Contemporary Mathematics
Volume 47, 1985

78

*

SEMISTABILITY FACTORS AND SEMIFACTORS

Daniel Hershkowitz and Hans Schneider*

ABSTRACT. A (semistability) factor [semifactor] of a matrix $A \in \mathbb{C}^{nn}$ is a positive definite [positive semidefinite] matrix H such that $AH + HA^*$ is positive semidefinite. We give three proofs to show that if A has a semistability factor then it cannot be unique. We give necessary and sufficient conditions for a matrix H to be a (semi)factor of a given matrix. We also determine the dimension of the cone of semistability factors.

1. INTRODUCTION. A (square) complex matrix A is said to be (positive) stable [semistable] if all its eigenvalues lie in the open [closed] right halfplane. The matrix A is said to be near stable if A is semistable but not stable. A very well known characterization of stable matrices is essentially due to Lyapunov [8], see also [5, vol. II, p. 189], [9], [10], [4] and the references there.

THEOREM 1.1 (Lyapunov-Gantmacher): A complex matrix A is stable if and only if there exists a positive definite (hermitian) matrix H such that

$$AH + HA^* > 0. \tag{1.2}$$

We have here written $H > 0$ to denote a positive definite (hermitian) matrix. We also write $H \geq 0$ to denote a positive semidefinite matrix.

For near stable matrices the situation is more complicated, as shown by the following result [4, Corollary III.1].

THEOREM 1.3. Let A be a complex matrix. Then there exists a positive definite matrix H such that

$$AH + HA^* \geq 0. \tag{1.4}$$

if and only if A is semistable and

(1.5) the elementary divisors of all pure imaginary eigenvalues of A are linear.

*Research of this author was supported in part by NSF grant DMS-8320189.

We shall call the matrix H in (1.2) or (1.4) a semistability factor of the matrix A.

An interesting special case occurs when a semistability factor may be chosen to be diagonal. In this case we call the matrix Lyapunov diagonally stable [semistable] if there exists a positive definite diagonal H satisfying (1.2) [(1.4)], and the corresponding diagonal H is called a Lyapunov scaling factor of A, e.g. [6] and [7]. At the Bowdoin meeting on "Linear algebra and its role in systems theory" (July 1984) one of us discussed necessary conditions and sufficient conditions for the uniqueness (up to positive scalar multiplication) of a Lyapunov scaling factor of a given Lyapunov diagonally semistable matrix (see [6]). At the end of the talk Dale Olesky raised the question whether there are cases where a semistability factor of a given semistable matrix of order greater than 1 is unique. In this paper we show that the answer to that question is negative and we give three different proofs.

Our third proof is a consequence of results that go considerably beyond the question of uniqueness of stability factors. If $A \in \mathbb{C}^{nn}$ we call a matrix H a (semistability) semifactor of A if $H \geq 0$ and (1.4) holds. In Section 4 we determine necessary and sufficient conditions for H to be a semifactor. We give our principal results in two forms. The first (Theorem 4.7) is for a matrix A in block diagonal form. The second (Theorem 4.12) is an invariant form which holds for all $A \in \mathbb{C}^{nn}$. We show that our theorem generalizes a part of Theorem III of [4] which characterizes the ranks of semifactors of A.

In Section 5 we define the (semistability) factor cone $\mathcal{S}(A)$ and the semifactor cone $\mathcal{S}_0(A)$ of $A \in \mathbb{C}^{nn}$. We show that $\mathcal{S}_0(A)$ is the closure of the set of semifactors of A of maximal rank (Lemma 5.2) and we deduce a necessary and sufficient condition on A for $\mathcal{S}_0(A)$ to be the closure of $\mathcal{S}(A)$ (Corollary 5.9). We also give a formula for the dimension of $\mathcal{S}_0(A)$ (Theorem 5.10) and as a corollary (Corollary 5.11) we obtain our third proof of the nonuniqueness of semistability factors.

Our paper is related in spirit to the papers by Carlson [3] and Avraham-Loewy [1], though there is little overlap in results. Those papers discuss the general problem of the relation of ranks and inertias of $A \in \mathbb{C}^{nn}$ and the hermitian matrices H and $K = AH + HA^*$, in [3] under the assumption that $K \geq 0$ and in [1] under the assumption that $H > 0$.

2. STATEMENT OF THE THEOREM AND PROOF NO. 1.

THEOREM 2.1. Let A be a complex matrix of order greater than 1. Then either A has no semistability factor or A has at least two linearly independent semistability factors.

PROOF. Suppose that A has at least one semistability factor. Then, by Theorem 1.3, A is semistable.

Suppose that A is stable. By Theorem 1.1, A has a semistability factor H such that (1.2) holds. Let K be any positive definite matrix such that H and K are linearly independent. Since the positive definite matrices form an open set in the topological space of hermitian matrices, it follows that for sufficiently small positive ε, H and $H + \varepsilon K$ are linearly independent semistability factors of A.

Now suppose that A is near stable. Then A has a pure imaginary eigenvalue $\lambda = i\alpha$. Let x be an eigenvector of A associated with the eigenvalue λ. So,

$$Ax = i\alpha x \tag{2.2}$$

and by multiplying both sides of (2.2) by x^* we obtain

$$Axx^* = i\alpha xx^*, \tag{2.3}$$

which is equivalent to

$$xx^*A^* = -i\alpha xx^*. \tag{2.4}$$

It follows from (2.3) and (2.4) that

$$Axx^* + xx^*A^* = 0. \tag{2.5}$$

Let H be a positive definite matrix such that (1.4) holds. Observe that $H' = H + xx^*$ is also a positive definite matrix and, further, by (2.5) and (1.4), H' is a semistability factor of A. Since H is nonsingular and since xx^* is a matrix of rank 1 it follows that xx^* is not a scalar multiple of H, and therefore H' is not a scalar multiple of H. Hence, H and H' are two linearly independent semistability factors of A. □

3. PROOF NO. 2. We begin with a simple lemma.

LEMMA 3.1. Let $B, H \in \mathbb{C}^{n,n}$ such that H is a positive definite matrix. If $BHB^* = 0$ then $B = 0$.

PROOF. Given that $BHB^* = 0$, it follows that

$$(B^*x)^*H(B^*x) = x^*BHB^*x = 0, \quad \forall x \in \mathbb{C}^n. \tag{3.2}$$

Since H is positive definite it follows from (3.2) that

$$B^*x = 0, \quad \forall x \in \mathbb{C}^n,$$

which implies $B^* = 0$ and $B = 0$. $\square$

LEMMA 3.3. Let B be a semistable matrix, let $H > 0$ and let α be any real number. Then H is a semistability factor of B if and only if H is a semistability factor of $B + i\alpha I$.

PROOF. Observe that

$$BH + HB^* = (B + i\alpha I)H + H(B + i\alpha I)^*. \quad \square$$

Proposition 3.4. Let B be a singular semistable matrix. Then B does not have a unique semistability factor.

PROOF. If $B = 0$ then the claim is clear. So we may assume that

$$B \neq 0. \tag{3.5}$$

Assume that B has a unique semistability factor H. Thus

$$C = BH + HB^* \geq 0. \tag{3.6}$$

Clearly,

$$B(BHB^*) + (BHB^*)B^* = BCB^* \geq 0. \tag{3.7}$$

Since H is positive definite, so is $H" = H + BHB^*$, and by adding (3.6) and (3.7) we obtain that $H"$ is a semistability factor of B. Our uniqueness assumption yields that $H"$ is a scalar multiple of H, and hence BHB^* is a scalar multiple of H. Since H is nonsingular and B is singular it follows that $BHB^* = 0$. By Lemma 3.1, $B = 0$, which is a contradiction to (3.5). Hence, the uniqueness assumption is false. $\square$

PROOF NO. 2. As in the first proof we may assume that A is a near stable matrix. As such, A has a pure imaginary eigenvalue. So, $A + i\alpha I$ is singular for some real number α. Our theorem now follows from Proposition 3.4 and Lemma 3.3. $\square$

4. SEMIFACTORS.

LEMMA 4.1. Let $A \in \mathbb{C}^{nn}$ be such that the spectrum of A consists of a single pure imaginary eigenvalue λ, and let $H > 0$. Then H is a semistability factor of A if and only if

$$(A - \lambda I)H = 0. \tag{4.2}$$

PROOF. If (4.2) holds then $AH = \lambda H$ and (1.4) follows. Conversely, suppose that (1.4) holds. Let U be a unitary matrix such that $A' = U^*AU$ is upper triangular. Let H' be the positive definite matrix U^*HU. Denote by B the matrix $A' - \lambda I$. Observe that $BH' + H'B^* \geq 0$. Clearly, the last row of BH' is zero. Assume that $BH' \neq 0$ and let K be the index of the last nonzero row of BH'. Partition B by

$$B = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix},$$

where B_{11} is $k \times k$, and partition H' and BH' conformably. Since $[BH']_{22} = 0$ and $[BH']_{21} = 0$ it follows from the fact that $BH' + H'B^* \geq 0$ that $[BH']_{12} = 0$.

Hence

$$(\mathrm{Row}_k B_{11})H'_{12} + (\mathrm{Row}_k B_{12})H'_{22} = \mathrm{Row}_k[BH']_{12} = 0. \tag{4.3}$$

Since

$$\mathrm{Row}_k B_{11} = 0 \tag{4.4}$$

it follows from (4.3) that

$$(\mathrm{Row}_k B_{12})H'_{22} = 0. \tag{4.5}$$

It is a well known property of positive semidefinite matrices, e.g. [7], that (4.5) implies

$$(\mathrm{Row}_k B_{12})H'_{21} = 0. \tag{4.6}$$

Therefore, by (4.4) and (4.6), we have

$$\mathrm{Row}_k[BH']_{11} = (\mathrm{Row}_k B_{11})H'_{11} + (\mathrm{Row}_k B_{12})H'_{21} = 0.$$

Since $[BH']_{12} = 0$ it follows that $\mathrm{Row}_k(BH') = 0$, which is a contradiction. Hence, $BH' = 0$ and so

$$(A - \lambda I)H = UBH'U^* = 0. \qquad \square$$

THEOREM 4.7. Let

$$A = \bigoplus_{i=1}^{q+2} A_i \in \mathbb{C}^{nn}$$

where

$$\operatorname{spec}(A_i) = \{\lambda_i\}, \quad \operatorname{Re} \lambda_i = 0, \quad i = 1,\ldots,q$$

and

$$\lambda_i \neq \lambda_j, \quad i \neq j, \quad 1 \leq i, j \leq q.$$

where A_{q+1} and $-A_{q+2}$ are stable.

Let $H \geq 0$ be partitioned conformably. Then H is a semifactor of A if and only if

$$H = \bigoplus_{i=1}^{q+2} H_{ii} \tag{4.8}$$

where

$$(A_i - \lambda_i I)H_{ii} = 0, \qquad i = 1,\ldots,q, \tag{4.9}$$

$$A_{q+1}H_{q+1,q+1} + H_{q+1,q+1}A^*_{q+1} \geq 0, \tag{4.10}$$

and

$$H_{q+2,q+2} = 0. \tag{4.11}$$

PROOF. If H satisfies (4.8) - (4.11) then clearly

$$AH + HA \geq 0.$$

Conversely, let $AH + HA^* \geq 0$ for some $H \geq 0$. By Lemma 4.1, condition (4.9) holds. Condition (4.10) is obvious and by Lemma 1 of [2], we obtain (4.11). To complete the proof we must show that $H_{ij} = 0$, $i \neq j$, $i,j = 1,\ldots,q + 2$. If $i = q + 2$ then $H_{ij} = 0$ and $H_{ji} = 0$, $j = 1,\ldots,q + 1$, since $H_{ii} = 0$ and $H \geq 0$. Let $i \neq j$, $1 \leq i, j \leq q + 1$. Without loss of generality we may assume that $1 \leq i \leq q$. By (4.9),

$$(AH + HA^*)_{ii} = A_iH_{ii} + H_{ii}A_i^* = 0.$$

Since $AH + HA^* \geq 0$ it follows that

$$A_iH_{ij} + H_{ij}A^*_j = A_iH_{ij} + H^*_{ji}A^*_j = (AH + HA^*)_{ij} = 0.$$

Hence, by [5, Vol. I, p. 220], $H_{ij} = 0$ since $\lambda_i + \bar{\mu} \neq 0$, whenever $\mu \in \operatorname{spec}(A_j)$. □

Let $A \in \mathbb{C}^{nn}$ and let α be a complex number. We define the <u>eigenspace</u> $E_\alpha(A)$ as the nullspace of $(A - \alpha I)$ and we define the <u>generalized eigenspace</u> $G_\alpha(A)$ as the nullspace of $(A - \alpha I)^n$. Observe that $E_\alpha(A) \neq \{0\}$ and $G_\alpha(A) \neq \{0\}$ if and only if α is an eigenvalue of A. We denote by $G_+(A)$ the vector space sum

$$\sum_{\substack{\alpha \in \operatorname{spec}(A) \\ \operatorname{Re} \alpha > 0}} G_\alpha(A).$$

The following theorem is an invariant formulation of the results of Theorem 4.7.

<u>THEOREM 4.12</u>. Let $A \in \mathbb{C}^{nn}$ and let $H \geq 0$. Then H is a semifactor of A if and only if

$$H = \sum_{\substack{\alpha \in \operatorname{spec}(A) \\ \operatorname{Re} \alpha > 0}} H_\alpha + H_+, \tag{4.13}$$

where

$$\operatorname{Range} H_\alpha \subseteq E_\alpha(A), \tag{4.14}$$

$$\operatorname{Range} H_+ \subseteq G_+(A), \tag{4.15}$$

and

$$AH_+ + H_+A^* \geq 0. \tag{4.16}$$

<u>PROOF</u>. First suppose that A satisfies the hypotheses of Theorem 4.7. Suppose that $AH + HA^* \geq 0$. Then, by Theorem 4.7, H satisfies (4.8) - (4.11). Let

$$H_{\lambda_i} = \bigoplus_{j=1}^{q+2} \delta_{ij} H_{jj}, \qquad i = 1,\ldots q, \tag{4.17}$$

where δ_{ij} is the Kronecker delta, and let

$$H_+ = \bigoplus_{j=1}^{q+2} \delta_{q+1,j} H_{jj}. \tag{4.18}$$

Then H has the form (4.13). Observe that (4.14) follows from (4.9) and (4.17), and that (4.15) and (4.16) follow from (4.10) and (4.18) since $G_+(A)$ is spanned by the columns of

$$\bigoplus_{j=1}^{q+2} \delta_{q+1,j} I .$$

Conversely, assume that (4.13) - (4.16) hold. In view of the form of A it follows that there exist H_{ii}, $i = 1,\dots q+1$ satisfying (4.8) - (4.11). Hence, by Theorem 4.7, $AH + HA^* \geq 0$.

Now suppose that A is a general matrix in $\mathbb{C}^{nn}$. Then there exists a nonsingular matrix T such that $A' = TAT^{-1}$ satisfies the hypotheses of Theorem 4.7. Let $H' = THT^*$. Then clearly $AH + HA^* \geq 0$ if and only if

$$A'H' + H'(A')^* \geq 0. \tag{4.19}$$

By the discussion above, (4.19) holds if and only if H' satisfies

$$H' = \sum_{\substack{\alpha \in \operatorname{spec}(A) \\ \operatorname{Re}\alpha=0}} H'_\alpha + H'_+ ,$$

where

$$\operatorname{Range} H_\alpha' \subseteq E_\alpha(A'), \tag{4.21}$$

$$\operatorname{Range} H_+' \subseteq G_+(A'), \tag{4.22}$$

and

$$A'H_+' + H_+'(A')^* \geq 0. \tag{4.23}$$

Since for any matrix B

$$\operatorname{Range}[T^{-1}B(T^*)^{-1}] = T^{-1}\operatorname{Range} B,$$

and since

$$E_\alpha(A) = T^{-1}E_\alpha(A'), \quad G_+(A) = T^{-1}G_+(A'),$$

(4.20) - (4.23) are equivalent respectively to (4.13) - (4.16) whenever

$$H_\alpha = T^{-1}H_\alpha'(T^*)^{-1}$$

and

$$H_+ = T^{-1}H_+'(T^*)^{-1}. \qquad \square$$

Remark. Observe that (4.14) is equivalent to $AH_\alpha = \alpha H_\alpha$.

COROLLARY 4.24. Let $A \in \mathbb{C}^{nn}$ and let $H \geq 0$. If $AH + HA^* \geq 0$ then

$$\text{Range } (AH + HA^*) \subseteq G_+(A).$$

PROOF. By Theorem 4.12, H satisfies (4.13) - (4.16). Hence

$$AH_\alpha + H_\alpha A^* = 0, \quad \alpha \in \text{spec}(A), \quad \text{Re } \alpha = 0,$$

and it follows that

$$AH + HA^* = AH_+ + H_+A^*. \tag{4.25}$$

Since

$$AG_+(A) \subseteq G_+(A),$$

it follows from (4.15) and (4.25) that

$$\text{Range } (AH + HA^*) \subseteq G_+(A). \qquad \square$$

The following corollaries are derived either from Theorem 4.7 or Theorem 4.12.

COROLLARY 4.26. Let $A \in \mathbb{C}^{n,n}$. Then $-A$ is semistable if and only if $H \geq 0$ and $AH + HA^* \geq 0$ imply that $AH + HA^* = 0$.

PROOF. If $-A$ is semistable then $G_+(A) = \{0\}$ and our claim follows from Corollary 4.24. Conversely, if $-A$ is not semistable then let $\alpha \in \text{spec}(A)$ such that $\text{Re } \alpha > 0$. Let x be an eigenvector of A associated with α. Observe that the positive semidefinite matrix $H = xx^*$ satisfies

$$AH + HA^* = 2 \text{ Re}(\alpha) xx^*$$

which is a nonzero positive semidefinite matrix. $\square$

COROLLARY 4.27. Let $B \in \mathbb{C}^{nn}$. Then B has no pure imaginary eigenvalue if and only if $K \geq 0$ and $BK + KB^* = 0$ imply that $K = 0$.

PROOF. There exists a nonsingular T such that $A = TBT^{-1}$ satisfies the hypotheses of Theorem 4.7. It is enough to prove that A has no pure imaginary eigenvalue if and only if $H \geq 0$ and $AH + HA^* = 0$ imply that

$H = 0$. If A has an imaginary eigenvalue and x is a corresponding eigenvector, then $xx^* \neq 0$ and $A(xx^*) + (xx^*)A^* = 0$. Conversely, suppose that A has no pure imaginary eigenvalue and that $H \geq 0$ satisfies $AH + HA^* = 0$. Then $A = A_1 \oplus A_2$, where the spectrum of A_1 and of $-A_2$ are in the open right halfplane. By Theorem 4.7, $H = H_{11} \oplus H_{22}$, where $H_{22} = 0$ and $A_1H_{11} + H_{11}A_1^* = 0$. But, since $\alpha + \bar{\beta} \neq 0$ for every $\alpha, \beta \in \operatorname{spec}(A_1)$, it follows from [5, Vol. I, p. 220] that $H_{11} = 0$. Hence $H = 0$. □

As a direct consequence of (4.13) - (4.15) we obtain the first part of Theorem III of [4].

COROLLARY 4.28. Let $A \in \mathbb{C}^{nn}$. If $H \geq 0$ and $AH + HA^* \geq 0$ then

$$\operatorname{rank} H \leq p(A) + \pi(A), \tag{4.29}$$

where $p(A)$ is the number of elementary divisors of the pure imaginary roots of A and $\pi(A)$ is the number of eigenvalues of A (counting multiplicities) which lie in the open right halfplane.

The following immediate corollary strengthens Corollary III.1 of [4], (our Theorem 1.3) by characterizing semistability factors.

COROLLARY 4.30. Let $A \in \mathbb{C}^{nn}$ and let $H \geq 0$. Then H is a semistability factor of A if and only if A is semistable, (1.5) holds and H satisfies (4.13), (4.16) and

$$\operatorname{Range} H_\alpha = E_\alpha(A),$$

$$\operatorname{Range} H_+ = G_+(A).$$

5. FACTOR CONES AND SEMIFACTOR CONES.

Let $\mathcal{H}$ be the vector space over the real field consisting of complex $n \times n$ hermitian matrices. As in [2, p. 2] a cone $\mathcal{C}$ in $\mathcal{H}$ is a nonempty subset $\mathcal{H}$ of which is closed under addition as well as nonnegative scalar multiplication. By span $\mathcal{C}$ we denote the subspace spanned by $\mathcal{C}$ and we write $\dim \mathcal{C}$ for the dimension of span $\mathcal{C}$.

The (semistability) factor cone $\mathcal{S}(A)$ is defined by

$$\mathcal{S}(A) = \{0\} \cup \{H > 0 : AH + HA^* \geq 0\}$$

and the (semistability) semifactor cone $\mathcal{S}_0(A)$ is given by

$$\mathcal{S}_0(A) = \{H \geq 0 : AH + HA^* \geq 0\}.$$

We observe that the closure of $\mathcal{S}(A)$ (in the usual Euclidean topology), denoted by cl $\mathcal{S}(A)$, satisfies

$$\text{cl } \mathcal{S}(A) \subseteq \mathcal{S}_0(A). \tag{5.1}$$

The reverse inclusion does not necessarily hold as demonstrated by

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

By Theorem 1.3 we have $\mathcal{S}(A) = \{0\}$, while the positive semidefinite matrix,

$$H = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

satisfies $AH + HA^* \geq 0$. However, let

$$\mathcal{T}_0^k(A) = \{H \in \mathcal{S}_0(A) : \text{rank } H = k\},$$

and let

$$r_A = \max_{H \in \mathcal{S}_0(A)} (\text{rank } H).$$

Observe that by Theorem III of [4] the set $\mathcal{T}_0^k(A)$ is nonempty if and only if $0 \leq k \leq r_A$. Then we have:

LEMMA 5.2. Let k be a nonnegative integer. Then

$$\mathcal{S}_0(A) = \text{cl } \mathcal{T}_0^k(A)$$

if and only if $k = r_A$.

PROOF. Clearly $\mathcal{S}_0(A)$ is a closed set. Hence, for all k, cl $\mathcal{T}_0^k(A) \subseteq \mathcal{S}_0(A)$.

If $k < r(A)$ then, since the limit of rank k matrices has rank less than or equal to k, and since $\mathcal{S}_0(A)$ contains a matrix of rank r_A, it follows that $\mathcal{S}_0(A) \neq$ cl $\mathcal{T}_0^k(A)$.

If $k = r(A)$ then let $H \in \mathcal{S}_0(A)$ and choose $K \in \mathcal{T}_0^k(A)$. Then K has a positive definite principal submatrix of order r_A. It follows that the corresponding submatrix of $H + \varepsilon K$ is nonsingular for all $\varepsilon > 0$. Hence

$$\text{(5.3)} \qquad \operatorname{rank}(H + \varepsilon K) \geq r_A, \qquad \forall \varepsilon > 0.$$

By the definition of R_A, equality holds in (5.3). Thus $H \in \operatorname{cl} \mathcal{T}_0^k(A)$. □

<u>LEMMA 5.4</u>. Let k be a nonnegative integer. Then

$$\text{(5.5)} \qquad \mathcal{S}_0(A) = \operatorname{cl}(\{0\} \cup \mathcal{T}_0^k(A))$$

if and only if either $\mathcal{S}_0(A) = \{0\}$ or $k = r_A$.

<u>PROOF</u>. If $\mathcal{T}_0^k(A) = \emptyset$ then (5.5) hold if and only if $\mathcal{S}_0(A) = \{0\}$. If $\mathcal{T}_0^k(A) \neq \emptyset$, then $\operatorname{cl}(\{0\} \cup \mathcal{T}_0^k(A)) = \operatorname{cl} \mathcal{T}_0^k(A)$ and by Lemma 5.2, (5.5) holds if and only if $k = r_A$. □

<u>PROPOSITION 5.6</u>. Let A be a complex $n \times n$ matrix. Then

$$\text{(5.7)} \qquad \mathcal{S}_0(A) = \operatorname{cl} \mathcal{S}(A)$$

if and only if either $-A$ is stable or A is semistable and the elementary divisors of all pure imaginary eigenvalues of A are linear.

<u>PROOF</u>. By Theorem III of [4],

$$\text{(5.8)} \qquad r_A = \pi(A) + p(A),$$

(where $\pi(A)$ and $p(A)$ are defined in Corollary 4.28). Since $\mathcal{S}(A) = \{0\} \cup \mathcal{T}_0^n(A)$ it follows from Lemma 5.4 that (5.7) holds if and only if either $\mathcal{S}_0(A) = \{0\}$ or $r_A = n$. By (5.8), $\mathcal{S}_0(A) = \{0\}$ if and only if $-A$ is stable, and, by Theorem 1.3, $r_A = n$ if and only if A is semistable and (1.5) holds. □

<u>COROLLARY 5.9</u>. Let A be a complex matrix. If $\mathcal{S}(A) \neq \{0\}$ then $\operatorname{cl} \mathcal{S}(A) = \mathcal{S}_0(A)$.

<u>PROOF</u>. If $\mathcal{S}(A) \neq \{0\}$ then $r_A = n$ and since $\mathcal{S}(A) = \mathcal{T}_0^n(A) \cup \{0\}$ our claim follows from Lemma 5.4. □

<u>THEOREM 5.10</u>. Let $A \in \mathbb{C}^{nn}$. Then

$$\dim \mathcal{S}_0(A) = \frac{1}{2}\Big[\pi(A)(\pi(A)+1) + \sum_{\substack{\alpha \in \operatorname{spec}(A) \\ \operatorname{Re}\alpha = 0}} e_\alpha(A)(e_\alpha(A)+1)\Big]$$

where $e_\alpha(A)$ is the dimension of the eigenspace $E_\alpha(A)$.

PROOF. Without loss of generality we may assume that A is in its Jordan canonical form and satisfies the hypotheses of Theorem 4.7. By Theorem 4.7, $\mathcal{S}_0(A)$ consists of all positive semidefinite matrices H for which (4.8) - (4.11) hold. Let $1 \leq i \leq q$. By (4.9), H_{ii} is any positive semidefinite matrix all of whose entries are 0, except possibly those which lie in the intersections of first rows and first columns of Jordan blocks which correspond to the eigenvalue λ_i. The number of such rows (columns) is $e_{\lambda_i}(A)$. Hence the number of such linear independent H_{ii} is $(e_{\lambda_i}(A)(e_{\lambda_i}(A) + 1)/2$.

We must still show that the number of linearly independent positive semidefinite matrices $H_{q+1,q+1}$ which satisfy (4.10) is $\pi(A)(\pi(A) + 1)/2$. By Theorem 1.1 there exists a positive definite matrix $H_{q+1,q+1}$ such that

$$A_{q+1}H_{q+1,q+1} + H_{q+1,q+1}A^*_{q+1} > 0.$$

Hence by continuity arguments, there exists an open set of such matrices which satisfy (4.10). As is well known, the span of these matrices (over the real field) is the whole set of hermitian matrices of order $\pi(A)$. Our theorem now follows. □

COROLLARY 5.11. (and proof No. 3). Let $A \in \mathbb{C}^{nn}$, $n > 1$, be such that $\mathcal{S}(A) \neq \{0\}$. Then

$$\dim \mathcal{S}(A) = \dim \mathcal{S}_0(A) > 1.$$

PROOF. Since $\mathcal{S}(A) \neq \{0\}$ it follows that $r_A = n$ and also, by Corollary 5.9, $\mathrm{cl}\ \mathcal{S}(A) = \mathcal{S}_0(A)$. Hence, as is well known, $\dim \mathcal{S}(A) = \dim \mathcal{S}_0(A)$. By (5.8) one of the following three possibilities holds:

(i) $\pi(A) \geq 2$.

(ii) $\pi(A) = 1$ and $p(A) \geq 1$.

(iii) $\pi(A) = 0$ and $p(A) \geq 2$.

In each case $\dim \mathcal{S}_0(A) > 1$ by Theorem 5.10. □

We remark that using essentially the same arguments we can show that there exist at least two linearly independent semifactors of A of rank r_A provided that $r_A > 1$.

BIBLIOGRAPHY

1. S. Avraham and R. Loewy, On the inertia of the Lyapunov transform $AH + HA^*$ for $H > 0$, Lin. Mult. Alg. 6(1978), 1-21.

2. A. Berman and R. J. Plemmons, "Nonnegative matrices in the mathematical sciences", Academic Press, N.Y. 1979.

3. D. Carlson, Rank and inertia bounds for matrices under $R(AH) \geq 0$, J. Math. Anal. Appl. 10(1965), 100-111.

4. D. Carlson and H. Schneider, Inertia theorems for matrices: the semidefinite case, J. Math. Anal. Appl. 6(1963), 430-446.

5. F. R. Gantmacher, "The Theory of Matrices", Vols. I,II. Chelsea, New York, 1959.

6. D. Hershkowitz and H. Schneider, Scalings of vector spaces and the uniqueness of Lyapunov scaling factors, Lin. Mult. Alg. (to appear).

7. D. Hershkowitz and H. Schneider, Lyapunov diagonal semistability of real H-matrices, Lin. Alg. Appl. (to appear).

8. A. Lyapunov, Problème Général de la stabilité du mouvement, Commun. Soc. Math. Kharkov (1892, 1893); Annals of Mathematical Studies, Vol. 17, Princeton Univ. Press, Princeton, N.J., 1947.

9. A. Ostrowski and H. Schneider, Some theorems on the inertia of general matrices, J. Math. Anal. Appl. 4(1962), 72-84.

10. O. Taussky, A generalization of a theorem by Lyapunov, J. Soc. Ind. Appl. Math. 9(1961), 640-643.

DEPARTMENT OF MATHEMATICS
UNIVERSITY OF WISCONSIN
MADISON, WI 53706