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Eigenvalue Interlacing for Certain Classes of Matrices with Real Principal Minors

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ABSTRACT

Two common properties of Z-matrices and Hermitian matrices are considered: (1) The eigenvalue interlacing property, i.e., the two smallest real eigenvalues of a matrix are interlaced by the smallest real eigenvalue of any principal submatrix of order one less. (2) The positive GLP property, i.e., if a matrix has a positive sequence of

*The research of these authors was supported by their joint grant No. 85-00153 from the United States—Israel Binational Science Foundation (BSF), Jerusalem, Israel.

†This work was carried out in part while this author was visiting the University of Wisconsin, Madison.

52 Vanderbilt Ave., New York, NY 10017 0024-3795/87/$3.50

generalized leading principal minors, then all the principal minors of the matrix are positive. The relationship between these properties as well as related properties is examined in general.

1. INTRODUCTION

The study of common properties of certain classes of matrices such as Z-matrices, Hermitian matrices, totally nonnegative matrices, and weakly sign symmetric matrices (see Section 2 for definitions) has been the subject of many papers. In this paper we shall concentrate on two common properties that are shared by Z-matrices and Hermitian matrices, and their relation to each other.

Let A be a complex square matrix. The two properties are:

(1) *The eigenvalue interlacing property.* Every principal submatrix of A has a real eigenvalue, and the two smallest real eigenvalues of a matrix are interlaced by the smallest real eigenvalue of every principal submatrix of order one less. (For precise definition see Definition 2.14.)

(2) *The positive GLP (generalized leading principal minors) property.* Either some generalized leading principal minor of A is nonpositive or all principal minors of A are positive. (For precise definition see Definition 6.2.)

The eigenvalue interlacing property for Hermitian matrices follows from the well-known Cauchy interlacing theorem [2] proved in 1829 (at least for symmetric matrices). In 1908 Frobenius [6] proved a result that implies eigenvalue interlacing for Z-matrices, which were introduced by Ostrowski [17] in 1937. The result was stated explicitly by Hall and Porshing [9] in 1968. The positive GLP property for Hermitian matrices is well known, e.g., Gantmacher and Krein [8, p. 40]. (We do not know to whom this result is due.) The corresponding result for Z-matrices is a consequence of a result of Koteljanskiĭ [14], who proved in 1953 that weakly sign symmetric matrices have the positive GLP property. The explicit statement for Z-matrices is a part of a theorem by Fiedler and Pták [4] in 1962. A relation between eigenvalue interlacing and the positive GLP property for Z-matrices was proved by Koteljanskiĭ [13] in 1953.

A research problem by Taussky [19] in 1958, which asked for a unification of the theory of Hermitian positive semidefinite matrices, totally nonnegative matrices, and M-matrices, has motivated a series of papers (for a list see [18]). In 1977, Engel and Schneider [3] introduced the class of ω-matrices, defined by eigenvalue monotonicity (see Definition 2.12), which contains the three classes mentioned in [19]. They posed the question whether ω-matrices have the positive GLP property. This question was answered negatively by Hershkowitz and Berman [10] in 1984. The ω-matrices in general also do not have the eigenvalue interlacing property. However, in a paper by Mehrmann [15] in 1984 it was essentially shown that the matrices in a subclass of the ω-matrices, namely the R-matrices (see Definition 2.16), including the Z-matrices and the Hermitian matrices, have the eigenvalue interlacing property. This property does not hold in general for totally nonnegative matrices.

We now describe the contents of our paper in more detail.

In Section 2 we introduce some of the notation used in the paper and give some definitions.

In Section 3 we introduce the concept of principal multiplicity for an eigenvalue of a given matrix, and define a principal eigenvalue to be an eigenvalue for which the principal multiplicity equals the algebraic multiplicity. We mainly concentrate on properties of such eigenvalues that are to be used in the following sections. Nevertheless, some of the results in this section are of independent interest.

Section 4 discusses the relation between eigenvalue monotonicity and eigenvalue interlacing. In particular we show (Theorem 4.32) that for a given matrix A, if $A+D$ is an ω-matrix for all positive diagonal matrices D, then A has the eigenvalue interlacing property.

The cases of strict eigenvalue monotonicity and strict eigenvalue interlacing are characterized in Section 5. We also discuss the interrelations between the two properties.

In Section 6 we study the relations between the eigenvalue interlacing property and the positive GLP property. We also discuss the stronger semipositive and nonnegative GLP properties, and their interrelations. In particular we show (Theorem 6.11) that a matrix A has the eigenvalue interlacing property if and only if A is an ω-matrix and every principal submatrix of A has the semipositive GLP property. A similar result holds for strict eigenvalue interlacing (Theorem 6.15). The paper is concluded with some open problems.

2. NOTATION AND DEFINITIONS

NOTATION 2.1. We denote

$|\alpha|$ = the cardinality of a set α;
$\mathbb{R}$ = the field of real numbers;
$\mathbb{C}$ = the field of complex numbers.

NOTATION 2.2. For a field F and a positive integer n, we denote

$\langle n\rangle$ = the set $\{1,2,\dots,n\}$;
$F^{n,n}$ = the set of all $n\times n$ matrices over F.

NOTATION 2.3. Let A be an $n\times n$ matrix and let $\alpha,\beta\subseteq\langle n\rangle$, $\alpha,\beta\neq\varnothing$. We denote

$A[\alpha|\beta]$ = the submatrix of A whose rows are indexed by α and whose columns are indexed by β in their natural orders;
$A[\alpha]=A[\alpha|\alpha]$;
$A(\alpha|\beta)=A[\langle n\rangle\setminus\alpha|\langle n\rangle\setminus\beta]$;
$A(\alpha)=A(\alpha|\alpha)$;
$\sigma(A)$ = the spectrum of the matrix A.

NOTATION 2.4. Let A be an $n\times n$ matrix and let $i,j\in\langle n\rangle$. We denote

$A[i|j]=A[\{i\}|\{j\}]$;
$A[i]=A[i|i]$;
$A(i|j)=A(\{i\}|\{j\})$;
$A(i)=A(i|i)$;
E_{ij} = the $n\times n$ matrix all of whose entries are zero except for the one in the (i,j) position, whose value is 1.

DEFINITION 2.5. Let $A\in\mathbb{C}^{n,n}$, and let $\lambda_1\leqslant\lambda_2\leqslant\cdots\leqslant\lambda_k$ be the real eigenvalues of A (repetitions are possible), where $k=0$ if $\sigma(A)\cap\mathbb{R}=\varnothing$. We define the numbers $l(A)$, $s(A)$, and $h(A)$ by

$$l(A)=\begin{cases}\lambda_1, & k>0,\\ \infty, & k=0,\end{cases}$$

$$s(A)=\begin{cases}\lambda_2, & k\geqslant 2,\\ \infty, & k<2,\end{cases}$$

and

$$h(A)=\min_{i\in\langle n\rangle}\{l(A(i))\}.$$

Note that $l(A[\varnothing])=s(A[\varnothing])=\infty$. We also define $\det A[\varnothing]=1$.

OBSERVATION 2.21. *In view of Definitions* 2.19 *and* 2.20, *an eigenvalue* λ *of* A *is a principal eigenvalue of* A *if and only if the following statement holds*: *If the sums of all principal minors of order* k *vanish for* $k \geqslant m$, *then all principal minors of* A *of order* k, $k \geqslant m$, *are zero.*

3. PRINCIPAL EIGENVALUES

In this section we make a few observations concerning principal eigenvalues. We mainly concentrate on properties that are to be used in the sequel.

Let A be a square matrix. It is easy to prove that

$$n(A) \leqslant p(A) \leqslant m(A). \tag{3.1}$$

We remark that if A has the principal submatrix rank property as defined in [11], then

$$n(A) = p(A) \leqslant m(A).$$

OBSERVATION 3.2. *Let* $\lambda \in \sigma(A)$. *If the elementary divisors of* λ *as an eigenvalue of* A *are linear, then* λ *is a principal eigenvalue of* A.

Proof. If the elementary divisors of λ as an eigenvalue of A are linear, then

$$n(A - \lambda I) = m(A - \lambda I).$$

Our claim now follows from (3.1) and Definition 2.20. ∎

COROLLARY 3.3. *If* A *is a Hermitian matrix, then every eigenvalue of* A *is a principal eigenvalue.*

DEFINITION 3.4. A set S of complex numbers is said to be of *single sign* if there exists a ray (half line) from the origin which contains all the elements of S.

PROPOSITION 3.5. *Let* $A \in \mathbb{C}^{n,n}$ *be a singular matrix. If for every* k, $n - m(A) + 1 \leqslant k \leqslant n$, *the set*

$$S_k = \{\det A[\alpha] : \alpha \subseteq \langle n \rangle,\ |\alpha| = k\}$$

is of single sign, then 0 *is a principal eigenvalue of* A.

Proof. Let

$$\det(A-\lambda I)=\sum_{j=0}^{n} a_j(-\lambda)^j$$

be the characteristic polynomial of A. As is well known,

$$a_j=\sum_{s\in S_{n-j}} s, \qquad j=0,\dots,n-1, \tag{3.6}$$

$$a_0=1.$$

Clearly,

$$m(A)=\min\{j:0\leqslant j\leqslant n,\ a_j\neq 0\}. \tag{3.7}$$

Since S_k, $k=n-m(A)+1,\dots,n$, are of single sign, and since by (3.7) $a_0=\cdots=a_{m(A)-1}=0$, it follows from (3.6) that

$$\det A[\alpha]=0 \qquad \forall\alpha\subseteq\langle n\rangle,\ |\alpha|\geqslant n-m(A)+1.$$

By Definition 2.19 we now have $p(A)\geqslant m(A)$, and by (3.1) we thus have $p(A)=m(A)$. ■

COROLLARY 3.8. *Let A be singular P°-matrix. Then 0 is a principal eigenvalue of A.*

Proof. Observe that here $S_1,\dots,S_n$ are all of single sign. ■

COROLLARY 9.9. *Let $A\in\omega_{\langle n\rangle}$. Then $l(A)$ is a principal eigenvalue of A.*

Proof. By [3], $A_{l(A)}$ is a P°-matrix. Our claim now follows from Corollary (3.8). ■

REMARK 3.10. By the well-known Perron-Frobenius theory concerning the spectral properties of nonnegative matrices, it follows from Corollary 3.8 that the spectral radius of a nonnegative matrix A is a principal eigenvalue of A.

PROPOSITION 3.11. *Let $A \in \mathbb{C}^{n,n}$, and let λ be a principal eigenvalue of A of multiplicity m, $m \geqslant 2$. Let k be a positive integer, $k < m$, and let $\alpha \subseteq \langle n \rangle$, $|\alpha| = n - k$.*

Then λ is an eigenvalue of $A[\alpha]$ of multiplicity at least $m - k$.

Proof. By Definitions 2.19 and 2.20 the order of the largest nonzero principal minor of $A - \lambda I$ is $n - m$. Thus the order of the largest nonzero principal minor of $(A - \lambda I)[\alpha]$ is at most $n - m$, and by Definition 2.19 we have

$$p(A - \lambda I) \geqslant m - k,$$

and our assertion follows by (3.1). ∎

We remark that in Proposition 3.11, λ is not necessarily a principal eigenvalue of $A[\alpha]$, as demonstrated by the following example.

EXAMPLE 3.12. Let

$$A = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 2 & 4 \\ -1 & -1 & -2 \end{bmatrix}.$$

Observe that 0 is a principal eigenvalue of A of multiplicity 2, since $n(A) = p(A) = m(A) = 2$. However, 0 is not a principal eigenvalue of $B = A[\{2,3\}]$, since $n(B) = p(B) = 1$, but $m(B) = 2$.

We conclude this section with an elementary observation. This will not be used in the sequel, but is given here for the sake of completeness.

OBSERVATION (3.13). *Let*

$$A = \begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1k} \\ 0 & A_{22} & \cdots & A_{2k} \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & A_{kk} \end{bmatrix},$$

where A_{ii} is square, $i = 1, \ldots, k$, and let $\lambda \in \sigma(A)$. Then λ is a principal eigenvalue of A if and only if for every i, $1 \leqslant i \leqslant k$, either λ is a principal eigenvalue of A_{ii} or $\lambda \notin \sigma(A_{ii})$.

Proof. The assertion follows immediately on observing that

$$p(A-\lambda I)=\sum_{i=1}^{k} p(A-\lambda I)$$

and that

$$m(A-\lambda I)=\sum_{i=1}^{k} m(A-\lambda I). \qquad \blacksquare$$

COROLLARY 3.14. *Every diagonal element of a triangular matrix A is a principal eigenvalue of A.*

4. EIGENVALUE INEQUALITIES

Before we examine the eigenvalue inequalities for various classes of matrices we list a few observations and known relationships between these classes.

Observe that

$$R_{\langle 1\rangle}=\omega_{\langle 1\rangle}=\underline{I}_{\langle 1\rangle}=R^{<}_{\langle 1\rangle}=\omega^{<}_{\langle 1\rangle}=\underline{I}^{<}_{\langle 1\rangle}=\mathbb{R}^{1,1}, \tag{4.1}$$

$$R_{\langle 2\rangle}=\omega_{\langle 2\rangle}=\underline{I}_{\langle 2\rangle}=\left\{A\in\mathbb{C}^{2,2}: a_{11},a_{22}\in\mathbb{R},\ a_{12}a_{21}\geqslant 0\right\}, \tag{4.2}$$

and

$$R^{<}_{\langle 2\rangle}=\omega^{<}_{\langle 2\rangle}=\underline{I}^{<}_{\langle 2\rangle}=\left\{A\in\mathbb{C}^{2,2}: a_{11,22}\in\mathbb{R},\ a_{12}a_{21}>0\right\}. \tag{4.3}$$

It is known that

$$Z_{\langle n\rangle}\subseteq R_{\langle n\rangle}\subseteq \omega_{\langle n\rangle} \tag{4.4}$$

and

$$T_{\langle n\rangle}\subseteq \omega_{\langle n\rangle}, \tag{4.5}$$

but

$$T_{\langle n\rangle}\not\subseteq R_{\langle n\rangle}; \tag{4.6}$$

see [3] and [16].

By Definition 2.14, we have

$$\underline{I}_{\langle n\rangle} \subseteq \omega_{\langle n\rangle}.$$

Also, for $n > 2$ we have

$$\underline{I}_{\langle n\rangle} \subsetneq \omega_{\langle n\rangle}, \tag{4.7}$$

as demonstrated by the following example, taken from [3, p. 174].

EXAMPLE 4.8. Let

$$A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 0 \\ 1 & 1 & 2 \end{bmatrix}.$$

Then $A \in \tau_{\langle 3\rangle} \subseteq \omega_{\langle 3\rangle}$, and $\sigma(A) = \{1, \frac{1}{2}(5 \pm \sqrt{5})\}$. But $s(A) = \frac{1}{2}(5 - \sqrt{5}) < 2 = l(A[\{2,3\}])$. Thus, $A \notin \underline{I}_{\langle 3\rangle}$. By using direct sums, we now obtain (4.7) for all $n > 2$.

LEMMA 4.9. *Let $A \in \omega_{\langle n\rangle}$ and let $t \in \mathbb{R}$. If $t < l(A)$ then* $\det A_t > 0$.

Proof. By [3], all principal minors of A are real. Since $\lim_{t \to -\infty} \det A_t = \infty$ and since $l(A)$ is the least real eigenvalue of A, we have $\det A_t > 0$ whenever $t < l(A)$.

PROPOSITION 4.10. *Let $A \in \omega_{\langle n\rangle}$, $n \geqslant 2$. If $s(A) = l(A)$ then*

$$l_i(A) = l(A) \qquad \forall i \in \langle n\rangle.$$

Proof. Our claim follows from Corollary 3.9 and Proposition 3.11. ■

PROPOSITION 4.11. *Let $A \in \omega_{\langle n\rangle}$, $n \geqslant$. If $l(A) < s(A)$ then*

$$h(A) = \min_{i \in \langle n\rangle} \{l_i(A)\} < s(A).$$

Proof. Assume that

$$h(A) \geqslant s(A). \tag{4.12}$$

Observe that

$$\frac{d}{dt}\det A_t = -\sum_{j=1}^{n}\det A_t(j). \tag{4.13}$$

By Lemma 4.9 it follows from (4.12) and (4.13) that

$$\frac{d}{dt}\det A_t < 0, \qquad l(A) \leqslant t < s(A),$$

in contradiction to Rolle's theorem. Therefore, our assumption (4.12) is false. ■

As a Corollary to Proposition (4.11) we obtain the following characterization of 3×3 ω-matrices.

THEOREM 4.14. *Let $A \in \mathbb{C}^{3,3}$. Then $A \in \omega_{\langle 3\rangle}$ if and only if*

$$a_{ii} \in R, \qquad i \in \langle 3\rangle, \tag{4.15}$$

$$a_{ij}a_{ji} \geqslant 0, \qquad i,j \in \langle 3\rangle, \quad i \neq j, \tag{4.16}$$

and

$$\det[A - h(A)I] \leqslant 0. \tag{4.17}$$

Proof. If (4.15) and (4.16) hold, then by (4.2) every 2×2 principal submatrix of A is an ω-matrix. If, further, (4.17) holds, then $l(A) < \infty$ and by Lemma 4.9 we have $l(A) \leqslant h(A)$, and thus by Definition 2.5 we have

$$l(A) \leqslant l_i(A), \qquad i \in \langle 3\rangle. \tag{4.18}$$

Therefore $A \in \omega_{\langle 3\rangle}$.

Conversely, if $A \in \omega_{\langle 3\rangle}$, then (4.15) and (4.16) hold by (4.2). By Propositions 4.10 and 4.11 we have $l(A) \leqslant h(A) \leqslant s(A)$, and hence, in view of Lemma 4.9, we have (4.17). ■

The following theorem provides a sufficient condition for a kind of "local" interlacing.

THEOREM 4.19. *Let $A \in \mathbb{C}^{n,n}$ and let $i \in \langle n \rangle$. If*

$$A + dE_{ii} \in \omega_{\langle n \rangle} \qquad \forall d \in \mathbb{R},\ d \geqslant 0, \tag{4.20}$$

then

$$l(A) \leqslant l_i(A) \leqslant s(A). \tag{4.21}$$

Proof. Let A satisfy (4.20), and assume that (4.21) does not hold. Hence, since $A \in \omega_{\langle n \rangle}$, it follows that

$$s(A) < l_i(A). \tag{4.22}$$

In view of Proposition 4.10 we have

$$l(A) < s(A), \tag{4.23}$$

and by Lemma 4.9, it follows from 4.22 and 4.23 that

$$\det A_t(i) > 0, \qquad t \leqslant s(A), \tag{4.24}$$

and

$$\det A_t < 0, \qquad l(A) < t < s(A). \tag{4.25}$$

Observe that

$$\det(A + dE_{ii})_t = \det A_t + d \det A_t(i). \tag{4.26}$$

By (4.24) it now follows that

$$\det(A + dE_{ii})_{l(A)}, \det(A + dE_{ii})_{s(A)} > 0 \qquad \forall d > 0. \tag{4.27}$$

Observe that in view of (4.24) and (4.26), by increasing d we elevate the graph of $\det(A + dE_{ii})_t$ between $l(A)$ and $s(A)$. Furthermore, let

$$a = -\frac{\min_{t \in [l(A), s(A)]} \{\det A_t\}}{\min_{t \in [l(A), s(A)]} \{\det A_t(i)\}} > 0.$$

Then, by (4.26) we have

$$\det(A + dE_{ii})_t > 0, \quad t \in [l(A), s(A)], \qquad \forall d > a. \tag{4.28}$$

Therefore, by continuity arguments it follows from (4.24), (4.25), (4.26), (4.27), and (4.28) that for some $b > 0$ the matrix $B = A + bE_{ii}$ satisfies

$$\min_{t \in [l(A), s(A)]} \{\det B_t\} = 0. \tag{4.29}$$

Observe that by (4.27) a point t where the minimum (4.29) is attained has to satisfy $l(A) < t < s(A)$. Thus necessarily

$$l(A) < l(B) = s(B) < s(A). \tag{4.30}$$

By (4.20) we have $B \in \omega_{\langle n \rangle}$ and hence, by (4.30) and Proposition 4.10, we have

$$l_i(A) = l_i(B) = l(B) < s(A),$$

which contradicts (4.22). Therefore, our assumption that (4.21) does not hold is false. ∎

The converse of Theorem 4.19 does not hold in general. The following example shows that (4.21) does not imply (4.20), even if A is given to be an $\underline{I}$-matrix.

EXAMPLE 4.31. Let

$$A = \begin{bmatrix} 2 & 2 & 2 \\ 0.1 & 0 & 1 \\ 1 & 1.1 & 0.1 \end{bmatrix}.$$

Observe that

$$\sigma(A) = \{-1.4, 0, 3.5\},$$

$$\sigma(A(1)) = \{-1, 1.1\},$$

$$\sigma(A(2)) = \{-0.6536725, 2.7536725\},$$

$$\sigma(A(3)) = \{-0.0954451, 2.0954451\}.$$

Hence we have $A \in \underline{I}^{<}_{\langle 3 \rangle}$. However, for

$$B = A + E_{33} = \begin{bmatrix} 2 & 2 & 2 \\ 0.1 & 0 & 1 \\ 1 & 1.1 & 1.1 \end{bmatrix}$$

we have

$$h(B) = l_1(B) = -0.6342719$$

and

$$\det[B - h(B)I] = 0.604602 > 0.$$

Thus, by Theorem 4.14 we have $B \notin \omega_{\langle 3 \rangle}$.

As a corollary to Theorem 4.19 we obtain the following sufficient condition for a matrix to be an I-matrix.

THEOREM 4.32. *Let $A \in \mathbb{C}^{n,n}$. If $A + dE_{ii} \in \omega_{\langle n \rangle}$ for all nonnegative numbers d and all $i \in \langle n \rangle$, then $A \in \underline{I}_{\langle n \rangle}$.*

Observe that in particular if $A + D \in \omega_{\langle n \rangle}$ for all nonnegative diagonal matrices D, then $A \in \underline{I}_{\langle n \rangle}$.

The following corollary to Theorem 4.32 is mostly known. The result for Hermitian matrices is a special case of Cauchy interlacing theorems in [2] (at least for symmetric matrices). The result for Z-matrices follows essentially form Frobenius [6], and is stated explicitly in [9].

COROLLARY 4.33. *The classes $Z_{\langle n \rangle}$, $R_{\langle n \rangle}$ and the class of all $n \times n$ Hermitian matrices are contained in $\underline{I}_{\langle n \rangle}$.*

Proof. Observe that the class of all $n \times n$ Hermitian matrices and $Z_{\langle n \rangle}$ are invariant under addition of real diagonal matrices. The same is true for $R_{\langle n \rangle}$ by [15]. ∎

Example 4.31 shows that $\underline{I}_{\langle n \rangle}$ is not invariant under addition of diagonal matrices. This class is also not closed under multiplication by (even positive) diagonal matrices.

EXAMPLE 4.34. Let

$$A=\begin{bmatrix}1&8&8\\1&8&9\\1&8&27\end{bmatrix}.$$

Here

$$\sigma(A)=\{0,5.31142,30.68857\},$$

$$\sigma(A(1))=\{4.76226,30.23773\},$$

$$\sigma(A(2))=\{0.69586,27.30413\},$$

$$\sigma(A(3))=\{0,9\}.$$

Thus, $A\in\underline{I}_{\langle 3\rangle}$. However, for $B=\operatorname{diag}(1,1,\frac{1}{27})A$ we obtain

$$\sigma(B)=\{0,0.6454,9.355\}$$

and

$$\sigma(B(1))=\{0.7695,8.231\},$$

so $B\notin\underline{I}_{\langle 3\rangle}$.

We remark that the matrix B in Example 4.34 is totally nonnegative. So it shows that for $n>2$, $T_{\langle n\rangle}\not\subseteq\underline{I}_{\langle n\rangle}$.

We conclude the section with two observations concerning further eigenvalue inequalities satisfied by classes we have discussed.

OBSERVATION 4.35. *Let $A\in Z_{\langle n\rangle}$, $n\geqslant 3$. It is known (e.g. [13, p. 14]) that if $s(A)$ then there exist $<\infty$ $i,j\in\langle n\rangle$, $i\neq j$, such that $l(A)\leqslant l_{ij}(A)\leqslant s(A)$. This property does not hold in general, either for R-matrices or for Hermitian matrices, as demonstrated by the matrix*

$$A=\begin{bmatrix}2&1&1\\1&1&1\\1&1&1\end{bmatrix}.$$

We have $\sigma(A)=\{0,2-\sqrt{2},2+\sqrt{2}\}$, so $s(A)<1$. However, all the diagonal elements of A are greater than or equal to 1.

OBSERVATION 4.36. *Consider the Cauchy interlacing property for a Hermitian matrix. Such a property does not hold in general for R-matrices, nor for Z-matrices A, even if the spectra of all principal submatrices of A are granted to be real.*

To see this consider the matrix

$$A = \begin{bmatrix} 5 & -1 & 0 \\ 0 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix},$$

which is discussed in [5]. We have

$$\sigma(A) = \{0.8851, 3.254, 4.861\}$$

and

$$\sigma(A(1)) = \{1,3\},$$

so

$$s(A(1)) \leqslant s(A).$$

5. STRICT EIGENVALUE INEQUALITIES

In this section we discuss the cases where the eigenvalue inequalities are strict. We start with a characterization of $\omega^{<}_{\langle n \rangle}$, which is of the same manner as the characterization of $\omega_{\langle n \rangle}$ given in [3].

THEOREM 5.1. *Let $A \in \mathbb{C}^{n,n}$, $n \geqslant 2$, have real principal minors. Then the following are equivalent:*

(i) $A \in \omega^{<}_{\langle n \rangle}$;

(ii) $\det A_t[\mu] < \det A_t[\mu \setminus \nu] \det A_t[\nu] \quad \forall \nu, \mu, \quad \varnothing \neq \nu \subsetneq \mu \subseteq \langle n \rangle$. *whenever $A_t[\mu]$ is a P°-matrix;*

(iii) $\det A_t[\mu] < \det A_t[\mu \setminus \{j\}] \det A_t[j] \quad \forall j, \mu, \quad j \in \mu \subseteq \langle n \rangle, \quad |\mu| > 1$, *whenever $A_t[\mu]$ is a P°-matrix;*

(iv) $A \in \omega_{\langle n \rangle}$, *and for every* $\mu \subseteq \langle n \rangle$, $|\mu| \geqslant 2$, *we have* $\det A_{l(A[\mu])}[\mu \setminus \{i\} | \mu \setminus \{j\}] \neq 0$ *for all* $i, j \in \mu$, $i \neq j$.

Proof. (i) $\Rightarrow$ (ii): Let $A \in \omega_{\langle n \rangle}^{<}$ and $\varnothing \neq \nu \subsetneq \mu \subseteq \langle n \rangle$. By Theorem 3.6 in [3] the matrix $A_t[\mu]$ is a P°-matrix if and only if $t \leqslant l(A[\mu])$. In this case, by Theorem 3.12 in [3] we have

$$\det A_t[\mu] \leqslant \det A_t[\mu \backslash \nu] \det A_t[\nu], \tag{5.2}$$

where, by Theorem 4.3 there, equality holds for some $t \leqslant l(A[\mu])$ if and only if

$$l(A[\mu]) = \min\{l(A[\nu]), l(A[\mu \backslash \nu])\},$$

which contradicts (i). Hence, the inequality in (5.2) is strict.

(ii) $\Rightarrow$ (iii): Obvious.

(iii) $\Rightarrow$ (i): Assume that (iii) holds. By Theorem 3.12 in [3] we have $A \in \omega_{\langle n \rangle}$. Hence $l(A[\mu]) \leqslant l(A[\mu \backslash \{j\}])$ for all $j, \mu,\ j \in \mu \subseteq \langle n \rangle, |\mu| > 1$. By (iii) we now have

$$\det A_t[\mu] < \det A_t[\mu \backslash \{j\}] \det A_t[j]$$

for $t = l(A[\mu])$. Thus, we have the inequality

$$l(A[\mu]) < l(A[\mu \backslash \{j\}])$$

for all $j \in \mu$, which implies (i).

(i) $\Rightarrow$ (iv): Let $A \in \omega_{\langle n \rangle}^{<}$. Then clearly $A \in \omega_{\langle n \rangle}$. Let $\mu \subseteq \langle n \rangle$, $|\mu| \geqslant 2$, and let $i, j \in \mu,\ i \neq j$. By Sylvester's identity (e.g. [7, p. 33]) we have (for a general matrix A)

$$\begin{aligned} &\det A_t[\mu \backslash \{i\}] \det A_t[\mu \backslash \{j\}] \\ &\quad - \det A_t[\mu \backslash \{i\} | \mu \backslash \{j\}] \det A_t[\mu \backslash \{j\} | \mu \backslash \{i\}] \\ &\quad = \det A_t[\mu \backslash \{i, j\}] \det A_t[\mu]. \end{aligned}$$

In particular for $t_0 = l(A[\mu])$ we obtain

$$\begin{aligned} &\det A_{t_0}[\mu \backslash \{i\}] \det A_{t_0}[\mu \backslash \{j\}] \\ &\quad = \det A_{t_0}[\mu \backslash \{i\} | \mu \backslash \{j\}] \det A_{t_0}[\mu \backslash \{j\} | \mu \backslash \{i\}]. \end{aligned} \tag{5.3}$$

Since $A \in \omega^{<}_{\langle n \rangle}$, it follows that

$$l_i(A[\mu]), l_j(A[\mu]) > t_0$$

and hence, by Lemma 4.9, the left-hand side of the equality (5.3) is positive. Our claim follows.

(iv) $\Rightarrow$ (i): Let A satisfy (iv), let $\mu \subseteq \langle n \rangle$, $|\mu| \geqslant 2$, and let $i, j \in \mu$, $i \neq j$. For $t_0 = l(A[\mu)]$ we have (5.2). Since the right-hand side of (5.3) is now nonzero, it follows that necessarily

$$l_i(A[\mu]), l_j(A[\mu]) > t_0. \qquad \blacksquare$$

We remark that the condition $A \in \omega_{\langle n \rangle}$ cannot be dropped from statement (iv) in Theorem 5.1, as demonstrated by Example 7.2 in [3]. Let

$$A = \begin{bmatrix} 5 & 2 & 1 \\ 1 & 5 & 2 \\ 2 & 1 & 5 \end{bmatrix}.$$

As shown in [3], the matrix A is not in $\omega_{\langle 3 \rangle}$. However, we have $l(A) = 8$, and it is easy to verify that all almost principal minors of A are nonzero.

We note that in statement (iv) of Theorem 5.1 we do not necessarily have $\det A_t[\mu \backslash \{i\} | \mu \backslash \{j\}] \neq 0$ for all $t \leqslant l(A)$, even if A is given to be a Hermitian $\underline{I}^{<}$-matrix. This is shown in the following example.

Example 5.4. Let

$$A = \begin{bmatrix} 43 & 6 & 1 \\ 6 & 36 & 6 \\ 1 & 6 & 31 \end{bmatrix}.$$

We have

$$\sigma(A) = \{26.5597, 36, 47.440\},$$

$$\sigma(A(1)) = \{27, 40\},$$

$$\sigma(A(2)) = \{30.917, 43.083\},$$

$$\sigma(A(3)) = \{32.553, 46.446\}.$$

Hence, A is a Hermitian $\underline{I}^{<}$-matrix. Nevertheless, we have

$$\det A(3|2) = \det A(2|3) = 0.$$

Observe that in statement (iv) of Theorem 5.1 we actually have

$$\det A_{t_0}[\mu\backslash\{i\}|A\backslash\{j\}]\det A_{t_0}[\mu\backslash\{j\}|\mu\backslash\{i\}] > 0 \tag{5.5}$$

for $t_0 = l(A[\mu])$. This follows, by Lemma 4.9, from the fact that $A \in \omega_{\langle n\rangle}$, and from (5.3). Applying (5.5) for sets μ of cardinality 2, we obtain from Theorem 5.1 that if $A \in \omega^{<}_{\langle n\rangle}$ then $a_{ij}a_{ji} > 0$ for all $i, j \in \langle n\rangle$, $i \neq j$. This, was already observed in (4.3).

As a corollary to Theorem (5.1) we obtain the following:

PROPOSITION 5.6. *Let $A \in Z_{\langle n\rangle}$. Then $A \in \omega^{<}_{\langle n\rangle}$ if and only if*

$$a_{ij} \neq 0 \qquad \forall i, j \in \langle n\rangle,\ i \neq j. \tag{5.7}$$

Proof. Let $A \in Z_{\langle n\rangle}$. If $A \in \omega^{<}_{\langle n\rangle}$, then by Theorem 5.1 we have (5.7). Conversely, if (5.7) holds, then it follows from the Perron-Frobenius Theorem (e.g., [1, p. 27]) that $A \in \omega^{<}_{\langle n\rangle}$.

REMARK 5.8. *If A is a Hermitian $n \times n$ matrix, then the condition* (5.7) *is not sufficient for $A \in \omega^{<}_{\langle n\rangle}$. To see this, consider the matrix*

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix},$$

which is not in $\omega^{<}_{\langle n\rangle}$.

The following theorem provides a sufficient condition for a matrix A to be in $\underline{I}^{<}_{\langle n\rangle}$.

THEOREM 5.9. *Let $A \in \mathbb{C}^{n,n}$. If $A + dE_{ii} \in \omega^{<}_{\langle n\rangle}$ for all nonnegative numbers d and all $I \in \langle n\rangle$, then $A \in \underline{I}^{<}_{\langle n\rangle}$.*

Proof. By Theorem 4.32, a matrix A which satisfies the conditions of our theorem is in $\underline{I}_{\langle n\rangle}$. So, choosing $i \in \langle n\rangle$, we have

$$l(A) < l_i(A) \leqslant s(A). \tag{5.10}$$

Assume that

$$l_i(A) = s(A). \tag{5.11}$$

By (4.26) it now follows from (5.11) and Lemma 4.9 that for positive d we have

$$l(A) < l(A + dE_{ii}) \leqslant s(A), \tag{5.12}$$

and also

$$(A + dE_{ii})_{s(A)} = 0. \tag{5.13}$$

Since $A \in \omega^{<}_{\langle n \rangle}$ it follows that

$$l_i(A) < l_{ij}(A) \qquad \forall j \in \langle n \rangle,\ j \neq i.$$

Therefore, since

$$\frac{d}{dt} \det A(i)_t = -\sum_{\substack{j=1 \\ j \neq i}}^{n} \det A_t(\{i, j\}),$$

it follows from (4.26) and Lemma 4.9 that for positive d sufficiently large we have

$$\frac{d}{dt} \det(A + dE_{ii})_t < 0, \qquad l(A) \leqslant t \leqslant s(A). \tag{5.14}$$

Together with (5.12) and (5.13), (5.14) yields that

$$l(A + dE_{ii}) = s(A).$$

Since $A + dE_{ii} \in \omega^{<}_{\langle n \rangle}$, it now follows that

$$s(A) < l_i(A + dE_{ii}) = l_i(A),$$

which contradicts (5.11). Hence, our assumption (5.11) is false and so we have

$$l(A) < l_i(A) < s(A). \qquad \blacksquare$$

COROLLARY 5.15. *Let* $A \in Z_{\langle n \rangle}$. *Then* $A \in \underline{I}^{<}_{\langle n \rangle}$ *if and only if* (5.7) *holds.*

Proof. Let $A \in \underline{I}^{<}_{\langle n \rangle}$. Then $A \in \omega^{<}_{\langle n \rangle}$, and (5.7) follows by Proposition 5.6. Conversely, if A satisfies (5.7), then $A + dE_{ii}$ satisfies (5.7) for all real numbers d and all $i \in \langle n \rangle$. Thus, by Proposition 5.6 we have $A + dE_{ii} \in \omega^{<}_{\langle n \rangle}$, and our assertion follows from Theorem 5.9. ■

The converse of Theorem 5.9 holds for $n \leqslant 2$ by (4.3), but does not hold in general for $n > 2$, as demonstrated by Example 4.31. That example shows that although $A \in \underline{I}^{<}_{\langle 3 \rangle}$, the matrix $A + E_{33}$ is not even in $\omega_{\langle 3 \rangle}$. The following example shows that even if we restrict ourselves to symmetric matrices, the converse of Theorem 5.9 still does not hold.

EXAMPLE 5.16. Let J be an $n \times n$ matrix all of whose entries are 1, and let

$$A = J + \operatorname{diag}(d_1, \ldots, d_n),$$

where $d_1, \ldots, d_n$ are distinct. Let $k \in \langle n \rangle$. Subtracting any row but the kth one from the others, one can obtain that

$$\det A_t = (d_k - t) \det A_t(k) + \prod_{\substack{i=1 \\ i \neq k}}^{n} (di - t). \tag{5.17}$$

It now follows from (5.17) that an eigenvalue λ of A is also an eigenvalue of $A(k)$ if and only if $\lambda \in \{d_1, \ldots, d_n\} \backslash \{d_k\}$. It is easy to verify that if $n = 2$, then A does not have a common eigenvalue either with $A(1)$ or with $A(2)$. Hence, using induction one can prove that

$$\sigma(A) \cap \sigma(A(k)) = \varnothing \qquad \forall k \in \langle n \rangle. \tag{5.18}$$

Since A is symmetric, we have $A \in \underline{I}_{\langle n \rangle}$, and with (5.18) we obtain $A \in \underline{I}^{<}_{\langle n \rangle}$. However, if

$$d_j = \min_{i \in \langle n \rangle} \{d_i\}$$

and

$$d_k = \min_{\substack{i \in \langle n \rangle \\ i \neq j}} \{d_i\},$$

then

$$B = A + (d_k - d_j)E_{jj} \notin \omega^{<}_{\langle n \rangle},$$

since for any $l \in \langle n \rangle \backslash \{k, j\}$ we have

$$l(B[\{j,k,l\}]) = l(B[\{j,k\}]) = d_k - 1.$$

REMARK 5.19. In Example 5.16 we discussed a Hermitian matrix A in $\underline{I}^{<}_{\langle n \rangle}$ such that $A + D \notin \omega^{<}_{\langle n \rangle}$ for some real diagonal matrix D. We remark that there exist Hermitian matrices A such that $A + D \in \omega^{<}_{\langle n \rangle}$ for all real diagonal matrices D. As an example for such a matrix consider

$$A = \begin{bmatrix} 0 & i & 1 \\ -i & 0 & i \\ 1 & -i & 0 \end{bmatrix}.$$

It is easy to verify that whenever $B = A + D$, we have

$$\det B_{l_i(B)} < 0, \qquad i = 1,2,3,$$

and hence $B \in \omega^{<}_{\langle n \rangle}$.

6. EIGENVALUE INTERLACING AND GENERALIZED LEADING PRINCIPAL MINOR SEQUENCES

In [3] the authors posed the question whether an ω-matrix which has positive leading principal minors is a τ-matrix. This question was answered negatively in [10]. In order to treat this question for the classes of matrices discussed above, we introduce the following definitions and notation.

DEFINITION 6.1. Let $A \in \mathbb{C}^{n,n}$, and let $(i_1, \ldots, i_n)$ be a permutation of $(1, \ldots, n)$. The sequence

$$\det A[\{i_1\}], \det A[\{i_1, i_2\}], \ldots, \det A$$

is called a *GLP sequence* (generalized leading principal minor sequence) of A. A GLP-sequence of A is said to be *nonnegative* [*positive*] if all its elements are nonnegative [positive]. If the first $n-1$ elements of the sequence are positive and the nth one is nonnegative, then the sequence is said to be *semipositive*.

DEFINITION 6.2. Let $A \in \mathbb{C}^{n,n}$. The matrix A is said to have the *nonnegative* [*semipositive*] (*positive*) *GLP property* if the following statement holds for A: For all $t \in \mathbb{R}$, if A_t has a nonnegative [semipositive] (positive) GLP sequence, then $A_t \in P^0_{\langle n \rangle}$.

It is proved in [14] that the weakly sign symmetric matrices have the semipositive GLP property (it is claimed there only that these matrices have the positive GLP property, but the proof actually shows the above). The semipositive GLP property is also shared by Z-matrices, as proven in [4], and by Hermitian matrices, e.g. [7, p. 337]. The latter two results also follow from Theorem 6.11 below.

PROPOSITION 6.3. *Let $A \in \mathbb{C}^{n,n}$. Then the following are equivalent:*

(i) *A has the positive GLP property;*
(ii) *For all $t \in \mathbb{R}$, if A_t has a positive GLP sequence, then $A \in P_{\langle n \rangle}$.*

Proof. (i) $\Rightarrow$ (ii): Let $A \in \mathbb{R}$ be such that A_t has a positive GLP sequence. By continuity argument, for $\epsilon > 0$ sufficiently small the matrix $A - (t+\epsilon)I$ has a positive GLP sequence too. Thus, by (i) we have $B = A - (t+\epsilon)I \in P^0_{\langle n \rangle}$, and hence $A_t = B + \epsilon I \in P_{\langle n \rangle}$.
(ii) $\Rightarrow$ (i): Obvious. ∎

An interesting relation between the positive and the semipositive GLP properties is given in the following theorem.

THEOREM 6.4. *Let $A \in \mathbb{C}^{n,n}$, and assume that all the real eigenvalues of A are principal eigenvalues. Then A has the positive GLP property if and only if A has the semipositive GLP property.*

Proof. The "if" part is obvious. Conversely, assume that A has the positive GLP property. Let $t \in \mathbb{R}$, and let $(i_1, \dots, i_n)$ be a permutation of $(1, \dots, n)$ such that

$$\det A_t[\{i_1, \dots, i_k\}] > 0, \qquad k = 1, \dots, n-1, \tag{6.5}$$

and

$$\det A_t \geqslant 0. \tag{6.6}$$

We have to show that

$$A_t \in P^0_{\langle n \rangle}. \tag{6.7}$$

If the inequality (6.6) is strict, then, since A has the positive GLP property, (6.7) follows from (6.5) and (6.6). So we can assume that $\det A_t = 0$. By the conditions of the theorem, t is a principal eigenvalue of A. Since by (6.5) we have $\det A_t(i_n) > 0$, it follows that t is a simple eigenvalue of A. Therefore, choosing $\epsilon > 0$ sufficiently small, we have either

$$\det A_{t-\epsilon} > 0 \tag{6.8}$$

or

$$\det A_{t+\epsilon} > 0. \tag{6.9}$$

Suppose that (6.8) holds. By continuity arguments, it follows from (6.5) that for ϵ sufficiently small we have

$$\det A_{t-\epsilon}[\{i_1, \ldots, i_k\}] > 0, \qquad k = 1, \ldots, n-1. \tag{6.10}$$

Since A has the positive GLP property, it follows from (6.8) and (6.10) that

$$A_{t-\epsilon} \in P^0_{\langle n \rangle}$$

for all $\epsilon > 0$ sufficiently small. Our assertion (6.7) now follows using continuity arguments. In case (6.9) holds, the proof is similar. ■

The semipositive GLP property is related to eigenvalue interlacing. Such a relation is expressed in the following characterization for $\underline{I}_{\langle n \rangle}$.

THEOREM 6.11. *Let $A \in \mathbb{C}^{n,n}$. Then the following are equivalent*:

(i) $A \in \underline{I}_{\langle n \rangle}$;

(ii) $A \in \omega_{\langle n \rangle}$, *and every principal submatrix of A has the semipositive GLP property.*

Proof. (i) $\Rightarrow$ (ii): Let $A \in \underline{I}_{\langle n \rangle}$. We prove (ii) by induction on n. For $n \leqslant 2$ the claim is easy. Assume that our assertion holds for $n < m$, and let $n = m > 2$. Since $\underline{I}_{\langle n \rangle} \subseteq \omega_{\langle n \rangle}$, we have $A \in \omega_{\langle n \rangle}$. Also, by the inductive assumption, every proper principal submatrix of A has the semipositive GLP property. Therefore, all we have to prove is that if A_t has a semipositive GLP sequence, then $A_t \in P^0_{\langle n \rangle}$. So, let $(i_1, \dots, i_n)$ be a permutation of $(1, \dots, n)$ such that

$$(6.12) \qquad \det A_t[\{i_1, \dots, i_k\}] > 0, \qquad k = 1, \dots, n-1,$$

and

$$(6.13) \qquad \det A_t \geqslant 0.$$

Assume that

$$(6.14) \qquad A_t \notin P^0_{\langle n \rangle}.$$

Then $A_t \notin \tau_{\langle n \rangle}$ and hence $l(A_t) < 0$. By (6.13) we have $s(A_t) \leqslant 0$. Since $A \in \underline{I}_{\langle n \rangle}$, it now follows that

$$(6.15) \qquad l_i(A_t) \leqslant 0 \qquad \forall i \in \langle n \rangle.$$

However, by the inductive assumption we have $A_t(i_n) \in P^0_{\langle n \rangle}$, and hence $A_t(i_n) \in \tau_{\langle n \rangle}$, which means that $l_{i_n}(A_t) \geqslant 0$. Furthermore, it follows from (6.12) that $l_{i_n}(A_t) > 0$, in contradiction to (6.15). Therefore, our assumption (6.14) is false.

(ii) $\Rightarrow$ (i): Let A satisfy (ii). Clearly, it is enough to show that

$$(6.16) \qquad l(A) \leqslant l_i(A) \leqslant s(A) \qquad \forall i \in \langle n \rangle.$$

Assume that (6.16) does not hold, namely that there exists $j \in \langle n \rangle$ such that

$$(6.17) \qquad s(A) < l_j(A).$$

Let $(i_1, \dots, i_n)$ by any permutation of $(1, \dots, n)$ such that $i_n = j$. Since $A \in \omega_{\langle n \rangle}$, it follows from (6.17) and Lemma 4.9 that

$$(6.18) \qquad \det A_{s(A)}[\{i_1, \dots, i_k\}] > 0 \qquad k = 1, \dots, n-1,$$

and

$$\det A_{s(A)} = 0. \tag{6.19}$$

By (ii), (6.18) and (6.19) yield that $A_{s(A)} \in P^0_{\langle n \rangle}$. Thus, by (6.19) we have $l(A_{s(A)}) = 0$, and hence

$$l(A) = l(A_{s(A)}) + s(A) = s(A). \tag{6.20}$$

By Proposition 4.10 it now follows from (6.20) that $l_j(A) = s(A)$, in contradiction to (6.17). Therefore, our assumption (6.17) is false and (6.16) follows. ■

We remark that condition (ii) in Theorem 6.11 cannot be weakened by requiring that only A (and not necessarily the proper principal submatrices of A) have the semipositive GLP property, as demonstrated by the following example.

EXAMPLE 6.21. Let

$$A = \begin{bmatrix} -4 & 0 & 0 & 0 \\ 0 & 1 & 12 & 6 \\ 0 & 1 & 1 & 2 \\ 0 & 2 & 8 & 1 \end{bmatrix}.$$

As shown in [10], we have $A(1) \in \omega_{\langle 3 \rangle}$, but $A(1)$ does not have the semipositive GLP property. Hence, by Theorem 6.11, $A \notin \underline{I}_{\langle 4 \rangle}$. However, the matrix A has the semipositive GLP property. To see this, observe that A_t might have a semipositive GLP sequence only for $t \leqslant -4$. But since $l(A) = -4$, it follows then that $A_t \in \tau_{\langle n \rangle}$ and hence $A_t \in P^0_{\langle n \rangle}$.

We remark that by Corollary 4.33 it follows from Theorem 6.11 that R-matrices, Z-matrices, and Hermitian matrices have the semipositive GLP property.

Similarly to Theorem 6.11, we obtain the following characterization for $\underline{I}^{<}_{\langle n \rangle}$.

THEOREM 6.22. *Let $A \in \mathbb{C}^{n,n}$. Then the following are equivalent:*

(i) $A \in \underline{I}^{<}_{\langle n \rangle}$;

(ii) $A \in \omega^{<}_{\langle n \rangle}$, *and every principal submatrix of A has the nonnegative GLP property.*

Proof. The proof is very similar to the proof of Theorem 6.11. Nevertheless, it is presented here in detail because of some differences.

(i) ⇒ (ii): Let $A \in \underline{I}^{<}_{\langle n\rangle}$. We prove (ii) by induction on n. For $n \leqslant 2$ the claim is easy. Assume that our assertion holds for $n < m$, and let $n = m > 2$. Since $\underline{I}^{<}_{\langle n\rangle} \subseteq \omega^{<}_{\langle n\rangle}$, we have $A \in \omega^{<}_{\langle n\rangle}$. Also, by the inductive assumption every proper principal submatrix of A has the nonnegative GLP property. Therefore, all we have to prove is that if A_t has a nonnegative GLP sequence, then $A_t \in P^0_{\langle n\rangle}$. So, let $(i_1,\dots,i_n)$ be a permutation of $(1,\dots,n)$ such that

$$\det A_t[\{i_1,\dots,i_k\}] > 0, \qquad k = 1,\dots,n-1. \tag{6.23}$$

Assume that

$$A_t \notin P^0_{\langle n\rangle}. \tag{6.24}$$

Then $A_t \notin \tau_{\langle n\rangle}$ and hence $l(A_t) < 0$. By (6.23) (for $k = n$) we have $s(A_t) \leqslant 0$. Since $A \in \underline{I}^{<}_{\langle n\rangle}$, it now follows that

$$l_i(A_t) < 0 \qquad \forall i \in \langle n\rangle. \tag{6.25}$$

However, by the inductive assumption we have $A_t(i_n) \in P_{\langle n-1\rangle}$, and hence $A_t(i_n) \in \tau_{\langle n-1\rangle}$, which means that $l_{i_n}(A_t) \geqslant 0$, in contradiction to (6.25). Therefore, our assumption (6.24) is false.

(ii) ⇒ (i): Let A satisfy (ii). Clearly it is enough to show that

$$l(A) < l_i(A) < s(A) \qquad \forall i \in \langle n\rangle. \tag{6.26}$$

Since $A \in \omega^{<}_{\langle n\rangle}$, we have the left inequality in (6.26). Assume that the right inequality in (6.26) does not hold, namely that there exists $j \in \langle n\rangle$ such that

$$s(A) \leqslant l_j(A). \tag{6.27}$$

Let $(i_1,\dots,i_n)$ be any permutation of $(1,\dots,n)$ such that $i_n = j$. Since $A \in \omega^{<}_{\langle n\rangle}$, it follows from (6.27) and Lemma 4.9 that

$$\det A_{s(A)}[\{i_1,\dots,i_k\}] \geqslant 0, \qquad k = 1,\dots,n. \tag{6.28}$$

By (ii), (6.28) yields that $A_{s(A)} \in P^0_{\langle n\rangle}$. Thus, we have $l(A_{s(A)}) = 0$, and hence

$$l(A) = l(A_{s(A)}) + s(A) = s(A). \tag{6.29}$$

By Proposition 4.10 it now follows from (6.29) that $l_j(A) = l(A)$, in contradiction to $A \in \omega^{<}_{\langle n \rangle}$. Therefore, our assumption (6.27) is false and (6.26) follows. ■

We remark that statement (ii) in Theorem 6.22 is not equivalent to the weaker statement $A \in \omega^{<}_{\langle n \rangle}$, not even for Hermitian matrices.

EXAMPLE 6.30. Let

$$A = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}.$$

We have

$$\sigma(A) = \{-1.3722, 0, 4.3722\},$$

$$\sigma(A(1)) = \sigma(A(2)) = \{-1, 3\},$$

$$\sigma(A(3)) = \{0, 2\}.$$

Hence we have $A \in \omega^{<}_{\langle 3 \rangle}$, but $A \notin \underline{I}^{<}_{\langle 3 \rangle}$.

As a corollary to Theorem 6.22 we obtain the following result, which was already proven in [3].

COROLLARY 6.31. *Let A be a Z-matrix with nonzero off-diagonal elements. Then A has the nonnegative GLP property.*

Proof. By Corollary 5.15, A is an $\underline{I}^{<}$-matrix. By Theorem 6.22, A has the nonnegative GLP property. ■

Corollary 6.31 does not hold in general if we eliminate the requirement that A have nonzero off-diagonal elements, even if we require that A be irreducible. This is shown in the following example.

EXAMPLE 6.32. Let

$$A = \begin{bmatrix} 1 & -1 & 0 & -10 \\ -1 & 1 & 0 & -10 \\ 0 & 0 & 0 & -10 \\ -10 & -10 & -10 & -1 \end{bmatrix}.$$

The matrix A is an irreducible Z-matrix with nonnegative leading principal minors. However, A does not have the nonnegative GLP property, since $\det A(1) = -100$.

The above discussion raises interesting problems. We conclude the paper by posing some.

PROBLEM 6.33. Characterize the Hermitian matrices which have the nonnegative GLP property.

PROBLEM 6.34. Characterize the Hermitian matrices in $\underline{I}_{\langle n\rangle}^{<}$.

PROBLEM 6.35. As shown in Example 4.34, $T_{\langle n\rangle} \not\subseteq \underline{I}_{\langle n\rangle}$. Characterize the totally nonnegative matrices in $\underline{I}_{\langle n\rangle}$.

PROBLEM 6.36. Is there any relation between a matrix having the nonnegative GLP property and its principal submatrices having similar properties?

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Received 20 March 1986; revised 27 April 1986