, \tag{3.12}$$

or

$$\frac{2\pi k + \alpha}{2\pi - \alpha}\, r_{m-1} < r_m < \frac{2\pi(k+1) - \alpha}{\alpha}\, r_{m-2} \tag{3.13}$$

for some positive integer k (which depends on m). Then R is an α-forcing sequence.

Furthermore, let $T \subseteq \mathbb{C}$, $T \nsubseteq \mathbb{R}_+^0$, be such that $v(T) > 0$, and suppose that t is the (smallest) positive integer such that $r_t\, v(T) > \alpha$. Then the sequence $(r_1, r_2, \ldots, r_{t+1})$ is (T, α)-forcing.

Theorem 3.14. *Let $T \subseteq \mathbb{C}$, $T \nsubseteq \mathbb{R}_+^0$, be such that $v(T) > 0$, let $0 \leqq \alpha \leqq \pi$ and let $R = (r_1, r_2, \ldots, r_t)$ be a finite sequence of positive integers such that $r_t\, v(T) > \alpha$. Suppose that $r_1 = 1$ and that*

$$\alpha r_{m-1} < (2\pi - \alpha)\, r_{m-2}\,, \quad m = 3, 4, \ldots, t+1\,.$$

Then R is a (T, α)-forcing sequence.

The conditions in Theorem 3.11 cannot be weakened by replacing an arbitrary strict inequality in (3.12) by equality, as demonstrated by Example 4.8.

Let $0 \leqq \alpha \leqq \pi$ and let $T \subseteq \mathbb{C}$, $T \nsubseteq \mathbb{R}^0_+$. If $\nu(T) > 0$ then the theorems above show that there are finite (T, α)-(semi)forcing sequences. On the other hand, if $\nu(T) = 0$ then it easy to prove that every (T, α)-(semi)forcing sequence is infinite.

The following is a corollary to Theorems 3.4 and 3.11.

Corollary 3.15. *Let p be a positive integer, $p > 1$. The sequence $R = (1, p, p^2, \ldots)$ is α-forcing [α-semiforcing] if and only if $\alpha < 2\pi/(p + 1)$ [$\alpha \leqq 2\pi/(p + 1)$].*

Proof. Let c be a nonzero complex number such that $\arg(c) = 2\pi/(p + 1)$, and let m be a nonnegative integer. Since $p + 1$ divides $p^m - 1$ [$p^m + 1$] when m is even [odd], it follows that

$$\arg\left(c^{(p^m)}\right) = \begin{cases} 2\pi/(p + 1)\,, & m \text{ even}\,, \\ -2\pi/(p + 1)\,, & m \text{ odd}\,. \end{cases}$$

Thus, if $\alpha \geqq 2\pi/(p + 1)$ [$\alpha > 2\pi/(p + 1)$] then $c \in S[R, \alpha, \infty]$ [$c \in S(R, \alpha, \infty)$], and so R is not an α-forcing [α-semiforcing] sequence.

Conversely, notice that if $\alpha < 2\pi/(p + 1)$ [$\alpha \leqq 2\pi/(p + 1)$] then R satisfies (3.12) [(3.6)] for every m, $m = 3, 4, \ldots$, and hence, by Theorem 3.11 [3.4], R is α-forcing [α-semiforcing]. □

Another interesting Corollary to Theorem 3.11 is:

Corollary 3.16. *Let p be a positive integer, $p \geqq 3$, and let the sequence $R = (r_1, r_2, \ldots)$ be defined by $r_1 = 1$, $r_2 = p$, and $r_m = pr_{m-1} + 1$, $m = 3, 4, \ldots$. Then R is a $2\pi/(p + 1)$-forcing sequence.*

Proof. Since $p \geqq 3$ we have

$$(3.17) \qquad p^2 = pr_2 < r_3 = p^2 + 1 < (p^2 + p - 1)\, r_1 = p^2 + p - 1$$

and

$$(3.18) \qquad pr_{m-1} < r_m = pr_{m-1} + 1 = p^2 r_{m-2} + p + 1 < (p^2 + p - 1) r_{m-2}\,, \quad m = 4, 5, \ldots.$$

Observe that inequalities (3.17) and (3.18) are exactly (3.13) for $\alpha = 2\pi/(p + 1)$, choosing $k = p - 1$. Therefore, by Theorem 3.11, R is a $2\pi/(p + 1)$-forcing sequence.

In view of Corollary 3.15, Corollary 3.16 is somewhat surprising. The i-th element and the ratio between the i-th and the $(i - 1)$-th elements of the sequence in Corollary 3.16 are greater than the corresponding quantities in Corollary 3.15 for $i > 2$ (the first two elements are identical in both sequences). But still, the sequence in Corollary 3.15 is only $2\pi/(p + 1)$-semiforcing while the one in Corollary 3.16 is $2\pi/(p + 1)$-forcing.

We now observe that the fact that in theorem 3.4 [3.11] we have either (3.6) (3.12)] or (3.7) [(3.13)] for *every* m may limit the selection of the k's possible in (3.7) [(3.13)].

Observation 3.19. (i) Assume that (3.7) is satisfied for $m = n$ and $m = n + 1$, namely

$$(3.20) \qquad \frac{2\pi k + \alpha}{2\pi - \alpha} r_{n-1} \leqq r_n \leqq \frac{2\pi(k+1) - \alpha}{\alpha} r_{n-2},$$

and

$$(3.21) \qquad \frac{2\pi k' + \alpha}{2\pi - \alpha} r_n \leqq r_{n+1} \leqq \frac{2\pi(k'+1) - \alpha}{\alpha} r_{n-1},$$

where k and k' are positive integers.

It follows from (3.20) and (3.21) that

$$(3.22) \qquad \frac{2\pi k + \alpha}{2\pi - \alpha} r_{n-1} \leqq r_n \leqq \frac{2\pi - \alpha}{\alpha} f(k') r_{n-1},$$

where

$$f(x) = \frac{2\pi(x+1) - \alpha}{2\pi x + \alpha}.$$

Inequalities (3.22) imply that

$$(3.23) \qquad 2\pi k + \alpha \leqq \frac{(2\pi - \alpha)^2}{\alpha} f(k').$$

Since $\alpha \leqq \pi$, the function $f(x)$ is a monotonic decreasing function for $x > 0$ and hence, (3.23) yields

$$(3.24) \qquad k \leqq \frac{(2\pi - \alpha)^2}{2\pi\alpha} f(1) - \frac{\alpha}{2\pi} = \frac{(2\pi - \alpha)^2 (4\pi - \alpha)}{2\pi\alpha(2\pi + \alpha)} - \frac{\alpha}{2\pi}.$$

For example, for $\alpha = \pi/2$ we obtain from (3.24) that $k \leqq 2.9$, so the possible values for k in (3.20) are 1 and 2. For $\alpha = 2\pi/3$ we obtain $k \leqq 4/3$, so the only possible value for k in (3.20) is 1.

(ii) Assume that (3.7) is satisfied for $m = n$ while (3.6) is satisfied for $m = n + 1$, namely, we have (3.20) and

$$(3.25) \qquad \alpha r_n \leqq (2\pi - \alpha) r_{n-1}.$$

It now follows from (3.20) and (3.25) that

$$\frac{2\pi k + \alpha}{2\pi - \alpha} r_{n-1} \leqq r_n \leqq \frac{2\pi - \alpha}{\alpha} r_{n-1},$$

which implies that

$$(3.26) \qquad k \leqq \frac{(2\pi - \alpha)^2}{2\pi\alpha} - \frac{\alpha}{2\pi}.$$

Since $\alpha \leqq \pi$, we have $(4\pi - \alpha)/(2\pi + \alpha) \geqq 1$, and so (3.26) yields (3.24). Thus, in this case too we have (3.24).

(iii) It follows from (i) and (ii) above that if we have either (3.6) or (3.7) for every m, $m = 3, 4, \ldots$, then (3.24) provides an upper bound for the k in (3.7).

(iv) If one replaces (3.6) and (3.7) in the discussion above by (3.12) and (3.13) respectively, then one can obtain the upper bound (3.24) for k with the weak inequality replaced by strict inequality.

Another sufficient condition for (semi)forcing sequences is the following

Proposition 3.27. *Let $0 \leqq \alpha \leqq \pi$ and let $R = (r_1, r_2, \ldots)$ be an α-forcing [α-semiforcing] sequence. Let p be a positive integer and let R^* be a (T, α)-forcing [(T, α)-semiforcing] sequence, where T is the set of p-th roots of unity. Then the sequence (obtained by reordering) $(pr_1, pr_2, \ldots) \cup R^*$ is an α-forcing [α-semiforcing] sequence.*

Proof. We give the proof for forcing sequences. Denote by pR the sequence $(pr_1, pr_2, \ldots)$, and by R' the sequence $pR \cup R^*$. Clearly, $S[R', \alpha, |R'|] = S[pR, \alpha, |R|] \cap S[R^*, \alpha, |R^*|]$. Let $c \in S[pR, \alpha, |R|]$. Since R is an α-forcing sequence, it follows that $c^p > 0$. Without loss of generality we may assume that $|c| = 1$, and hence $c \in T$. Since R^* is a (T, α)-forcing sequence, it now follows that if $c \in S[R^*, \alpha, |R^*|]$ then $c > 0$. □

As mentioned in Remark 3.2. (iv), there exists no $2\pi/3$-forcing sequence. However, for $\alpha < 2\pi/3$ we obtain the following, using Proposition 3.27.

Corollary 3.28. *Let p be a positive integer and let $0 \leqq \alpha < 2\pi/3$ [$0 \leqq \alpha \leqq 2\pi/3$]. Then the infinite sequence $(p, p + 1, p + 2, \ldots)$ is an α-forcing [α-semiforcing] sequence.*

Proof. Observe that by Theorem 3.11, the sequence $R = (1, 2, \ldots)$ is α-forcing [α-semiforcing]. Thus, we may assume that $p > 1$. Let c be a nonpositive p-th root of unity. Since $0 \leqq \alpha < 2\pi/3$ [$0 \leqq \alpha \leqq 2\pi/3$], some k-th power of c, $1 \leqq k \leqq p - 1$, is outside $W[\alpha]$ [$W(\alpha)$]. Hence, since $c^p = 1$, the sequence $R^* = (p + 1, p + 2, \ldots \ldots, 2p - 1)$ is a (T, α)-forcing [(T, α)-semiforcing] sequence, where T is the set of p-th roots of unity. The result follows from Proposition 3.19. □

We remark that Corollary 3.28 can be proved directly using arguments similar to those in parts (ii) and (iv) of Remark 3.2.

We conclude the section with a discussion of an interesting relation between (semi)forcing sequences and continued fractions. For a real number d we shall use the notation $\|d\|$ for the distance between d and the nearest integer, viz. $\|d\| = \min\{|d - k|: k \text{ an integer}\}$. Let $(2, 3, 5, 8, 13, \ldots)$ be the sequence of Fibonacci numbers (omitting 1) obtained by setting $r_1 = 2$, $r_2 = 3$ and recursively $r_k = r_{k-2} + r_{k-1}$, $k = 3, 4, \ldots$. Then it is well known that the r_k are the denominators of the regular continued fraction for $d = (\sqrt{5} - 1)/2$, [5, p. 125]. Hence, by a theorem due to Lagrange, see [6, p. 37, Formula (9) and Satz 2.10]or [5. p. 74, Theorem 3.8], we have

$$\|r_k d\| < 1/r_k, \quad k = 1, 2, \ldots, \tag{3.29}$$

and hence

$$\|r_k d\| < 1/4 \tag{3.30}$$

for $k = 3, 4, \ldots$. Since $d = .6180\ldots$, we also have (3.30) for $k = 1, 2$ and hence

$$(3.31) \qquad c^{r_k} \in W(\pi/2), \quad k = 1, 2, \ldots,$$

where $c = e^{2\pi i d}$. Thus, the Fibonacci sequence satisfies condition (3.6) for $m = 3, 4, \ldots$ where $\alpha = \pi/2$, but is not forcing (nor even semiforcing). (This is another example showing that condition (3.5) cannot be omitted from Theorem 3.4). More generally, by [6, p. 3 and p. 33, Satz 2.6], [5, p. 21, Theorem 1.3], a sequence $(r_1, r_2, \ldots)$ of positive integers is a sequence of regular denominators of a continued fraction expansion of some irrational number d if and only if

$$(3.32) \qquad (r_{k+2} - r_k)/r_{k+1} \text{ is a positive integer}, \quad k = 0, 1, 2, \ldots,$$

where $r_0 = 1$. Therefore, no sequence satisfying (3.32) and $r_1 \geqq 4$ can be a semiforcing sequence for then, by (3.29), we have (3.30) for $k = 1, 2, \ldots$, and hence (3.31) holds for $c = e^{2\pi i d}$.

4. EXAMPLES

This section contains examples to illustrate the results of the previous sections. The relevant assertions are referred to in each example.

Example 4.1. (for Proposition 2.14, Proposition 2.39 and Theorem 3.4). Let $R = (2, 4, 6, \ldots)$ and let α be any number $0 < \alpha < \pi$. Then for n sufficiently large we have $\alpha r_n \leqq (2\pi - \alpha)/r_{n-1}$. However,

$$-1 \in S(R, \alpha, n) \setminus W(\alpha, r_n), \quad n = 1, 2, \ldots.$$

Example 4.2. (for Corollary 2.26). Let $R = (2, 3, 6, 18, \ldots)$ where $r_m = 2(3^{m-2})$, $m = 3, 4, \ldots$, let $\alpha = \pi/2$ and let $n = 3$. Observe that

$$S(R, \alpha, n-1) = W(\pi/6) \cup W(3\pi/4, 5\pi/6) \cup W(-5\pi/6, -3\pi/4) \nsubseteq W(\alpha/r_{n-1}),$$

but

$$S(R, \alpha, n) = W(\pi/12) = W(\alpha/r_n).$$

Furthermore, applying Corollary 2.26 repeatedly we obtain

$$S(R, \alpha, m) = W(\alpha/r_m)$$

for all m, $m \geqq 3$. Thus, R is a semiforcing sequence.

Example 4.3. (for Proposition 2.39). Let $\alpha = \pi/2$, let $n = 3$, and let $r_1 = 1$, $r_2 = 3$, $r_3 = 12$. Observe that $2\pi - \alpha = 3\alpha$. By Theorem 2.13 and Corollary 2.26 we have

$$S(R, \alpha, 3) \subseteq S(R, \alpha, 2) = W(\alpha/r_2).$$

Further, since $r_3 > r_2$ it follows from Proposition 2.14 that

$$W(\alpha/r_3) \subsetneqq S(R, \alpha, 3).$$

Hence, (2.40) is satisfied. Clearly, (2.41) holds. However, there exists no integer k such that

$$4k + 1 = \frac{2\pi k + \alpha}{2\pi - \alpha} r_2 \leqq r_3 = 12 \leqq \frac{2\pi(k+1) - \alpha}{\alpha} r_1 = 4k + 3 .$$

Example 4.4. (for Proposition 2.39). Let $\alpha = \pi/2$, let $n = 4$, and let $r_1 = 1$, $r_2 = 3$, $r_3 = 10$, $r_4 = 31$. By Theorem 2.13 and Proposition 2.22 we have

$$\mu(S(R, \alpha, 4)) \leqq \mu(S(R, \alpha, 2)) = \alpha/r_2 . \tag{4.5}$$

Also observe that (2.41) does not hold. Nevertheless, (2.42) holds for $k = 2$:

$$30 = \frac{2\pi k + \alpha}{2\pi - \alpha} r_3 \leqq r_4 = 31 \leqq \frac{2\pi(k+1) - \alpha}{\alpha} r_2 = 33 . \tag{4.6}$$

By Theorem 2.53 it follows from (4.5) and (4.6) that (2.40) holds.

Example 4.7. (for Proposition 2.39 and Theorem 3.4). Let $\alpha = \pi/2$ and let $R = (r_1, r_2, \ldots)$ be defined by

$$r_1 = 4 ,$$
$$r_m = 3r_{m-1} + 2 , \quad m = 2, 3, \ldots .$$

Since $2\pi - \alpha = 3\alpha$ it follows that for every n, $n > 2$, (2.41) does not hold. However, (2.42) does hold for every n, $n > 2$, choosing $k = 2$:

$$3r_{n-1} = \frac{2\pi k + \alpha}{2\pi - \alpha} r_{n-1} \leqq r_n = 9r_{n-2} + 8 \leqq \frac{2\pi(k+1) - \alpha}{\alpha} r_{n-2} = 11r_{n-2} .$$

Since r_k is always even, it follows that

$$-1 \in S(R, \alpha, n) , \quad n = 1, 2, \ldots ,$$

so (2.40) does not hold in this case.

Example 4.8. (for Theorem 3.4 and 3.11). Let $T = \mathbb{C}$ and let $\alpha = \pi/2$. The sequence R is defined by $r_1 = 1$, $r_2 = 3$, and $r_m = 2^{m-1}$, $m = 3, 4, \ldots$. Notice that (3.13) is satisfied for $m = 4, 5, \ldots$ and that equality holds for $m = 3$. This sequence is not forcing since $i \in S[R, \pi/2, \infty]$. However, the sequence R is semiforcing by Theorem 3.4.

Example 4.9. (for Theorem 3.4). Let $\alpha = 2\pi/3$ and let $R = (1, 2, 4, 9, 19, \ldots)$, where $r_m = 2r_{m-1} + 1$, $m = 4, 5, \ldots$. It is easy to verify that inequalities (3.7) are satisfied for $m = 3, 4, \ldots$, choosing $k = 1$. Hence, by Theorem 3.4, R is a $2\pi/3$-semiforcing sequence. Also, by Remark 3.2. (v), R is a β-forcing sequence whenever $0 \leqq \beta < 2\pi/3$.

Example 4.10. (for Theorem 3.10). Let T be the set of all 150-th roots of unity. By Theorem 3.10, the finite sequence $(1, 5, 25)$ is $(T, \pi/3)$-semiforcing and the sequence $(1, 3, 9, 27, 81)$ is T-semiforcing.

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