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MATRICES WITH SIGN SYMMETRIC DIAGONAL SHIFTS OR SCALAR SHIFTS*

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Abstract. We generalize the concepts of sign symmetry and weak sign symmetry by defining k-sign symmetric matrices. For a positive integer k, we show that all diagonal shifts of an irreducible matrix are k-sign symmetric if and only if the matrix is diagonally similar to a Hermitian matrix. A similar result holds for scalar shifts, but requires an additional condition in the case $k = 1$. Extensions are given to reducible matrices.

Key words. matrix, Hermitian, sign symmetric, diagonal shift, scalar shift, diagonal similarity, graph, cordless circuit

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1. Introduction. A square complex matrix is said to be sign symmetric (weakly sign symmetric) if it has nonnegative products of symmetrically located minors (almost principal minors) (for detailed definition see Definition 2.11).

Weakly sign symmetric matrices were studied first by Gantmacher and Krein [8, p. 111] and by Koteljanskii [13]. That is why these matrices are also called GKK-matrices, e.g., Fan [5]. One reason for the interest in these classes of matrices is that they contain the important classes of the Hermitian matrices, the totally nonnegative matrices and the M-matrices. Another reason is the strong linkage between weak sign symmetry and the Fischer–Hadamard determinantal inequalities. This connection is studied in Gantmacher and Krein [8], Koteljanskii [12], Carlson [1], Green [9] and Hershkowitz and Berman [10].

A sufficient condition for positivity of the principal minors of a weakly sign symmetric matrix in terms of leading principal minors is given by Koteljanskii [13].

Relations between weakly sign symmetric matrices and ω-matrices are discussed in Engel and Schneider [4] and in Hershkowitz and Berman [11].

Sign symmetry and weak sign symmetry are also related to stability. It was proved by Carlson [2] that sign symmetric matrices whose principal minors are positive are stable, i.e., their spectra lie in the open right half plane. The same result is conjectured to hold for weakly sign symmetric matrices too.

In this paper we generalize the concepts of sign symmetry and weakly sign symmetry. We define k-sign symmetric matrices, where k is a nonnegative integer (see Definition 2.11). In view of our definition an $n \times n$ sign symmetric matrix is a k-sign symmetric matrix whenever $k \geqq (n-1)/2$. The 1-sign symmetric matrices are those weakly sign symmetric matrices whose principal minors are real. Since reality of principal minors is assumed in all the results on weakly sign symmetric matrices quoted above, one may as well consider those as assertions on 1-sign symmetric matrices.

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After giving graph theoretic preliminaries in § 3, we characterize in § 4 the matrices all of whose diagonal shifts are k-sign symmetric, that is matrices A such that $A + D$ is k-sign symmetric for every real diagonal matrix D. Given a positive k, we show that an irreducible matrix satisfies this condition if and only if it is diagonally similar to a Hermitian matrix. Thus, a matrix satisfies the above shift condition for some positive k if and only if it satisfies the condition for every positive k.

For $k \geqq 2$, we prove in § 5 a similar result for a matrix A all of whose scalar shifts $A + tI$, where t is real, are k-sign symmetric. If $k = 1$ then we need an additional graph theoretic hypothesis, namely the reversibility of the chordless directed circuits of even length in the directed graph of A.

The extensions of our results to reducible matrices follow from a theorem in § 6 that a matrix A is k-sign symmetric if and only if every diagonal block in the Frobenius normal form of A is k-sign symmetric.

2. Definitions and notation.

Notation 2.1. We denote

- $|\alpha|$: the cardinality of a set α.
- $\mathbb{R}$: the field of real numbers.
- $\mathbb{C}$: the field of complex numbers.
- $[x]$: the maximal integer which is less than or equal to the real number x.

Notation 2.2. For a positive integer n we denote

- $\langle n \rangle$: the set $\{1, 2, \cdots, n\}$.
- $F^{n,n}$: the set of all $n \times n$ matrices over a field F.

Notation 2.3. For a (nondirected, simple) graph Γ we denote

- $V(\Gamma)$: the vertex set of Γ.
- $E(\Gamma)$: the edge set of Γ.
- $[i, j]$: an edge between i and j, $i, j \in V(\Gamma)$. Observe that $[i, j] = [j, i]$.

DEFINITION 2.4. Let Γ be a graph. A sequence of edges in Γ which leads from i to j, $[i, p_1], [p_1, p_2], \cdots, [p_{m-1}, p_m], [p_m, j]$, is called a *path* in Γ between i and j and is denoted by $[i, p_1, p_2, \cdots, p_m, j]$. A path $[i_1, \cdots, i_l]$ in Γ is said to be a *closed path* if $i_l = i_1$. A closed path $[i_1, \cdots, i_k, i_1]$ is said to be a *circuit* if $i_1, \cdots, i_k$ are distinct. A circuit is said to be of length k, or a *k-circuit*, if it consists of k edges.

Notation 2.5. For a (simple) directed graph (or digraph) Δ we denote

- $V(\Delta)$: the vertex set of Δ.
- $E(\Delta)$: the arc set of Δ.
- (i, j): an arc from i to j, $i, j, \in V(\Delta)$. Observe that $(i, j) = (j, i)$ if and only if $i = j$.

DEFINITION 2.6. Let Δ be a digraph. A sequence of arcs in Δ from i to j, (i, p_1), $(p_1, p_2), \cdots, (p_{m-1}, p_m), (p_{m,}j)$, is called a *directed path* in Δ from i to j and is denoted by $(i, p_1, p_2, \cdots, p_m, j)$. A directed path $(i_1, \cdots, i_l)$ in Δ is said to be a *closed directed path* if $i_l = i_1$. A closed directed path $(i_1, \cdots, i_k, i_1)$ is said to be a *directed circuit* (or *dicircuit*) if $i_1, \cdots, i_k$ are distinct. A dicircuit is said to be of length k, or a *k-dicircuit*, if it consists of k arcs.

DEFINITION 2.7. A digraph Δ is said to be *strongly connected* if either $|V(\Delta)| = 1$ or for every $i, j \in V(\Delta)$ there exists a directed path in Δ from i to j.

DEFINITION 2.8. A dicircuit $(i_1, \cdots, i_k, i_1)$, $k \geqq 3$, in a digraph Δ is said to have a *chord* if $E(\Delta)$ contains an arc (i_l, i_t) where

$$t \notin \begin{cases} \{l-1, l+1\}, & 1 < l < k, \\ \{2, k\}, & l = 1, \\ \{k-1, 1\}, & l = k. \end{cases}$$

A dicircuit of length greater than 2 in Δ is said to be *chordless* if it has no chord.

DEFINITION 2.9. (i) A directed path $\alpha = (i_1, \cdots, i_k)$ in a digraph Δ is said to be *reversible* in Δ if $(i_k, \cdots, i_1)$ is also a directed path in Δ. In this case we denote the directed path $(i_k, \cdots, i_1)$ by α^*.

(ii) A digraph Δ is said to be *reversible* or *symmetric* if every directed path in Δ is reversible. Observe that Δ is reversible if and only if

$$(i,j) \in E(\Delta) \Rightarrow (j,i) \in E(\Delta).$$

Notation 2.10. Let A be an $n \times n$ matrix and let $\alpha, \beta \subseteq \langle n \rangle$, $\alpha, \beta \neq \phi$. We denote

$A[\alpha|\beta]$: the submatrix of A whose rows are indexed by α and whose columns are indexed by β in their natural orders.

$A[\alpha] = A[\alpha|\alpha]$,

$A(\alpha|\beta) = A[\langle n \rangle \backslash \alpha | \langle n \rangle \backslash \beta]$,

$A(\alpha) = A(\alpha|\alpha)$.

DEFINITION 2.11. (i) Let $A \in \mathbf{C}^{n,n}$ and let $\alpha, \beta \subseteq \langle n \rangle$, $|\alpha| = |\beta| > 0$. The submatrix $A[\alpha|\beta]$ of A is said to have *dispersion* k whenever $k = |\alpha| - |\alpha \cap \beta|$ (see also [12]). Submatrices with dispersion 1 are called *almost principal* submatrices.

(ii) Let k be a nonnegative integer. A square matrix A is said to be *k-sign symmetric* if it satisfies

$$\det A[\alpha|\beta] \det A[\beta|\alpha] \geqq 0 \tag{2.12}$$

for all submatrices $A[\alpha|\beta]$ of A with dispersion less than or equal to k. The set of all k-sign symmetric matrices in $\mathbf{C}^{n,n}$ is denoted by $SS^k_{\langle n \rangle}$.

(iii) A square matrix is called *sign symmetric* if (2.12) holds for all square submatrices $A[\alpha|\beta]$ of A (see also [13]). The set of all sign symmetric matrices in $\mathbf{C}^{n,n}$ is denoted by $SS_{\langle n \rangle}$.

(iv) A square matrix is called *weakly sign symmetric* if (2.12) holds for all submatrices $A[\alpha|\beta]$ of A with dispersion exactly 1 (see also [13]). The set of all weakly sign symmetric matrices in $\mathbf{C}^{n,n}$ is denoted by $WSS_{\langle n \rangle}$.

Remark 2.13. (i) Observe that for nonnegative integers k and m, the inequality $m > k$ implies $SS^m_{\langle n \rangle} \subseteq SS^k_{\langle n \rangle}$.

(ii) Let $\alpha, \beta \subseteq \langle n \rangle$, $|\alpha| = |\beta| > 0$, and let $k = |\alpha| - |\alpha \cap \beta|$. Since

$$|\alpha| + |\beta| - |\alpha \cap \beta| = |\alpha \cup \beta| \leqq n$$

and since $k \leqq |\alpha|$ it follows that $k \leqq n/2$. Thus, the dispersion of a square submatrix of an $n \times n$ matrix cannot exceed $n/2$. It now follows that for a nonnegative integer m, $m \geqq (n-1)/2$ we have $SS^m_{\langle n \rangle} = SS_{\langle n \rangle}$.

(iii) Since submatrices of a given matrix have dispersion 0 if and only if they are principal submatrices, it follows from Definition 2.11(ii) that the 0-sign symmetric matrices are just the matrices all of whose principal minors are real. Also, a k-sign symmetric matrix has real principal minors for every positive integer k.

(iv) Observe that $SS^1_{\langle n \rangle}$ is the set of those matrices in $WSS_{\langle n \rangle}$ that have real principal minors.

DEFINITION 2.14. Let A be an $n \times n$ matrix. The *graph* $\Gamma(A)$ of A and the *digraph* $\Delta(A)$ of A are defined by

$$V(\Gamma(A)) = V(\Delta(A)) = \langle n \rangle,$$

$$E(\Gamma(A)) = \{[i,j], i,j \in \langle n \rangle : a_{ij} \neq 0 \text{ or } a_{ji} \neq 0\},$$

$$E(\Delta(A)) = \{(i,j), i,j \in \langle n \rangle : a_{ij} \neq 0\}.$$

DEFINITION 2.15. Let A be an $n \times n$ matrix and let $\alpha = (i_1, \cdots, i_k)$ be a directed path in $\Delta(A)$. The corresponding *path product* is defined to be

$$\Pi_\alpha(A) = \prod_{j=1}^{k-1} a_{i_j i_{j+1}}.$$

DEFINITION 2.16. An $n \times n$ matrix A is said to be *combinatorially symmetric* if $\Delta(A)$ is reversible.

DEFINITION 2.17. Let $A.B. \in \mathbf{C}^{n,n}$. The matrices A and B are said to be *diagonally similar* if there exists a nonsingular diagonal matrix D such that

$$B = D^{-1}AD.$$

The matrices A and B are said to be *permutationally similar* if there exists a permutation matrix P such that

$$B = P^T AP.$$

DEFINITION 2.18. (i) A square matrix A is said to be in *Frobenius normal form* if A may be written in the block form

$$A = \begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1k} \\ 0 & A_{22} & & \vdots \\ \vdots & & \ddots & \vdots \\ 0 & \cdots & 0 & A_{kk} \end{bmatrix}$$

where A_{ii} is an irreducible square matrix, $i = 1, \cdots, k$.

(ii) Let $A, B \in \mathbf{C}^{n,n}$. The matrix B is said to be a *Frobenius normal form of A* if B is in Frobenius normal form and if A and B are permutationally similar.

Remark 2.19. Observe that by Definition 2.18 the Frobenius normal form of a square matrix A is unique up to permutation similarity, and so Frobenius normal forms of A have the same diagonal blocks up to permutation similarity. Also, since, as is well known, a square matrix is irreducible if and only if its digraph is strongly connected, it follows that the diagonal blocks of the Frobenius normal form of A are the principal submatrices of A that correspond to the maximal strongly connected subgraphs (components) of $\Delta(A)$.

DEFINITION 2.20. Let $A \in \mathbf{C}^{n,n}$. A *diagonal shift* of A is a matrix $A + D$ where D is a real diagonal $n \times n$ matrix. A *scalar shift* of A is a matrix $A + tI$ where t is a real number.

3. Reversible digraphs.

PROPOSITION 3.1. *Let Δ be a digraph. Then every dicircuit in Δ is reversible if and only if every chordless dicircuit in Δ is reversible.*

Proof. The "only if" part is trivial. Conversely we prove our assertion by induction on the length of the dicircuits. Clearly, all dicircuits in Δ of length 1 and 2 are reversible. Also all 3-dicircuits are chordless and hence reversible. Assume that all dicircuits in Δ of length less than n $(n > 3)$ are reversible, and let $\alpha = (i_1, \cdots, i_n, i_1)$ be an n-dicircuit in Δ. If α is chordless then it is reversible by the conditions of the proposition. Assume that α is not chordless. Without loss of generality we may assume that $(i_1, i_l) \in E(\Delta)$ where $l \neq 1, 2, n$. Observe that $\beta = (i_1, i_l, i_{l+1}, \cdots, i_n, i_1)$ is a dicircuit in Δ of length less than n and therefore, by the inductive assumption, β is reversible. Thus we have

$$(i_k, i_{k-1}) \in E(\Delta), \qquad k = l+1, \cdots, n, \tag{3.2}$$

$$(i_1, i_n) \in E(\Delta), \tag{3.3}$$

and

$$(i_l, i_1) \in E(\Delta). \tag{3.4}$$

By (3.4), $\gamma = (i_1, \cdots, i_l, i_1)$ is also a dicircuit in Δ. Since the length of γ is less than n, it follows from the inductive assumption that γ is reversible. Hence we have

$$(i_k, i_{k-1}) \in E(\Delta), \qquad k = 2, \cdots, l. \tag{3.5}$$

It now follows from (3.2), (3.3) and (3.5) that the dicircuit α is reversible. □

COROLLARY 3.6. *Let Δ be a strongly connected digraph. Then Δ is reversible if and only if every chordless dicircuit in Δ is reversible.*

Proof. The "only if" part is again trivial. Conversely, since Δ is strongly connected it follows that every arc (i, j) of Δ lies on some dicircuit α in Δ. By Proposition 3.1 the dicircuit α is reversible and hence $(j, i) \in E(\Delta)$. □

COROLLARY 3.7. *Let $A \in \mathbb{C}^{n,n}$. Then every diagonal block in the Frobenius normal form of A is combinatorially symmetric if and only if every chordless dicircuit in $\Delta(A)$ is reversible.*

Proof. Our claim follows immediately from Corollary 3.6 and Remark 2.19. □

4. Irreducible matrices with sign symmetric diagonal shifts.

LEMMA 4.1. *Let $A \in \mathbb{C}^{n,n}$ be diagonally similar to a Hermitian matrix. Then $A \in SS^k_{\langle n \rangle}$ for every nonnegative integer k.*

Proof. Let D be a diagonal matrix and B be a Hermitian matrix such that

$$A = D^{-1}BD.$$

For all $\alpha, \beta \subseteq \langle n \rangle$, $|\alpha| = |\beta| > 0$ we have

$$\begin{aligned}
&\det A[\alpha|\beta] \det A[\beta|\alpha] \\
&\quad = \det D[\alpha] \det B[\alpha|\beta] \det D^{-1}[\beta] \det D[\beta] \det B[\beta|\alpha] \det D^{-1}[\alpha] \\
&\quad = \det B[\alpha|\beta] \det B[\beta|\alpha] = \det B[\alpha|\beta] \overline{\det B[\alpha|\beta]} \geqq 0.
\end{aligned}$$
□

LEMMA 4.2. *Let $a, b \in \mathbb{C}$ and let*

$$p(t) = (t+a)(t+b).$$

(i) *If $p(t_1), p(t_2) \in \mathbb{R}$ for two distinct real numbers t_1 and t_2 then either $a = \bar{b}$ or $a, b \in \mathbb{R}$.*

(ii) *If $b > a$ then $p(t) < 0$ for all t, $-b < t < -a$.*

Proof. (i) If $p(t_1), p(t_2) \in \mathbb{R}$ for two distinct real numbers t_1 and t_2 then necessarily $a + b$, $ab \in \mathbb{R}$. Therefore, $p(t)$ is a polynomial with real coefficients. Since the roots of $p(t)$ are $-a$ and $-b$ our claim follows.

(ii) Immediate, since for $-b < t < -a$ we have $t + a < 0$ and $t + b > 0$. □

COROLLARY 4.3. *Let $a, b \in \mathbb{C}$. If $(t + a)(t + b) \geqq 0$ for all $t \in \mathbb{R}$ then $a = \bar{b}$.*

Proof. By Lemma 4.2(i) we have either $a = \bar{b}$ or $a, b \in \mathbb{R}$. In the latter case, by Lemma 4.2(ii) we have $a = b$. Hence, in each case, $a = \bar{b}$. □

In the following results we discuss k-sign symmetric matrices, $k \geq 1$. As observed in Remark 2.13(iii), a matrix A is 1-sign symmetric if and only if A is weakly sign symmetric matrix with real principal minors. Note that a matrix $A \in \mathbb{C}^{n,n}$ may have nonreal principal minors even if all its diagonal shifts are in $WSS_{\langle n\rangle}$. This assertion can easily be verified for $n = 1, 2$. However it holds for higher orders too as demonstrated by the following irreducible 3×3 matrix

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 1 & i & 1 \\ 0 & 1 & 0 \end{bmatrix}.$$

The following theorem relates weakly sign symmetric matrices to 1-sign symmetric matrices.

THEOREM 4.4. *Let $A \in \mathbb{C}^{n,n}$ be a weakly sign symmetric matrix and suppose that all the principal submatrices of A of order less than or equal to $n-2$ are nonsingular. Then A has real principal minors if and only if the diagonal entries of A are real.*

Proof. The "only if" direction is obvious. Conversely, assume that A is a weakly sign symmetric matrix with real diagonal entries and nonsingular principal submatrices of order less than or equal to $n-2$. We prove that the principal minors A are real by induction on the order of A. The claim is clear for matrices in $WSS_{\langle 1\rangle}$ and $WSS_{\langle 2\rangle}$. Assume it holds for weakly sign symmetric matrices of order less than n, $n \geq 3$, and let $A \in WSS_{\langle n\rangle}$. Since every principal submatrix of a weakly sign symmetric matrix is also weakly sign symmetric, it follows from the inductive assumption that all principal minors of A of order less than n are real. Thus, all we have to prove is that $\det A$ is real.

Let $\alpha_1 = \langle n\rangle \backslash \{n\}$ and $\alpha_2 = \langle n\rangle \backslash \{n-1\}$, and define a 2×2 matrix B by

$$b_{ij} = \det A[\alpha_i|\alpha_j], \qquad i,j = 1,2.$$

Since $A \in WSS_{\langle n\rangle}$ it follows that $B \in WSS_{\langle 2\rangle}$. Furthermore, b_{11} and b_{22} are principal minors of A of order $n-1$, and hence b_{11} and b_{22} are real by the inductive assumption. Therefore, the determinant of B is real. By Sylvester's identity, e.g., [7, Vol. I, p. 33], we have

$$\det B = \det A[\langle n-2\rangle] \det A. \tag{4.5}$$

Since $\det A[\langle n-2\rangle] \neq 0$ and by the inductive assumption $\det A[\langle n-2\rangle]$ is real, it now follows from (4.5) that $\det A$ is real. $\square$

The assumption of nonsingularity of the principal minors of A cannot be dropped from Theorem 4.4 as demonstrated by the matrix

$$A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & i \\ 2 & -i & 0 \end{bmatrix}.$$

It is easy to verify that $A \in WSS_{\langle 3\rangle}$. However, $\det A = i$.

LEMMA 4.6. *Let $A \in \mathbb{C}^{n,n}$, $n \geq 3$. Assume that α is an n-dicircuit in $\Delta(A)$ and that $\Gamma(A)$ consists of a single circuit. If $A + D \in SS^1_{\langle n\rangle}$ for all real diagonal matrices D then*

$$\Pi_\alpha(A) = \overline{\Pi_{\alpha^*}(A)}.$$

Proof. Without loss of generality assume that $\alpha = (1, \cdots, n, 1)$. Notice that $\Gamma(A)$ consists of the single circuit $[1, \cdots, n, 1]$. Since $A + D \in SS^1_{\langle n\rangle}$ for all real diagonal matrices D, it follows that

$$y(D) = [\det(A+D)(1|2)][\overline{\det(A+D)(2|1)}] \tag{4.7}$$
$$= [a_{21}z(D) + (-1)^{n-2}a_{n1}p][\overline{a_{12}z(D) + (-1)^{n-2}a_{1n}q}] \geqq 0$$

where

$$z(D) = \det(A+D)(1,2),$$

$$p = \prod_{j=2}^{n-1} a_{j,j+1},$$

and

$$q = \prod_{j=2}^{n-1} a_{j+1,j}.$$

Since α is a dicircuit in $\Delta(A)$, it follows that

$$a_{12}, p, a_{n1} \neq 0.$$

Since, as observed in Remark 2.13(iii), A has real principal minors, it follows that $z(D)$ attains every real value for suitable choices of D. Therefore, if $a_{21} = 0$ then for an appropriate choice of D we have $y(D) < 0$, which contradicts (4.7). Thus we have $a_{21} \neq 0$. Similarly we show that $a_{j+1,j} \neq 0, j = 1, \cdots, n-1$ and $a_{1n} \neq 0$. Since $A \in SS^{1}_{\langle n \rangle}$ we now have $a_{12}a_{21} > 0$. Dividing (4.7) by $a_{12}a_{21}$, we obtain

$$[z(D) + (-1)^{n-2}a_{n1}p/a_{21}][z(D) + \overline{(-1)^{n-2}a_{1n}q/a_{12}}] \geqq 0. \tag{4.8}$$

Since $z(D)$ attains every real value, it follows from (4.8) and Corollary 4.3 that

$$a_{n1}p/a_{21} = \overline{a_{1n}q/a_{12}}. \tag{4.9}$$

Notice that since $a_{12}a_{21} > 0$ we have $a_{12}a_{21} = \overline{a_{12}a_{21}}$. Hence, by multiplying the left and the right sides of (4.9) by $a_{12}a_{21}$ and $\overline{a_{12}a_{21}}$, respectively, we obtain

$$\Pi_\alpha(A) = \overline{\Pi_{\alpha^*}(A)}. \qquad \square$$

LEMMA 4.10. *Let $A \in \mathbb{C}^{n,n}$ have real diagonal entries and assume that*

$$a_{ij}a_{ji} \in \mathbb{R} \quad \text{for all } i,j \in \langle n \rangle, \quad i \neq j. \tag{4.11}$$

If the equality

$$\Pi_\alpha(A) = \overline{\Pi_{\alpha^*}(A)} \tag{4.12}$$

holds for all chordless dicircuits α in $\Delta(A)$ then it holds for all dicircuits α in $\Delta(A)$.

Proof. Since A has real diagonal entries, it follows that (4.12) holds for 1-dicircuits. Also it follows from (4.11) that (4.12) holds for 2-dicircuits. Assume by induction that (4.12) holds for dicircuits of length less than m, $m \geqq 3$, and let $\alpha = (i_1, \cdots, i_m, i_1)$ be an m-dicircuit in $\Delta(A)$. If α is chordless then by the lemma's conditions (4.12) holds. If α is not chordless, then necessarily $m > 3$, and without loss of generality we may assume that $(i_1, i_l) \in E(\Delta(A))$, where $l \neq 1, 2, m$. Since $\Delta = \Delta(A[i_1, \cdots, i_m])$ is strongly connected and since by the conditions of the lemma every chordless dicircuit in Δ is reversible, it follows from Corollary 3.6 that Δ is reversible. Hence, $(i_l, i_1) \in E(\Delta(A))$ and hence $\beta = (i_1, i_l, i_{l+1}, \cdots, i_m, i_1)$ and $\gamma = (i_1, \cdots, i_l, i_1)$ are dicircuits in $\Delta(A)$ with length less than m. By the inductive assumption we have

$$\Pi_\beta(A) = \overline{\Pi_{\beta^*}(A)}, \tag{4.13}$$

and

$$\Pi_\gamma(A) = \overline{\Pi_{\gamma^*}(A)}. \tag{4.14}$$

Observe that

$$\prod_\alpha(A) = \prod_\beta(A)\prod_\gamma(A)/a_{i_1 i_l}a_{i_l i_1}, \tag{4.15}$$

and

$$\prod_{\alpha^*}(A) = \prod_{\beta^*}(A)\prod_{\gamma^*}(A)/a_{i_1 i_l}a_{i_l i_1}. \tag{4.16}$$

Since we have (4.11), it now follows from (4.13), (4.14), (4.15) and (4.16) that

$$\prod_\alpha(A) = \overline{\prod_{\alpha^*}(A)}. \qquad \square$$

We remark that Lemma 4.10 may be generalized. One can similarly prove the same conclusion under the assumptions that (4.11) holds and that (4.12) holds for all the dicircuits in an integral basis for the flow space of $\Delta(A)$, see [14].

THEOREM 4.17. *Let $A \in \mathbf{C}^{n,n}$ be an irreducible matrix and let k be a positive integer. Then the following are equivalent.*

(i) *$A + D \in SS^k_{\langle n\rangle}$ for all real diagonal matrices D.*

(ii) *The matrix A is diagonally similar to a Hermitian matrix.*

Proof. (i) $\Rightarrow$ (ii). In view of Remark 2.13(i) it is enough to show this implication for $k = 1$. Assume that $A + D \in SS^1_{\langle n\rangle}$ for all real diagonal matrices D. Observe that since A is irreducible, the digraph $\Delta(A)$ is strongly connected. Let $\alpha = (i_1, \cdots, i_m, i_1)$ be a chordless m-dicircuit in $\Delta(A)$. By Definition 2.8 we have $m \geqq 3$. Let $B = A[i_1, \cdots, i_m]$. Notice that $\Gamma(B)$ consists of a single circuit. By Lemma 4.6 we have

$$\prod_\alpha(A) = \overline{\prod_{\alpha^*}(A)}. \tag{4.18}$$

It now follows from (4.18) that the chordless dicircuit α is reversible. By Corollary 3.6 the strongly connected digraph $\Delta(A)$ is reversible. Thus, since A is in $SS^1_{\langle n\rangle}$ it follows that

$$a_{ij} \neq 0 \Rightarrow a_{ij}a_{ji} > 0 \quad \text{for all } i,j \in \langle n\rangle. \tag{4.19}$$

Furthermore, by Lemma 4.10 we have

$$\prod_\alpha(A) = \overline{\prod_{\alpha^*}(A)}, \tag{4.20}$$

for every dicircuit in $\Delta(A)$. Therefore, by Corollary 4.20 of [3] it follows from (4.19) and (4.20) that A is diagonally similar to a Hermitian matrix.

(ii) $\Rightarrow$ (i). Assume that A satisfies (ii). Since $A + D$ is diagonally similar to a Hermitian matrix for all real diagonal matrices D, it follows by Lemma 4.1 that $A + D$ is in $SS^k_{\langle n\rangle}$. $\square$

5. Irreducible matrices with sign symmetric scalar shifts. In this section we discuss matrices A all of whose scalar shifts are k-sign symmetric. Although the condition here is weaker than $A + D \in SS^k_{\langle n\rangle}$ for all real diagonal matrices D, the results are similar to those of the previous section.

The following lemma is well known and may be found in [8, p. 79, Remark 6^0].

LEMMA 5.1. *Let $A \in \mathbf{C}^{n,n}$ be a tridiagonal matrix such that*

$$a_{ii} \in \mathbf{R}, \qquad i = 1, \cdots, n,$$

and

$$a_{i,i+1}a_{i+1,i} > 0, \qquad i = 1, \cdots, n-1.$$

Then A has distinct real eigenvalues. Furthermore, if $\lambda_1 < \cdots < \lambda_n$ are the eigenvalues of A and $\mu_1 < \cdots < \mu_{n-1}$ are the eigenvalues of $A(n)$ or of $A(1)$, then

$$\lambda_1 < \mu_1 < \lambda_2 < \cdots < \mu_{n-1} < \lambda_n.$$

An easy well-known consequence of Lemma 5.1 is:

LEMMA 5.2. *Let $A \in \mathbb{C}^{n,n}$ be a tridiagonal matrix such that*

$$a_{ii} \in \mathbb{R}, \qquad i = 1, \cdots, n,$$

and

$$a_{i,i+1} a_{i+1,i} \geqq 0, \qquad i = 1, \cdots, n-1.$$

Then A has real eigenvalues.

LEMMA 5.3. *Let $A \in \mathbb{C}^{n,n}$, $n \geqq 3$, and suppose that if n is even then A is combinatorially symmetric. Assume that α is an n-dicircuit in $\Delta(A)$ and that $\Gamma(A)$ consists of a single circuit. If $A + tI \in SS^1_{\langle n \rangle}$ for all $t \in \mathbb{R}$ then*

$$\Pi_\alpha(A) = \overline{\Pi_{\alpha^*}(A)}.$$

Proof. Without loss of generality assume that $\alpha = (1, \cdots, n, 1)$. Notice that $\Gamma(A)$ consists of the single circuit $[1, \cdots, n, 1]$. Since $A + tI \in SS^1_{\langle n \rangle}$ for all $t \in \mathbb{R}$, it follows that

$$\begin{aligned} (5.4) \qquad f(t) &= \det(A+tI)(1|2) \det(A+tI)(2|1) \\ &= [a_{21} g(t) + (-1)^{n-2} a_{n1} p][a_{12} g(t) + (-1)^{n-2} a_{1n} q] \geqq 0 \quad \text{for all } t \in \mathbb{R}, \end{aligned}$$

where

$$g(t) = \det(A+tI)(1,2),$$

$$p = \prod_{j=2}^{n-1} a_{j,j+1},$$

and

$$q = \prod_{j=2}^{n-1} a_{j+1,j}.$$

Observe that since A has real principal minors, if n is odd then $g(t)$ attains every real value and our proof follows as the proof of Lemma 4.6 where (5.4), $f(t)$ and $g(t)$ replace (4.7), $y(D)$ and $z(D)$, respectively. If n is even then, since $a_{12} \neq 0$ and since A is combinatorially symmetric 1-sign symmetric matrix, it follows that $a_{12}a_{21} > 0$. Dividing (5.4) by $a_{12}a_{21}$, we obtain

$$(5.5) \qquad [g(t) + a][g(t) + b] \geqq 0 \quad \text{for all } t \in \mathbb{R}$$

where

$$a = \frac{a_{n1} p}{a_{21}}$$

and

$$b = \frac{a_{1n} q}{a_{12}}.$$

Since $g(t)$ attains infinitely many real values, it follows from Lemma 4.2(i) that either

$$(5.6) \qquad a = \bar{b},$$

or

$$(5.7) \qquad a, b \in \mathbb{R}.$$

If (5.6) holds, then we have (4.9) and we complete our proof as we do for Lemma 4.6.

If (5.6) does not hold, then we have (5.7) where $a \neq b$. Without loss of generality we may assume that

$$b > a. \tag{5.8}$$

Observe tnat if $g(t)$ attains the value x then $g(t)$ attains every value which is greater than x. Thus, it follows from (5.5), (5.8) and Lemma 4.2(ii) that

$$g(t) \geqq -a > -b \quad \text{for all } t \in \mathbb{R}. \tag{5.9}$$

Given that $A + tI \in SS^{1}_{\langle n \rangle}$ for all $t \in \mathbb{R}$ we have

$$\begin{aligned} h(t) &= [\det (A + tI)(1|n)][\det (A + tI)(n|1)] \\ &= [a_{n1}r(t) + a_{21}q][a_{1n}r(t) + a_{12}p] \geqq 0 \quad \text{for all } t \in \mathbb{R} \end{aligned} \tag{5.10}$$

where

$$r(t) = \det (A + tI)(1, n).$$

Dividing (5.10) by the positive number $a_{1n}a_{n1}$, we obtain

$$[r(t) + c][r(t) + d] \geqq 0 \quad \text{for all } t \in \mathbb{R} \tag{5.11}$$

where

$$c = \frac{a_{21}q}{a_{n1}},$$

and

$$d = \frac{a_{12}p}{a_{1n}}.$$

Observe that (5.8) implies that

$$c > d. \tag{5.12}$$

As before, by Lemma 4.2(ii) it follows from (5.11) and (5.12) that

$$r(t) \geqq -d > -c \quad \text{for all } t \in \mathbb{R}. \tag{5.13}$$

Observe that $(A + tI)(1, 2)$ and $(A + tI)(1, n)$ are tridiagonal matrices which satisfy the conditions of Lemma 5.1. Hence by Lemma 5.1 their eigenvalues are simple. Thus, for appropriate choices of t, the determinants of these matrices, which are $g(t)$ and $r(t)$ respectively, attain negative values. Hence, it follows from (5.9) and (5.13) that

$$a, b, c, d > 0. \tag{5.14}$$

Let $\alpha_1 = \langle n \rangle \backslash \{1, n\}$ and $\alpha_2 = \langle n \rangle \backslash \{1, 2\}$, and define a 2×2 matrix B by

$$b_{ij} = \det (A + tI)[\alpha_i|\alpha_j], \qquad i, j = 1, 2.$$

Observe that

$$b_{11} = r(t), \quad b_{22} = g(t), \quad b_{12} = p, \quad b_{21} = q. \tag{5.15}$$

By Sylvester's identity we have

$$\det B = [\det (A + tI)(1, 2, n)][\det (A + tI)(1)] \quad \text{for all } t \in \mathbb{R}. \tag{5.16}$$

By Lemma 5.1 let λ be the minimal eigenvalue of $A(1, 2, n)$, and choose $t_0 = -\lambda$. Thus

$$\det (A + t_0 I)(1, 2, n) = 0. \tag{5.17}$$

Furthermore, by Lemma 5.1 we have

$$(5.18) \qquad r(t_0), g(t_0) < 0.$$

By (5.9), (5.13), (5.14) and (5.18) we now obtain

$$(5.19) \qquad r(t_0)g(t_0) < ac = pq.$$

On the other hand, by (5.15), (5.16) and (5.17) we obtain

$$r(t_0)g(t_0) = pq,$$

which is a contradiction to (5.19). Therefore, our assumption that (5.6) does not hold is false, and our proof is completed. □

Lemma 5.3 does not hold for even n when we omit the combinatorial symmetry requirement as demonstrated by the following example.

Example 5.20. Let

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}.$$

Let $\alpha, \beta \subseteq \langle 4 \rangle$, $|\alpha| = |\beta| = |\alpha \cap \beta| + 1$. To see that

$$(5.21) \qquad \det(A + tI)[\alpha|\beta] \det(A + tI)[\beta|\alpha] \geqq 0 \quad \text{for all } t \in \mathbb{R},$$

observe that the left side of (5.21) is equal to zero whenever $|\alpha| \leqq 2$, and is equal to t^2 whenever $|\alpha| = 3$.

We remark that it is possible that a condition which is somewhat weaker than combinatorial symmetry will do in Lemma 5.3.

However, for matrices with k-sign symmetric scalar shifts, $k > 1$, we do not need to state the condition of combinatorial symmetry.

LEMMA 5.22. *Let $A \in \mathbb{C}^{n,n}$, $n \geqq 3$, and let k be a positive integer, $k > 1$. Assume that α is an n-dicircuit in $\Delta(A)$ and that $\Gamma(A)$ consists of a single circuit. If $A + tI \in SS^k_{\langle n \rangle}$ for all $t \in \mathbb{R}$ then α is reversible in $\Delta(A)$.*

Proof. Without loss of generality assume that $\alpha = (1, \cdots, n, 1)$. Thus $\Gamma(A)$ consists of the single circuit $[1, \cdots, n, 1]$. Assume that α is not reversible. Without loss of generality we may assume that $a_{1n} = 0$. In view of Lemma 5.3 it is enough to consider the case where n is even. Hence we may assume that $n \geqq 4$. Recall that

$$(5.23) \qquad A + tI \in SS^k_{\langle n \rangle} \quad \text{for all } t \in \mathbb{R}$$

yields that A has real principal minors. Also, it follows from (5.23), that

$$h(t) = \det(A + tI)(1|n) \det(A + tI)(n|1)$$
$$= [\tilde{p} + a_{n1} r(t)]\tilde{q} \geqq 0 \quad \text{for all } t \in \mathbb{R}$$

where

$$r(t) = \det(A + tI)(1, n),$$

$$\tilde{p} = \prod_{j=2}^{n} a_{j,j-1},$$

and

$$\tilde{q} = \prod_{j=2}^{n} a_{j-1,j}.$$

Observe that $h(t)$ is a polynomial in t of degree $n-2$. Since it is nonnegative for all $t \in \mathbb{R}$ it follows that the leading coefficient $a_{n1}\tilde{q}$ must be nonnegative. In fact, since α is a dicircuit in $\Delta(A)$ we have

$$a_{n1}\tilde{q} > 0. \tag{5.24}$$

We distinguish between two cases:

Case 1. $n = 4$. By (5.23) we have

$$\begin{aligned} f(t) &= \det(A+tI)(1|2)\det(A+tI)(2|1) \\ &= [a_{21}g(t) + a_{41}a_{23}a_{34}]a_{12}g(t) \\ &= [a_{21}a_{12}g(t) + a_{41}\tilde{q}]g(t) \geqq 0 \quad \text{for all } t \in \mathbb{R} \end{aligned} \tag{5.25}$$

where

$$g(t) = \det(A+tI)(1,2).$$

If $a_{43} \neq 0$ then, since $a_{34} \neq 0$, $g(t)$ attains also negative values (for example for $t = -a_{33}$). Thus, in view of (5.24) we can choose t_0 such that $g(t_0) < 0$ and

$$|a_{21}a_{12}g(t_0)| < a_{41}\tilde{q}.$$

But then $f(t_0) < 0$ in contradiction to (5.25). Therefore we must assume that $a_{43} = 0$. Since $k > 1$ we now obtain by (5.23) that

$$\det(A+tI)(1,3|2,4)\det(A+tI)(2,4|1,3) = -a_{41}a_{23}a_{12}a_{34} \geqq 0,$$

which is a contradiction to (5.24).

Case 2. $n > 4$. By (5.23) we have

$$\begin{aligned} \tilde{f}(t) &= \det(A+tI)(1,n-1|2,n)\det(A+tI)(2,n|1,n-1) \\ &= [a_{21}a_{n,n-1}\tilde{g}(t) - a_{n1}\tilde{q}/a_{12}a_{n-1,n}][a_{12}a_{n-1,n}\tilde{g}(t)] \\ &= [a_{12}a_{21}a_{n-1,n}a_{n,n-1}\tilde{g}(t) - a_{n1}\tilde{q}]\tilde{g}(t) \geqq 0 \quad \text{for all } t \in \mathbb{R} \end{aligned} \tag{5.26}$$

where

$$\tilde{g}(t) = \det(A+tI)(1,2,n-1,n).$$

By Lemma 5.2 $\tilde{g}(t)$ attains every nonnegative value. Thus, in view of (5.24) we can choose t_0 such that $\tilde{g}(t_0) > 0$ and

$$|a_{12}a_{21}a_{n-1,n}a_{n,n-1}\tilde{g}(t_0)| < a_{n1}\tilde{q}.$$

But then $\tilde{f}(t_0) < 0$ in contradiction to (5.26).

In each case we obtain a contradiction, which means that our assumption that α is not reversible is false. □

We now state the theorem for the irreducible case.

THEOREM 5.27. *Let $A \in \mathbb{C}^{n,n}$ be an irreducible matrix and let k be a positive integer, $k \geqq 2$. Then the following are equivalent.*

(i) *$A + tI \in SS^k_{\langle n\rangle}$ for all $t \in \mathbb{R}$.*

(ii) *$A + tI \in SS^1_{\langle n\rangle}$ for all $t \in \mathbb{R}$ and every chordless dicircuit in $\Delta(A)$ is reversible.*

(iii) *$A + tI \in SS^1_{\langle n\rangle}$ for all $t \in \mathbb{R}$ and every chordless dicircuit of even length in $\Delta(A)$ is reversible.*

(iv) *The matrix A is diagonally similar to a Hermitian matrix.*

Proof. (i) ⇒ (ii). Lemma 5.22 yields that every chordless dicircuit in $\Delta(A)$ is reversible. The rest of the implication is trivial.

(ii) ⇒ (iii). Obvious.

(iii) $\Rightarrow$ (iv). The proof follows exactly as the proof of the part (i) $\Rightarrow$ (ii) in Theorem 4.17, where D is replaced by tI, and where Lemma 5.3 is used instead of Lemma 4.6.

(iv) $\Rightarrow$ (i). Since $A + tI$ is diagonally similar to a Hermitian matrix for all $t \in \mathbb{R}$, it follows by Lemma 4.1 that $A + tI \in SS^k_{\langle n \rangle}$. □

6. Reducible matrices with sign symmetric shifts.

THEOREM 6.1. *Let $A \in \mathbb{C}^{n,n}$ have the block form*

$$A = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix}$$

where A_{11} and A_{22} are square, and let k be a nonnegative integer. Then A is k-sign symmetric if and only if A_{11} and A_{22} are k-sign symmetric.

Proof. Clearly, if A is k-sign symmetric then so are A_{11} and A_{22}. Conversely, assume that A_{11} and A_{22} are k-sign symmetric and let $\alpha, \beta \subseteq \langle n \rangle$ be such that $q = |\alpha| = |\beta| > 0$ and

$$q - |\alpha \cap \beta| \leqq k. \tag{6.2}$$

We shall show that

$$\det A[\alpha|\beta] \det A[\beta|\alpha] \geqq 0. \tag{6.3}$$

Let m be the order of A_{11}. Denote by

$$\alpha' = \alpha \cap \langle m \rangle, \quad \alpha'' = \alpha \backslash \alpha', \quad \beta' = \beta \cap \langle m \rangle, \quad \beta'' = \beta \backslash \beta'.$$

Observe that

$$|\alpha'| + |\alpha''| = |\beta'| + |\beta''| = q, \tag{6.4}$$

and hence

$$|\alpha'| + |\beta''| + |\beta'| + |\alpha''| = 2q. \tag{6.5}$$

In view of (6.5) we need to consider only the following two cases.

Case 1. $|\alpha'| + |\beta''| > q$ or $|\alpha''| + |\beta'| > q$. Assume that

$$|\alpha'| + |\beta''| > q. \tag{6.6}$$

By (6.4) we have $|\alpha'|, |\beta''| > 0$. Since $A[\beta''|\alpha'] = 0$ it follows from (6.6) by the easy direction of the Frobenius–König theorem [6] that $A[\beta|\alpha]$ is singular and hence

$$\det A[\alpha|\beta] \det A[\beta|\alpha] = 0.$$

Case 2. $|\alpha'| + |\beta''| = |\alpha''| + |\beta'| = q$. If $|\alpha'| = q$ [$|\beta''| = q$] then $|\alpha''| = 0$ [$|\beta'| = 0$] and hence $|\beta'| = q$ [$|\alpha''| = q$]. In this case $A[\alpha|\beta]$ and $A[\beta|\alpha]$ are submatrices of A_{11} [A_{22}] and (6.3) follows. If $|\alpha'|, |\beta''| < q$ then observe that $A[\alpha|\beta]$ and $A[\beta|\alpha]$ are reducible. Furthermore, we have

$$\det A[\alpha|\beta] = \det A_{11}[\alpha'|\beta'] \det A_{22}[\alpha''|\beta''] \tag{6.7}$$

and

$$\det A[\beta|\alpha] = \det A_{11}[\beta'|\alpha'] \det A_{22}[\beta''|\alpha'']. \tag{6.8}$$

By (6.2), the sets α' and α'' contain at most k indices which are not in β' and β'', respectively. Hence, since A_{11} and A_{22} are k-sign symmetric, inequality (6.3) follows from (6.7) and (6.8). □

In view of Remark 2.13(ii) we obtain the following immediate corollary to Theorem 6.1.

COROLLARY 6.9. *Let $A \in \mathbb{C}^{n,n}$ have the block form*

$$A = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix}$$

where A_{11} and A_{22} are square. Then A is sign symmetric if and only if A_{11} and A_{22} are sign symmetric.

We remark that the "only if" part of Theorem 6.1 holds trivially also when we replace "k-sign symmetric" by "weakly sign symmetric." On the other hand, weak sign symmetry of A_{11} and A_{22} does not imply in general the weak sign symmetry of A for matrices with nonreal principal minors, as demonstrated by the following example.

Example 6.10. Let

$$A = \begin{bmatrix} i & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

where A_{11} is a 1×1 matrix. Obviously, the matrices A_{11} and A_{22} are weakly sign symmetric. However, the matrix A is not in $WSS_{\langle 3 \rangle}$ since

$$\det A(3|2) \det A(2|3) = -1.$$

Since the class $SS^k_{\langle n \rangle}$ is invariant under permutation similarity, the following is a corollary to Theorem 6.1.

COROLLARY 6.11. *Let k be a nonnegative integer. A square matrix A is k-sign symmetric if and only if every diagonal block in the Frobenius normal form of A is k-sign symmetric.*

Let A be a square matrix. Observe that every dicircuit in $\Delta(A)$ is a dicircuit in $\Delta(B)$ where B is some diagonal block in the Frobenius normal form of A. Thus, the following theorem for the general case follows directly from Theorems 4.17 and 5.27 and Corollary 6.11.

THEOREM 6.12. *Let $A \in \mathbb{C}^{n,n}$ and let k and m be positive integers, $m \geq 2$. Then the following are equivalent.*

(i) *$A + D \in SS^k_{\langle n \rangle}$ for all real diagonal matrices D.*
(ii) *$A + tI \in SS^m_{\langle n \rangle}$ for all $t \in \mathbb{R}$.*
(iii) *$A + tI \in SS^1_{\langle n \rangle}$ for all $t \in \mathbb{R}$ and every chordless dicircuit in $\Delta(A)$ is reversible.*
(iv) *$A + tI \in SS^1_{\langle n \rangle}$ for all $t \in \mathbb{R}$ and every chordless dicircuit of even length in $\Delta(A)$ is reversible.*
(v) *Every diagonal block in the Frobenius normal form of A is diagonally similar to a Hermitian matrix.*

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