Europ. J. Combinatorics (1987) **8**, 297–302

79

On $2k$-Twisted Graphs

DANIEL HERSHKOWITZ AND HANS SCHNEIDER*

We define the concept of a k-twisted chain in a (directed) graph and the concept of a $2k$-twisted graph. We show that for a $2k$-twisted graph the set of algebraic $2k$-twisted cycles is an integral spanning set for the integral flow module of G. Since a graph is 0-twisted if and only if it is strongly connected, our result generalizes the well-known theorem that there is a basis for the flow space of a strongly connected graph consisting of algebraic circuits.

1. INTRODUCTION

It is the purpose of this note to formalize the notion of the number of twists in a chain in a directed graph and then to generalize a theorem on strongly connected graphs to graphs we call $2k$-twisted.

Intuitively, a chain in a directed graph is obtained by putting a pointer at a vertex and moving it in the direction or against the direction of a connected sequence of arcs to another vertex. Each change of direction is a *twist*. In this paper we formalize the notion of chains with at most k twists, in other words we define the concept of a k-twisted chain. In particular, a circuit (directed cycle) is a 0-twisted cycle.

We also formally define a $2k$-twisted graph as a graph such that for every pair of vertices there exists a $2k$-twisted closed chain through the given vertices. Thus a graph is 0-twisted if and only if it is strongly connected.

In many papers in graph theory a chain is identified with an algebraic chain (i.e. a vector) corresponding to it. However, as we show by an example, two chains with different numbers of twists may correspond to the same algebraic chain and thus we must maintain a distinction between chains and algebraic chains.

Our main result asserts that for a $2k$-twisted graph the set of algebraic $2k$-twisted cycles is an integral spanning set for the integral flow module of G, i.e. every algebraic closed chain is an integral linear combination of algebraic cycles corresponding to $2k$-twisted cycles. As a corollary, we obtain the existence of a basis for the flow space consisting of algebraic $2k$-twisted cycles. This corollary generalizes the well-known theorem that there exists a basis for the flow space of a strongly connected graph consisting of algebraic circuits, e.g. Berge [1, p. 29].

There is a known converse for our main result in the case $k = 0$. However for $k > 0$ the converse fails as is shown by an example. In fact, one would not expect all results for strongly connected graphs to generalize to $2k$-twisted graphs since a chain obtained by concatenation of two k-twisted chains with positive initial signs may not be k-twisted if $k > 0$. So the natural transitive relation which occurs in the case $k = 0$ has no generalization.

The proof in [1] of the result quoted above shows more: For a strongly connected graph there exists a set of *linearly independent* algebraic circuits such that every element of the integral flow module is an integral linear combination of these algebraic circuits. We do not know if our result for $2k$-twisted graphs can be strengthened in a similar manner.

We were led to these investigations by an application of 2-twisted graphs to a problem concerning the extremal case of a classical inequality on the inverse of H-matrices due to Ostrowski [4], which has been studied by Neumaier [3]. The application will appear elsewhere.

*The research of this author was supported in part by NSF grants DMS-8320189 and DMS-8521521, by the Office of Naval Research and the Lady Davis Foundation of Israel.

The research of both authors was supported by their joint grant No. 85-00153 from the United States–Israel Binational Science Foundation (BSF), Jerusalem, Israel.

0195-6698/87/030297+06 $02.00/0

2. Definitions and Preliminaries

Definition 2.1. A (simple, directed) *graph* G is a pair (V, E) of finite sets with $E \subseteq V \times V$. An element of V is called a *vertex* of G, and an element of E is called an *arc* of G.

Definition 2.2. Let G be a graph. A *chain* in G of length s from a vertex i_0 to a vertex i_s of G is a sequence

$$\gamma = (i_0, e_1, i_1, e_2, i_2, \ldots, i_{s-1}, e_s, i_s) \tag{2.3}$$

where $i_p \in V$, $p = 0, \ldots, s$, and either $e_p = 1$ and (i_{p-1}, i_p) is an arc of G or $e_p = -1$ and (i_p, i_{p-1}) is an arc of G, $p = 1, \ldots, s$. The arc (i_{p-1}, i_p), $[(i_p, i_{p-1})]$, $1 \leqslant p \leqslant s$, is said to *lie on* γ if $e_p = 1$ $[e_p = -1]$. The chain γ is called *closed* if $i_0 = i_s$, and γ is called a *cycle* if it is closed and the vertices $i_1, \ldots, i_s$ are distinct. A chain given by (2.3) such that $e_1 = \cdots e_s = 1$ is called a *path*. A cycle that is a path is called a *circuit*. A closed chain of the form

$$\gamma = (i_0, e_1, i_1, \ldots, i_s, -e_s, i_{s-1}, \ldots, -e_1, i_0)$$

will be called *trivial*. The empty chain $\varnothing$ will be considered a chain of length 0 from any vertex to itself and is defined to be trivial.

A similar notation is used in Engel–Schneider [2].

Definition 2.4. (a) Let γ_1 and γ_2 be chains $(i_0, \ldots, i_s)$ and $(i_s, \ldots, i_t)$, respectively. Then $\gamma_1\gamma_2$ is defined to be the *concatenated chain* $(i_0, \ldots, i_s, \ldots, i_t)$.

(b) Let γ be the chain given by (2.3). Then γ^* is defined to be the *reverse chain* $(i_s, -e_s, i_{s-1}, \ldots, -e_1, i_0)$.

(c) We call e_1 the *initial sign* of γ.

Definition 2.5. Let γ be a chain given by (2.3).

(a) If $e_p \neq e_{p+1}$, $1 \leqslant p \leqslant s$, then we say that γ *has a twist at* p. If γ is a closed chain then in addition we say that γ has a twist at 0 if $e_s \neq e_1$.

(b) If γ has exactly k twists then γ is said to be *exactly k-twisted* and we write $t(\gamma) = k$.

(c) If $t(\gamma) \leqslant m$ for an integer m then γ is said to be *m-twisted.*

Note that if γ is not closed then $t(\gamma)$ is equal to the number of sign changes in the sequence $e_1, \ldots, e_s$. If γ is closed then $t(\gamma)$ is equal to the number of sign changes in the sequence $e_1, \ldots, e_s, e_1$.

Observe that a chain [cycle] is 0-twisted if and only if it is a path [circuit] or a reverse path [reverse circuit], and that a closed chain has an even number of twists. Further, if γ and δ are chains that may be concatenated then

$$t(\gamma) + t(\delta) - 2 \leqslant t(\gamma\delta) \leqslant t(\gamma) + t(\delta) + 2,$$

and, if $\gamma\delta$ does not have a twist at 0 (and in particular if $\gamma\delta$ is not closed) then

$$t(\gamma) + t(\delta) - 1 \leqslant t(\gamma\delta) \leqslant t(\gamma) + t(\delta) + 1.$$

Definition 2.6. A graph G is said to be *connected* [*strongly connected*] if for every two vertices i, j there exists a chain [path] from i to j.

Definition 2.7. A graph G is called *$2k$-twisted* if through each two vertices i, j there is a $2k$-twisted closed chain (i.e. i and j are vertices of the chain).

We note that a graph is 0-twisted if and only if it is strongly connected.

DEFINITION 2.8. Let G be a graph. If γ is a chain in G then the *algebraic chain corresponding to* γ (or *arising from* γ) is a vector $a(\gamma)$ indexed by the arc set E and with integer components defined as follows:

(a) $a(\varnothing) = 0$, the zero vector.

(b) If $\gamma = (h, 1, k)$ then

$$a(\gamma)_{(i,j)} = \begin{cases} 1, & \text{if } (i, j) = (h, k), \\ 0, & \text{if } (i, j) \in E, \text{ otherwise.} \end{cases}$$

(c) If $\gamma = (h, -1, k)$ then

$$a(\gamma)_{(i,j)} = \begin{cases} -1, & \text{if } (i, j) = (k, h), \\ 0 & \text{if } (j, j) \in E, \text{ otherwise.} \end{cases}$$

(d) If $\gamma = \gamma_1\gamma_2 \cdots \gamma_s$, where $\gamma_1, \ldots, \gamma_s$ are chains of length 1, then

$$a(\gamma) = a(\gamma_1) + \cdots + a(\gamma_s).$$

We shall speak of an 'algebraic [$2k$-twisted] cycle' when we mean an algebraic chain which arises from a [$2k$-twisted] cycle, etc. Note that

$$a(\gamma) = 0 \text{ if } \gamma \text{ is a trivial closed chain,}$$

$$a(\gamma\delta) = a(\gamma) + a(\delta),$$

$$a(\gamma^*) = -a(\gamma),$$

where γ and δ are chains.

DEFINITION 2.9. Let $G = (V, E)$ be a graph and let $\mathbb{Q}$ be the field of rational numbers. Let E have m elements. We define:

(a) The *integral flow module* $I(G)$ of G is the set of all integral linear combinations (i.e. linear combinations with integer coefficients) of all algebraic closed chains in G, considered as a module over the integers.

(b) The *flow space* $F(G)$ of G is the subspace of $\mathbb{Q}^m$ spanned by $I(G)$.

(c) Let X be a subset of $I(G)$. Then X is called an *integral spanning set* for $I(G)$ if every element of $I(G)$ is an integral linear combination of elements of X.

(d) Let X be an integral spanning set for $I(G)$. Then X is called an *integral basis* for $F(G)$ if the elements of X are linearly independent over the rationals.

An example of a strongly connected graph G and a basis of algebraic circuits for $F(G)$ which is not an integral basis is given in Saunders–Schneider [5, Comment 5.2].

Normally, one does not distinguish between chains and algebraic chains, e.g. Berge [1, p. 12]. However, we now give an example of two chains with different numbers of twists whose algebraic chains coincide. Thus it is necessary to distinguish between chains and algebraic chains when considering $2k$-twisted graphs.

In examples below we write $i \to j$ for the chain $(i, 1, j)$ and $i \leftarrow j$ for the chain $(i, -1, j)$ and we extend this notation to chains of length greater than 1. Intuitively, $i \to j$ is a step from i to j along arc (i, j) and $i \leftarrow j$ is a step from i to j along arc (j, i).

EXAMPLE 2.10. Let G be a graph with vertex set $\{1, \ldots, 5\}$ and arcs $(i, 5)$ and $(5, i)$, $i = 1, 2, 3, 4$. Consider the chains $\gamma_1 = 5 \to 1 \to 5$, $\gamma_2 = 5 \leftarrow 2 \leftarrow 5$, $\gamma_3 = 5 \to 3 \to 5$, $\gamma_4 = 5 \leftarrow 4 \leftarrow 5$. Then $\mu = \gamma_1\gamma_2\gamma_3\gamma_4$ is exactly 4-twisted while $\nu = \gamma_1\gamma_3\gamma_2\gamma_4$ is exactly 2-twisted. But $a(\mu) = a(\nu) = [1, -1, 1, -1, 1, -1, 1, -1]$ where the arcs in E are ordered lexicographically.

3. Results

Lemma 3.1. *Let G be a $2k$-twisted graph and let i and j be distinct vertices of G. Then* either *there exists an m-twisted chain from i to j where $m < k$,* or *there exist two exactly k-twisted chains*

$$\pi = (i, e_1, \ldots, e_s, j), \tag{3.2}$$

$$\nu = (i, e'_1, \ldots, e'_{t,j}) \tag{3.3}$$

from i to j where the initial signs of π and ν are opposite.

Proof. By definition, there exists a $2k$-twisted closed chain γ through i and j which we may write as $\gamma = \pi\nu^*$ where π and ν are given by (3.2) and (3.3), respectively. Observe that

$$t(\pi) + t(\nu) \leqslant t(\gamma) \leqslant 2k. \tag{3.4}$$

We distinguish two cases:
Case 1. The chain γ has a twist at 0. Since neither π nor ν are closed chains, they do not have a twist at 0. Hence we have

$$t(\pi) + t(\nu) + 1 \leqslant t(\gamma) \leqslant 2k,$$

and it follows that either $t(\pi) < k$ or $t(\nu) < k$.
Case 2. The chain γ does not have a twist at 0. In this case we have

$$e_1 \neq e'_1. \tag{3.5}$$

If $t(\pi) < k$ or $t(\nu) < k$ then the result holds. Otherwise, by (3.4), we have $t(\pi) = t(\nu) = k$ and by (3.5) the initial signs of π and ν are opposite.

Remark 3.6. Let

$$\gamma = (i_0, e_1, i_1, \ldots, e_q, i_0) \tag{3.7}$$

and let

$$\gamma' = (i_s, e_{s+1}, \ldots, e_q, i_0, e_1, i_1, \ldots, e_s, i_s),$$

where $0 \leqslant s < q$. Then observe that $a(\gamma) = a(\gamma')$ and $t(\gamma) = t(\gamma')$. Thus when dealing with algebraic closed chains we may replace γ by γ' without loss of generality.

Lemma 3.8. *Let γ be a $2k$-twisted closed chain in G. Then $a(\gamma)$ is an integral linear combination of algebraic $2k$-twisted cycles.*

Proof. Let γ be given by (3.7). The proof is by inducation on the length q of γ. If $q = 1$, 2 the result is immediate. Assume that $q > 2$ and that our assertion holds for closed chains of length less than q. If γ is a cycle then the result is again immediate. Otherwise, in view of Remark 3.6 we may assume without loss of generality that there exists t, $0 < t < q$ such that $\delta = (i_0, e_1, i_1, \ldots, e_t, i_t)$ is a cycle. We may write $a(\gamma) = a(\delta) + a(\nu)$, where $\nu = (i_t, e_{t+1}, i_{t+1}, \ldots, e_q, i_0)$ is a closed chain of length less than q. Since ν has a twist at p, $1 \leqslant p < q - t$, if an only if γ has a twist at $p + t$, it follows that $t(\nu) \leqslant t(\gamma) + 1$. But, since a closed chain has an even number of twists, we deduce that $t(\nu) \leqslant t(\gamma) \leqslant 2k$. Similarly we show that $t(\delta) \leqslant t(\gamma) \leqslant 2k$. By the inductive assumption $a(\nu)$ is an integral linear combination of algebraic chains which arise from $2k$-twisted cycles. Our claim now follows.

Theorem 3.9. *Let G be a $2k$-twisted graph. Then the set of algebraic $2k$-twisted cycles is an integral spanning set for the integral flow module $I(G)$.*

PROOF. In view of Lemma 3.8, it suffices to show that every algebraic cycle $a(\alpha)$ with $t(\alpha) = 2(k + m)$, $m > 0$, is the difference of algebraic closed chains which arise from chains with fewer than $2(k + m)$ twists. Let $a(\alpha)$ be an algebraic closed chain with $\alpha = (i_0, e_1, i_1, \ldots, i_{r-1}, e_r, i_0)$, $t(\alpha) = 2(k + m)$, $m > 0$. Suppose that α has twists at $p(q), q = 1, \ldots, 2(k + m)$ where $p(1) < p(2) < \cdots < p(2k + 2m)$. By Remark 3.6, we may assume without loss of generality that $p(1) = 0$. Let $i = i_0$ and $j = i_{p(k+m+1)}$. Note that $2 \leqslant k + m + 1 \leqslant 2k + 2m$ and hence $1 \leqslant p(k + m + 1) \leqslant r - 1$, so that $i \neq j$. We may write $\alpha = \varrho_1\varrho_2^*$ where ϱ_1 and ϱ_2 are (non-closed) chains from i to j, $t(\varrho_1) = t(\varrho_2) = k + m - 1$, and further the initial signs of ϱ_1 and ϱ_2 agree. By Lemma 3.1 there are two cases.

Case 1. There is an s-twisted chain π from i to j with $s < k$. In this case $\delta_q = \varrho_q\pi^*$, $q = 1$, 2, are closed chains with

$$t(\delta_q) \leqslant k + m - 1 + s + 2 = k + m + s + 1 \leqslant 2k + m < 2k + 2m, \qquad q = 1, 2$$

Case 2. There exist two exactly k-twisted chains from i to j whose initial signs are opposite. Hence we may choose a k-twisted chain π from i to j whose initial sign is opposite to the (common) initial signs of ϱ_1 and ϱ_2. Again we let $\delta_q = \varrho_q\pi^*$, $q = 1, 2$. Since δ_1 and δ_2 do not have twists at 0 we have

$$t(\delta_q) \leqslant k + m - 1 + k + 1 \leqslant 2k + m < 2k + 2m, \qquad q = 1, 2.$$

In either case we have $a(\alpha) = a(\delta_1) - a(\delta_2)$ and the result is proved.

The following converse of Theorem 3.9 for the case $k = 0$ is well-known and can be proved easily.

PROPOSITION 3.10. *Let G be a connected graph such that every arc of G lies on a non-trivial cycle. If the set of algebraic circuits spans I(G) then G is strongly connected.*

However, for $k > 0$ we have no similar converse. Intuitively, we may tie together two exactly $2k$-twisted cycles to form a connected graph which is t-twisted only for $t \geqslant 4k$. We construct an explicit example only for the case $k = 1$, but there is no difficulty in modifying it for the case of general positive k.

EXAMPLE 3.11. We label the vertices of G by $-3, -2, -1, 0, 1, 2, 3$. The arcs of the graph are $(-3, -2)$, $(-2, -1)$, $(0, -1)$, $(-3, 0)$, $(3, 2)$, $(2, 1)$, $(0, 1)$, $(3, 0)$. Then $\gamma = -2 \to -1 \leftarrow 0 \leftarrow -3 \to -2$ and $\delta = 2 \to 1 \leftarrow 0 \leftarrow 3 \to 2$ are exactly 2-twisted cycles, but there is no 2-twisted closed chain of G through the vertices -2 and 2.

We remark that the set of all algebraic *exactly* $2k$-twisted cycles may be a non-spanning set for the integral flow module (or the flow space) of a $2k$-twisted graph G, as demonstrated by the following example.

EXAMPLE 3.12 Let the vertices of G be 1, 2, 3, 4, 5, 6, 7, and let the arcs of G be

(1, 3), (2, 1), (3, 2), (3, 4), (3, 5), (4, 5), (5, 6), (6, 7), (7, 5).

Observe that G is a 2-twisted graph (but not 0-twisted). Let α be the cycle $3 \to 4 \to 5 \leftarrow 3$ and let β be the cycle $1 \to 3 \to 2 \to 1$. Since $a(\alpha)$ and $a(\alpha^*)$ are the only algebraic exactly 2-twisted cycles, it follows that $a(\beta)$ is not a linear combination of algebraic exactly 2-twisted cycles.

By a standard result in linear algebra, the following corollary is an immediate consequence of Theorem 3.9.

COROLLARY 3.13. *Let G be a $2k$-twisted graph. Then there exists a basis for the flow space $F(G)$ consisting of algebraic $2k$-twisted cycles.*

However, the question whether there always exists an integral basis for $F(G)$ consisting of algebraic $2k$-twisted cycles is still open.

ACKNOWLEDGEMENT

The authors are grateful to the referee for his helpful comments.

REFERENCES

1. C. Berge, *Graphs and Hypergraphs*, North-Holland, 1973.
2. G. M. Engel and H. Schneider, Algorithms for testing the diagonal similarity of matrices and related problems, *SIAM J. Alg. Disc. Meth.* **4** (1982), 429–438.
3. A. Neumaier, The extermal case of some matrix inequalities. *Archiv. Math.* **43** (1984), 137–141.
4. A. M. Ostrowski, Über die Determinanten mit überwiegender Hauptdiagonale, *Comment. Math. Helv.* **10** (1937), 69–96.
5. B. D. Saunders and H. Schneider, Flows on graphs applied to diagonal similarity and diagonal equivalence for matrices, *Disc. Math.* **24** (1978), 205–220.

Received 6 April 1985

DANIEL HERSHKOWITZ
Mathematics Department, Technion-Israel Institute of Technology, Haifa 32000, Israel
and
HANS SCHNEIDER
Department of Mathematics, University of Wisconsin, Madison, Madison, WI 53706, U.S.A.