



Scalings of Vector Spaces and the Uniqueness of Lyapunov Scaling Factors

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Given two subspaces V_1 and V_2 of \mathbb{C}^n we prove necessary and sufficient conditions for the existence and for the uniqueness of a diagonal matrix D such that $V_1 = DV_2$. We apply these to prove a necessary condition and a sufficient condition for the uniqueness of a Lyapunov scaling factor of a given Lyapunov diagonally semistable matrix and we conjecture a necessary and sufficient condition.

1. INTRODUCTION

A real $n \times n$ matrix A is said to be Lyapunov diagonally semistable if there exists a positive diagonal matrix D (called a Lyapunov scaling factor of A) such that

$$AD + DA^T \text{ is positive semidefinite.} \quad (1.1)$$

Such matrices are also called Volterra–Lyapunov stable [6] and diagonally stable [1]. Lyapunov diagonally semistable matrices play an important role in some problems in ecology and economics, see [1], [6] and [7] and the references there. There have been several recent matrix theoretical papers devoted to their study, see among others [2, 3, 4, 5], in addition to the references above.

In this paper we discuss the following problems:

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PROBLEM 1 *Let A be a Lyapunov diagonally semistable matrix. Is the Lyapunov scaling factor of A unique?*

It will be shown that this problem is related to the following one.

PROBLEM 2 *Given two subspaces V_1 and V_2 of \mathbb{C}^n , does there exist a diagonal $n \times n$ matrix D such that $V_1 = DV_2$? If there exists such a D , is it unique?*

The connection between these problems is established in Lemmas 6.3 and 6.6, where we prove that if D satisfies (1.1) then the null space $N(A)$ of A equals $DN(A')$.

We now describe the contents of our paper in more detail. In section 2 we introduce some of the notations used in our paper and give some definitions.

In section 3 we characterize pairs of vector spaces V_1 and V_2 for which there exists a diagonal matrix D such that

$$V_1 = DV_2, \quad (1.2)$$

see Theorem 3.4. Our results are stated in terms of the column reduced echelon forms $E(V_1)$ and $E(V_2)$ of the given vector spaces. In particular we prove, Theorem 3.6, that there exists a nonsingular diagonal matrix D such that (1.2) holds if and only if $E(V_1)$ and $E(V_2)$ are diagonally equivalent.

In section 4 we characterize those cases where D is unique in (1.2). We show, Theorem 4.5, that uniqueness holds if and only if V_2 has full support and the bipartite graph of a certain submatrix of $E(V_1)$ is connected.

These results are applied in section 5. We answer the questions when, for given $m \times n$ matrices A and B , there exists a nonsingular matrix D such that $N(AD) = N(B)$, and when such a D is unique, see Theorem 5.2. We also state some corollaries.

The uniqueness of Lyapunov scaling factors is discussed in section 6. Suppose A is a Lyapunov diagonally semistable matrix. By means of Lemmas 6.3 and 6.6 we make use of the results of section 5 to identify principal submatrices of A which have unique Lyapunov scaling factors, see Proposition 6.8. We then employ an extension process to find further principal submatrices for which the Lyapunov scaling factor is unique, see Lemmas 6.9, 6.14 and 6.16. We obtain, Theorem 6.18, a sufficient (but not necessary) condition for the uniqueness of the Lyapunov scaling

factor of A in terms of the connectedness of a certain graph $U(A)$ associated with A . In Theorem 6.20 a necessary (but not sufficient) condition for uniqueness involving $U(A)$ is proved.

We have not proved a necessary and sufficient condition for the uniqueness of the Lyapunov scaling factor of a Lyapunov diagonally semistable matrix. However, in Conjecture 6.33 we conjecture such a condition. Earlier, we raise open Question 6.31 and Conjecture 6.32. If the answer to Question 6.31 is affirmative and if Conjecture 6.32 holds then Conjecture 6.33 is proved.

2. NOTATIONS AND DEFINITIONS

In this section we introduce some of the notations used in the sequel, as well as some definitions.

Notations 2.1 For positive integers m and n we denote by: $\langle n \rangle$, the set $\{1, 2, \dots, n\}$. $\mathbb{R}^{m,n}$, the set of all $m \times n$ real matrices. $\mathbb{C}^{m,n}$, the set of all $m \times n$ complex matrices. \mathbb{C}^n , the set $\mathbb{C}^{n,1}$.

Notations 2.2 Let $A \in \mathbb{C}^{m,n}$ and let $\alpha \subseteq \langle m \rangle$ and $\beta \subseteq \langle n \rangle$, $\alpha, \beta \neq \emptyset$. We denote by: $r(A)$, the rank of A . $n(A)$, the nullity of A ($n(A) = n - r(A)$). $\text{span}(A)$, the subspace of \mathbb{C}^m which is spanned by the columns of A . $\det(A)$, the determinant of A (in case that $m = n$). $A[\alpha|\beta]$, the submatrix of A whose rows are indexed by α and whose columns are indexed by β in their natural orders. $A[\alpha]$, the principal submatrix $A[\alpha|\alpha]$ (in case that $m = n$).

Definitions 2.3

i) An $n \times n$ matrix is called a *scalar matrix* if it is a scalar multiple of the identity matrix.

ii) A diagonal $n \times n$ matrix is said to be a *positive diagonal matrix* if it has positive diagonal entries.

iii) An $n \times n$ matrix is said to be a *P-matrix* [P_0 -matrix] if all of its principal minors are positive [nonnegative].

iv) Let $A, B \in \mathbb{C}^{m,n}$. The matrices A and B are said to be *diagonally equivalent* if there exist a nonsingular diagonal $m \times m$ matrix D and a nonsingular diagonal $n \times n$ matrix F such that $A = DBF$.

v) Let $A, B \in \mathbb{C}^{m,n}$. The matrices A and B are said to have the same *zero pattern* if

$$a_{ij} = 0 \Leftrightarrow b_{ij} = 0, \quad i = 1, \dots, m; j = 1, \dots, n.$$

Remarks 2.4

i) Whenever we refer to a diagonal $n \times n$ matrix D we let $d_i = d_{ii}$, $i = 1, \dots, n$.

ii) Whenever we say "unique" in this paper we mean "unique up to a nonzero scalar multiplication", except in Theorems 3.4, 3.6 and 3.9.

3. SCALINGS OF VECTOR SPACES

Let V be a subspace of \mathbb{C}^n , and let D be a diagonal $n \times n$ matrix. We denote by DV the subspace of \mathbb{C}^n defined as the set of all the vectors Dx , where x is an element of V . The vector space DV is said to be a *scaling* of the vector space V , and the matrix D is said to be the *scaling factor* from V to DV .

Let V_1 and V_2 be subspaces of \mathbb{C}^n . In this section we discuss necessary and sufficient conditions for the existence of a scaling factor D satisfying $V_1 = DV_2$.

Recall that a matrix A is said to be in column reduced echelon form if it has the following properties:

1. The zero columns of A , if any, come last.
2. The first nonzero element of each nonzero column is 1, and this element is the only nonzero element in its row.
3. If the first nonzero elements in the nonzero columns occur in positions $(c_1, 1)$, $(c_2, 2)$, \dots , (c_r, r) , then $c_1 < c_2 < \dots < c_r$.

In the sequel we will restrict ourselves to column reduced echelon forms which contain no zero columns.

Let V be an r -dimensional subspace of \mathbb{C}^n . Let $A, B \in \mathbb{C}^{n,r}$ be such that the columns of each matrix form a basis for V . It is well known that A and B have the same column reduced echelon form. Therefore we may refer to this form as the column reduced echelon form of V , and we denote it by $E(V)$.

Observe that if $V = \{0\}$ then it has an empty basis and hence $E(V)$ is an $n \times 0$ (empty) matrix. By convention the term "nonsingular matrix" includes the (empty) 0×0 matrix.

Definition 3.1 Let $x = [x_1, x_2, \dots, x_n]^T \in \mathbb{C}^n$. We define the *support* of x as the set

$$\{i : 1 \leq i \leq n, x_i \neq 0\},$$

and we denote it by $\text{sup}(x)$. For a subset S of \mathbb{C}^n we define the *support* of S as

$$\bigcup_{x \in S} \text{sup}(x),$$

and we denote it by $\text{sup}(S)$.

Let V be a subspace of \mathbb{C}^n , and let α be a subset of $\langle n \rangle$. Define a diagonal matrix $D_\alpha = \text{diag}\{(d_\alpha)_1, (d_\alpha)_2, \dots, (d_\alpha)_n\}$ by

$$(d_\alpha)_i = \begin{cases} 1, & i \in \alpha \\ 0, & i \notin \alpha \end{cases} \quad i = 1, \dots, n.$$

We denote by $V|_\alpha$, the subspace $D_\alpha V$. Observe that the elements of $V|_\alpha$ are obtained from those of V by making the components not indexed by α zero.

LEMMA 3.2 *Let V_1 and V_2 be subspaces of \mathbb{C}^n , and let D be a diagonal $n \times n$ matrix such that $V_1 = DV_2$. Then*

- (i) $\text{sup}(V_1) \subseteq \text{sup}(V_2)$,
- (ii) for all $i \in \text{sup}(V_1)$, $d_i \neq 0$,
- (iii) for all $i \in \text{sup}(V_2) \setminus \text{sup}(V_1)$, $d_i = 0$,
- (iv) $V_1 = D'V_2$ for any diagonal matrix $D' = \text{diag}\{d'_1, \dots, d'_n\}$ such that $d'_i = d_i$ for $i \in \text{sup}(V_2)$,
- (v) $V_1 = D(V_2|_{\text{sup}(V_1)})$.

Proof Straightforward. ■

LEMMA 3.3 *Let V_1 and V_2 be subspaces of \mathbb{C}^n , and let D be a diagonal $n \times n$ matrix such that*

$$V_1 = D(V_2|_{\text{sup}(V_1)})$$

and

$$d_i = 0 \quad \text{for all } i \in \text{sup}(V_2) \setminus \text{sup}(V_1),$$

Then $V_1 = DV_2$.

Proof Immediate. ■

We now introduce the characterization of the cases where one vector space is a scaling of another one.

THEOREM 3.4 *Let V_1 and V_2 be subspaces of \mathbb{C}^n , and let D be a diagonal $n \times n$ matrix. Then the following are equivalent.*

- (i) $V_1 = DV_2$.
- (ii) D satisfies

$$d_i = 0, \quad \text{for all } i \in \text{sup}(V_2) \setminus \text{sup}(V_1), \tag{3.5}$$

and $E(V_1) = DE(V_2|_{\text{sup}(V_1)})F$ for some nonsingular diagonal matrix F .

Furthermore, if (ii) holds, F is unique.

Proof (i) \Rightarrow (ii). By Lemma 3.2(iii), D satisfies (3.5). By Lemma 3.2(ii), the dimensions of V_1 and $V_2|_{\text{sup}(V_1)}$ are the same, say r . By Lemma 3.2(v), the r columns of $DE(V_2|_{\text{sup}(V_1)})$ span V_1 , and thus they form a basis for V_1 . Let $(c_1, 1), (c_2, 2), \dots, (c_r, r)$ be the positions of the first nonzero elements in the columns of $E(V_2|_{\text{sup}(V_1)})$. Since

$$c_i \in \text{sup}(V_1), \quad i = 1, \dots, r,$$

by Lemma 3.2(ii), $d_{c_i} \neq 0, i = 1, \dots, r$. It is easy to verify that

$$E(V_1) = DE(V_2|_{\text{sup}(V_1)})F,$$

if and only if $F = \text{diag}\{f_1, \dots, f_r\}$ is defined by

$$f_i = \frac{1}{d_{c_i}}, \quad i = 1, \dots, r.$$

This completes the proof of (ii) and also shows the uniqueness of F .

(ii) \Rightarrow (i). Since F is nonsingular, the columns of $DE(V_2|_{\text{sup}(V_1)})$ form a basis for V_1 . Hence $V_1 = D(V_2|_{\text{sup}(V_1)})$. Since D also satisfies (3.5), it follows from Lemma 3.3 that $V_1 = DV_2$. ■

As a corollary of Theorem 3.4 we have

THEOREM 3.6 *Let V_1 and V_2 be subspaces of \mathbb{C}^n , and let D be a nonsingular diagonal $n \times n$ matrix. Then the following are equivalent.*

- (i) $V_1 = DV_2$.
 - (ii) $E(V_1) = DE(V_2)F$ for some nonsingular diagonal matrix F .
- Furthermore, if (ii) holds, F is unique.

Proof (i) \Rightarrow (ii). Since $V_1 = DV_2$ and D is nonsingular, it follows from parts (i) and (iii) of Lemma 3.2 that $\text{sup}(V_1) = \text{sup}(V_2)$. The rest follows from Theorem 3.4 since $V_2|_{\text{sup}(V_2)} = V_2$.

(ii) \Rightarrow (i). Since D and F are nonsingular, it follows from $E(V_1) = DE(V_2)F$ that $\text{sup}(V_1) = \text{sup}(V_2)$. The claim now follows from Theorem 3.4.

The uniqueness of F also follows from Theorem 3.4. ■

If D is a nonsingular diagonal matrix, then it follows from Theorems 3.4 and 3.6 that

$$E(V_1) = DE(V_2|_{\text{sup}V_1})F_1 \quad \text{for some nonsingular diagonal matrix } F_1 \tag{3.7}$$

is equivalent to

$$E(V_1) = DE(V_2)F_2 \quad \text{for some nonsingular diagonal matrix } F_2. \tag{3.8}$$

Observe that even if D is singular, (3.7) and (3.8) are equivalent (actually the same) whenever $\text{sup}(V_1) = \text{sup}(V_2)$. In fact, even if $\text{sup}(V_1) \neq \text{sup}(V_2)$, since (3.8) implies that $V_1 = DV_2$, it follows from Theorem 3.4 that (3.8) implies (3.7). However, without the assumption $\text{sup}(V_1) = \text{sup}(V_2)$, (3.7) does not imply (3.8). This is clear when V_1 and V_2 have different dimensions. But it may also happen that (3.7) holds but not (3.8) when the dimensions of V_1 and V_2 are the same, as is demonstrated by the following example. Let

$$V_2 = \{[a, b, a + b]^T : a, b \in \mathbb{C}\}$$

and

$$V_1 = \{[a, 0, a + b]^T : a, b \in \mathbb{C}\}.$$

Clearly $V_1 = DV_2$, where $D = \text{diag}\{1, 0, 1\}$. Observe that

$$E(V_1) = E(V_2|_{\text{sup}V_1}) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix},$$

but

$$E(V_2) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix}.$$

In fact there exist no diagonal matrices F and G such that $E(V_1) = FE(V_2)G$.

However, we have the following generalization of Theorem 3.6.

THEOREM 3.9 *Let V_1 and V_2 be subspaces of \mathbb{C}^n , and let D be a diagonal $n \times n$ matrix. Then the following are equivalent.*

(i) $V_1 = DV_2$, and $DE(V_2)$ has a zero pattern of a column reduced echelon form.

(ii) $E(V_1) = DE(V_2)F$ for some nonsingular diagonal matrix F .
Furthermore, if (ii) holds, F is unique.

Proof (i) \Rightarrow (ii). The columns of $DE(V_2)$ span the vector space V_1 . Since $DE(V_2)$ has a zero pattern of a column reduced echelon form, there exists a nonsingular diagonal matrix F (which is uniquely determined by D) such that $E(V_1) = DE(V_2)F$.

(ii) \Rightarrow (i). Clearly, V_1 is spanned by the columns of $DE(V_2)$. Hence $V_1 = DV_2$. Since F is a nonsingular diagonal matrix, $E(V_1)$ and $DE(V_2)$ have the same zero pattern. Thus $DE(V_2)$ has a zero pattern of a column reduced echelon form. ■

Observe that when applying Theorem 3.9 for nonsingular D we get Theorem 3.6.

It now follows from Theorems 3.4 and 3.9 that if D satisfies (3.5) and if $DE(V_2)$ has a zero pattern of a column reduced echelon form, then (3.7) and (3.8) are equivalent. However, we remark that in this case the matrices F_1 and F_2 are not necessarily identical. As an example of a case where $F_1 \neq F_2$ let

$$V_2 = \{[a, 2a, b]^T : a, b \in \mathbb{C}\}$$

and

$$V_1 = \{[0, 2a, b]^T : a, b \in \mathbb{C}\}.$$

Clearly $V_1 = DV_2$, where $D = \text{diag}\{0, 1, 1\}$. Observe that

$$E(V_1) = E(V_2|_{\cup_{i \in I} V_1}) = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad E(V_2) = \begin{bmatrix} 1 & 0 \\ 2 & 0 \\ 0 & 1 \end{bmatrix}. \quad (3.10)$$

By (3.10) it follows that

$$F_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad F_2 = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & 1 \end{bmatrix}.$$

4. UNIQUENESS OF SCALINGS OF VECTOR SPACES

Let V_1 and V_2 be subspaces of \mathbb{C}^n , and let D be a diagonal $n \times n$ matrix such that $V_1 = DV_2$. The scaling factor D is not necessarily unique. For

example let $V_2 = V_1 = \{[a, b, a]^T : a, b \in \mathbb{C}\}$. Observe that $V_1 = DV_2$ for $D = \text{diag}\{1, d, 1\}$ where d is any nonzero scalar.

In this section we characterize the cases where the scaling factor is unique. We recall some graph-theoretical definitions.

Definition 4.1 Let G be a (nondirected) graph and let i and j be two vertices of G . We say that there exists a *path* between i and j in G if there exists a sequence i_1, \dots, i_k of vertices of G such that $i_1 = i$, $i_k = j$ and there is an edge between i_t and i_{t+1} , $t = 1, \dots, k-1$, in G . A set S of vertices of G is said to be *connected* in G if either S consists of one vertex or for any two vertices i and j in S there exists a path between i and j in G . A connected set S is said to be a *connected component* of G if there exists no path in G between any vertex in S and any vertex outside S (i.e. S is a maximal connected set of vertices). The graph G is said to be *connected* if the whole set of vertices of G is connected in G .

Definition 4.2 Let A be an $m \times n$ matrix. The (nondirected) *bipartite graph of A* , denoted by $H(A)$, is the bipartite graph for which the two sets of vertices are $\{1, \dots, m\}$ and $\{m+1, \dots, m+n\}$, and where there is an edge between i and $j+m$ ($1 \leq i \leq m$, $1 \leq j \leq n$) if and only if $a_{ij} \neq 0$.

Observe that if A is an $m \times 0$ matrix then $H(A)$ is connected if and only if $m = 0$ or $m = 1$.

THEOREM 4.3 *Let V be an r -dimensional subspace of \mathbb{C}^n , and let D be a diagonal $n \times n$ matrix. Then $V = DV$ implies that $D[\text{sup}(V)]$ is a scalar matrix if and only if $H(E(V)[\text{sup}(V)]\langle r \rangle)$ is connected.*

Proof Since $V|_{\text{sup}(V)} = V$, and since for all $i \in \langle n \rangle \setminus \text{sup}(V)$ the i th row of $E(V)$ is a zero row, it follows from Theorem 3.4 that $V = DV$ if and only if there exists a nonsingular diagonal $m \times m$ matrix F such that

$$E(V)[\text{sup}(V)]\langle r \rangle = D[\text{sup}(V)]E(V)[\text{sup}(V)]\langle r \rangle F. \quad (4.4)$$

Since (4.4) holds where D and F are replaced by identity matrices, and since, by Theorem 3.4, F is uniquely determined by D , $D[\text{sup}(V)]$ has to be a scalar matrix if and only if there is a unique diagonal equivalence between $E(V)[\text{sup}(V)]\langle r \rangle$ and itself. Observe that $E(V)[\text{sup}(V)]\langle r \rangle$ has neither zero rows nor zero columns. Thus, by Proposition 3.3 of [9], the diagonal equivalence (4.4) is unique if and only if $H(E(V)[\text{sup}(V)]\langle r \rangle)$ is connected. \blacksquare

We now introduce the characterization of the cases of unique scaling factors.

THEOREM 4.5 *Let V_1 and V_2 be subspaces of \mathbb{C}^n , and let D be a diagonal $n \times n$ matrix such that*

$$V_1 = DV_2. \quad (4.6)$$

Then the following are equivalent.

(i) *D is the unique diagonal matrix satisfying (4.6).*

(ii) *$\text{sup}(V_2) = \langle n \rangle$, and $H(E(V_1)[\text{sup}(V_1)|\langle r \rangle])$ is connected, where r is the dimension of V_1 .*

Proof (i) \Rightarrow (ii). By Lemma 3.2(iv), if D is the unique scaling factor satisfying (4.6) then $\text{sup}(V_2) = \langle r \rangle$. Let \hat{D} be any diagonal $n \times n$ matrix such that $V_1 = \hat{D}V_1$. Hence the matrix $\hat{D}D$ satisfies

$$V_1 = \hat{D}DV_2. \quad (4.7)$$

By Lemma 3.2(ii), it follows from (4.7) that $(\hat{D}D)[\text{sup}(V_1)]$ is nonsingular. Hence, since D is the unique scaling factor satisfying (4.6), (4.7) implies that $\hat{D}[\text{sup}(V_1)]$ is a scalar matrix. By Theorem 4.3 $H(E(V_1)[\text{sup}(V_1)|\langle r \rangle])$ is connected.

(ii) \Rightarrow (i). Let D and $G = \text{diag}\{g_1, \dots, g_n\}$ be scaling factors satisfying (4.6). By Lemma 3.2(ii), the matrices $D[\text{sup}(V_1)]$ and $G[\text{sup}(V_1)]$ are nonsingular. Define a diagonal matrix $\hat{D} = \text{diag}\{d_1, \dots, d_n\}$ by

$$\hat{d}_i = \begin{cases} \frac{d_i}{g_i}, & i \in \text{sup}(V_1) \\ 0, & \text{otherwise.} \end{cases}$$

Since $\text{sup}(V_2) = \langle n \rangle$ it follows from Lemma 3.2(iii) that

$$\hat{D}G = D. \quad (4.8)$$

Since $V_1 = DV_2$ and $V_1 = GV_2$ it follows from (4.8) that

$$V_1 = \hat{D}V_1. \quad (4.9)$$

Given that $H(E(V)[\text{sup}(V_1)|\langle r \rangle])$ is connected, (4.9) yields that $\hat{D}[\text{sup}(V_1)]$ is a scalar matrix by Theorem 4.3, which implies, by the definition of \hat{D} and by Lemma 3.2(iii), that G is a scalar multiple of D . ■

Observe that if $H(E(V_1))$ is connected then either $n = 1$ and $\text{sup}(V_1) = \emptyset$ or $\text{sup}(V_1) = \langle n \rangle$. As a consequence of Theorems 3.6 and 4.5 we obtain

THEOREM 4.10 *Let V_1 and V_2 be subspaces of \mathbb{C}^n . There exists a nonsingular diagonal $n \times n$ matrix D such that $V_1 = DV_2$ if and only if the column reduced echelon forms of V_1 and V_2 are diagonally equivalent. The scaling factor D is unique if and only if $H(E(V_1))$ is connected.*

We remark that Theorem 4.10 covers also the case where $V_1 = \{0\}$. However, for the sake of clarity we describe this case explicitly.

COROLLARY 4.11 *Let $V_1 = \{0\}$ and V_2 be subspaces of \mathbb{C}^n . There exists a nonsingular $n \times n$ matrix D such that $V_1 = DV_2$ if and only if $V_2 = \{0\}$. The scaling factor D is unique if and only if $n = 1$.*

To see that Corollary 4.11 is a special case of Theorem 4.10 observe that $E(V_1)$ is the $n \times 0$ matrix and hence $E(V_1)$ and $E(V_2)$ are diagonally equivalent if and only if $V_2 = \{0\}$. Further, $H(E(V_1))$ is connected if and only if $n = 1$.

5. SCALINGS OF NULL SPACES OF MATRICES

Let A be an $m \times n$ matrix. Denote by $N(A)$ the null space of A , i.e.

$$N(A) = \{x \in \mathbb{C}^n : Ax = 0\}.$$

The results of this section are direct applications of the results in the previous section. We answer the questions when, for given matrices A and B , there exists a nonsingular diagonal matrix D such that $N(AD) = N(B)$, and when such a D is unique. We then apply our solutions to some particular cases.

Observe that if $A \in \mathbb{C}^{m,n}$ and if $E \in \mathbb{C}^{n,n}$ is nonsingular then

$$N(AE) = E^{-1}N(A). \quad (5.1)$$

In view of (5.1), the next theorem follows immediately from Theorem 4.10.

THEOREM 5.2 *Let $A, B \in \mathbb{C}^{m,n}$. Then there exists a nonsingular diagonal $n \times n$ matrix D such that the null spaces of AD and B are the same if and only if the column reduced echelon forms of $N(A)$ and $N(B)$ are diagonally*

equivalent. The diagonal matrix D is unique if and only if $H(E(N(A)))$ is connected.

Let $A \in \mathbb{C}^{m,n}$. Denote by $r(A)$ the rank of A , and by $n(A)$ the nullity of A (the dimension of $N(A)$). Recall that $n(A) = n - r(A)$.

The following claims are corollaries of Theorem 5.2.

COROLLARY 5.3 *Let $A, B \in \mathbb{C}^{m,n}$ be such that $r(A) = r(B) = n - 1$. There exists a nonsingular diagonal $n \times n$ matrix D such that $N(AD) = N(B)$ if and only if $\text{sup}(N(A)) = \text{sup}(N(B))$. The scaling factor D is unique if and only if $\text{sup}(N(A)) = \langle n \rangle$.*

Proof Observe that $E(N(A))$ and $E(N(B))$ are column matrices. ■

COROLLARY 5.4 *Let $A, B \in \mathbb{C}^{m,n}$ be such that $r(A) = r(B) = n - 1$. If, for all $\alpha \subseteq \langle n \rangle$ such that $|\alpha| = n - 1$, we have $r(A[\langle m \rangle|\alpha]) = r(B[\langle m \rangle|\alpha]) = n - 1$, then there exists a unique diagonal $n \times n$ matrix D such that $N(AD) = N(B)$.*

Proof Let x be a nonzero vector in $N(A)$. Since $r(A[\langle m \rangle|\alpha]) = n - 1$ for all $\alpha \subseteq \langle n \rangle$ such that $|\alpha| = n - 1$, the support of x is $\langle n \rangle$. Hence, $\text{sup}(N(A)) = \langle n \rangle$. Similarly, $\text{sup}(N(B)) = \langle n \rangle$. Our corollary now follows from the previous one. ■

The following corollary is a consequence of Corollary 5.4.

COROLLARY 5.5 *Let $A \in \mathbb{C}^{n,n}$ be a singular matrix. If for all $\alpha \subseteq \langle n \rangle$ such that $|\alpha| = n - 1$, $A[\alpha]$ is nonsingular, then there exists a unique diagonal $n \times n$ matrix D such that $N(AD) = N(A^T)$.*

Another consequence of Theorem 5.2 is

PROPOSITION 5.6 *Let $A, B \in \mathbb{C}^{m,n}$, and let D be a nonsingular diagonal matrix such that $N(AD) = N(B)$. Let $r(A) = k < n$, and assume that every set of k columns of A is linearly independent. Then D is the unique diagonal matrix satisfying $N(AD) = N(B)$.*

Proof Denote by $\text{Col}_i(A)$ the i th column of A . Since $r(A) = k$, and since any k columns of A are linearly independent, there exist $k(n - k)$ nonzero numbers c_{ij} , $i = 1, \dots, n - k$, $j = n - k + 1, \dots, n$, such that

$$\text{Col}_i(A) = \sum_{j=n-k+1}^n c_{ij} \text{Col}_j(A), \quad i = 1, \dots, n - k. \quad (5.7)$$

Define $n - k$ vectors $x^i, i = 1, \dots, n - k$, by

$$x_j^i = \begin{cases} 1, & j = i \\ 0, & 1 \leq j \leq n - k, j \neq i \\ -c_{ij}, & n - k + 1 \leq j \leq n. \end{cases}$$

By (5.7), $x^i, i = 1, \dots, n - k$ are in $N(A)$. Clearly, they are linearly independent. Furthermore, the matrix whose i th column is x^i ($i = 1, \dots, n - k$), is exactly $E(N(A))$. Since its last k rows contain no zero elements, $H(E(N(A)))$ is connected, and our proposition follows from Theorem 5.2. \blacksquare

Observe from the proof of Proposition 5.6 that the assumption that any k columns of A are linearly independent can be weakened to the existence of a set of $k + 1$ columns such that any k of them are linearly independent.

6. UNIQUENESS OF LYAPUNOV SCALING FACTORS

Definition 6.1 Let $A \in \mathbb{R}^{n,n}$. The matrix A is said to be *Lyapunov diagonally stable* [*Lyapunov diagonally semistable*] if there exists a positive diagonal matrix D such that $AD + DA^T$ is positive definite [positive semidefinite]. In this case the matrix D is called a *Lyapunov scaling factor* of A .

Definition 6.2 Let $A \in \mathbb{R}^{n,n}$. The matrix A is said to be *Lyapunov diagonally near stable* if A is Lyapunov diagonally semistable but not Lyapunov diagonally stable.

In this section we discuss Lyapunov diagonally semistable matrices for which the Lyapunov scaling factor is unique. Important tools are our Lemmas 6.3 and 6.6 which relate Lyapunov scaling factors and scalings of vector spaces, and which enable us to apply the results introduced in the previous section.

Let A be a Lyapunov diagonally stable matrix and let D be a positive diagonal matrix such that $AD + DA^T$ is positive definite. Using continuity arguments it follows that every positive diagonal matrix which is close enough to D is a Lyapunov scaling factor of A . Hence, a Lyapunov scaling factor of a Lyapunov diagonally stable matrix is not unique. The Lyapunov scaling factor is not necessarily unique even in case of

Lyapunov diagonally near stable matrices, as demonstrated by the zero matrix and the following more interesting example. Let

$$A = \begin{bmatrix} 2 & 2 & 3 \\ 2 & 2 & 3 \\ 1 & 1 & 2 \end{bmatrix},$$

and let $D = \text{diag}\{1, 1, d\}$. Observe that

$$C = AD + DA^T = \begin{bmatrix} 4 & 4 & 1 + 3d \\ 4 & 4 & 1 + 3d \\ 1 + 3d & 1 + 3d & 4d \end{bmatrix},$$

It can easily be verified that C is positive semidefinite for any d , $\frac{1}{4} \leq d \leq 1$. Since a Lyapunov diagonally stable matrix has positive principal minors, e.g. [6], the matrix A is Lyapunov diagonally near stable, and its Lyapunov scaling factor is not unique.

LEMMA 6.3 *Let $A \in \mathbb{R}^{n,n}$ be a singular matrix such that $A + A^T$ is positive semidefinite. Then the null space of A is the same as the null space of A^T and it is contained in the null space of $A + A^T$.*

Proof It is enough to prove that $N(A)$ is contained in $N(A^T)$ and $N(A + A^T)$. Let the vector x satisfying

$$Ax = 0. \tag{6.4}$$

Clearly,

$$x^T A^T = 0.$$

The last two equalities imply

$$x^T (A + A^T)x = 0. \tag{6.5}$$

Since $A + A^T$ is positive semidefinite, it follows from (6.5) that

$$(A + A^T)x = 0,$$

and by (6.4)

$$A^T x = 0. \quad \blacksquare$$

We remark that a matrix A for which the null spaces of A and A^T are the same is known as an *EP*-matrix, see [10, p. 130].

If we apply Lemma 6.3 to AD , and we observe that for a nonsingular D , $N(DA^T) = N(A^T)$, we obtain the following connection between Lyapunov scaling factors and scalings of null spaces.

LEMMA 6.6 *Let $A \in \mathbb{R}^{n,n}$ be a singular Lyapunov diagonally semistable matrix and let D be a Lyapunov scaling factor of A . Then*

$$N(AD) = N(A^T) \subseteq N(AD + DA^T).$$

As an immediate consequence of Lemma 6.6 and Theorem 5.2 we obtain the following necessary condition for Lyapunov diagonal semistability of a singular matrix.

PROPOSITION 6.7 *Let $A \in \mathbb{R}^{n,n}$ be a singular Lyapunov diagonally semistable matrix. Then the column reduced echelon forms of $N(A)$ and $N(A^T)$ are diagonally equivalent.*

Let A be a Lyapunov diagonally semistable matrix. By Lemma 6.6, if there is a unique diagonal matrix S such that $N(AS) = N(A^T)$ then the Lyapunov scaling factor of A is also unique. By Theorem 5.2 we have

PROPOSITION 6.8 *Let $A \in \mathbb{R}^{n,n}$ be a singular Lyapunov diagonally semistable matrix. If $H(E(N(A)))$ is connected then the Lyapunov scaling factor of A is unique.*

An application of these results will be introduced in [7]. It will be proven, using Corollary 5.5 and Lemma 6.6, that an irreducible singular real H -matrix with nonnegative diagonal elements has a unique Lyapunov scaling factor.

Observe that if D is a Lyapunov scaling factor of an $n \times n$ matrix A then $D[\alpha]$ is a Lyapunov scaling factor of $A[\alpha]$ for any $\alpha \subseteq \langle n \rangle$.

LEMMA 6.9 *Let $A \in \mathbb{R}^{n,n}$ be such that $A + A^T$ is positive semidefinite. Let α be a subset of $\langle n \rangle$ such that $|\alpha| = n - 1$ and such that the identity matrix is the unique Lyapunov scaling factor of $A[\alpha]$, and let $\{j\} = \langle n \rangle \setminus \alpha$. If the column matrix $A[\alpha j]$ is not in $\text{span}(A[\alpha] + A[\alpha]^T)$ then the identity matrix is the unique Lyapunov scaling factor of A .*

Proof Let D be a Lyapunov scaling factor of A . Since $D[\alpha]$ is a Lyapunov scaling factor of $A[\alpha]$, it follows from the conditions of the lemma that $D[\alpha]$ is a scalar matrix. Without loss of generality we may assume that

$$d_i = 1, \quad \text{for all } i \in \alpha. \quad (6.10)$$

Since the identity matrix is the unique Lyapunov scaling factor of $A[\alpha]$, $A[\alpha]$ is necessarily Lyapunov diagonally near stable. Thus

$$\det((A + A^T)[\alpha]) = \det(A[\alpha] + A[\alpha]^T) = 0. \quad (6.11)$$

It is well known that if C is an $n \times n$ positive semidefinite matrix, and if for some $\alpha \subseteq \langle n \rangle$ the matrix $C[\alpha]$ is singular, then

$$C[\alpha|k] \in \text{span}(C[\alpha]), \quad k = 1, \dots, n.$$

Hence, since by (6.11) $(A + A^T)[\alpha]$ is a singular principal submatrix of the positive semidefinite matrix $A + A^T$, we have

$$A[\alpha|j] + A[j|\alpha]^T = (A + A^T)[\alpha|j] \in \text{span}(A[\alpha] + A[\alpha]^T). \quad (6.12)$$

By (6.10) $(A + A^T)[\alpha]$ is also a principal submatrix of the positive semidefinite matrix $AD + DA^T$. Hence,

$$\begin{aligned} d_j A[\alpha|j] + A[j|\alpha]^T &= d_j A[\alpha|j] + D[\alpha]A[j|\alpha]^T \\ &= (AD + DA^T)[\alpha|j] \in \text{span}(A[\alpha] + A[\alpha]^T). \end{aligned} \quad (6.13)$$

It follows from (6.12) and (6.13) that

$$(d_j - 1)A[\alpha|j] \in \text{span}(A[\alpha] + A[\alpha]^T),$$

but since it is given that

$$A[\alpha|j] \notin \text{span}(A[\alpha] + A[\alpha]^T)$$

it follows that $d_j = 1$. ■

Let A be an $n \times n$ matrix and let $\alpha \subseteq \langle n \rangle$, $\alpha \neq \emptyset$. We define a set $s(A, \alpha)$ by

$$s(A, \alpha) = \alpha \cup \{j : 1 \leq j \leq n, A[\alpha|j] \notin \text{span}(A[\alpha] + A[\alpha]^T)\}.$$

The following Lemma is a generalization of Lemma 6.9.

LEMMA 6.14 *Let $A \in \mathbb{R}^{n,n}$ be such that $A + A^T$ is positive semidefinite, and let α be a nonempty subset of $\langle n \rangle$ such that the identity matrix is the unique Lyapunov scaling factor of $A[\alpha]$. Then the identity matrix is the unique Lyapunov scaling factor of $A[s(A, \alpha)]$.*

Proof Let D be a Lyapunov scaling factor of A . Since $D[\alpha]$ is a Lyapunov scaling factor of $A[\alpha]$ we may assume, without loss of generality, that $D[\alpha]$ is an identity matrix. For any $j \in s(A, \alpha) \setminus \alpha$ we apply Lemma 6.9 to the matrix $A[\alpha \cup \{j\}]$ and we obtain $d_j = 1$. ■

Let A and α satisfy the conditions of Lemma 6.14. By Lemma 6.14 we may find a set $\beta = s(A, \alpha)$, $\alpha \subseteq \beta$, which satisfies the same conditions. We can now apply Lemma 6.14 to A and β and so on. Motivated by this observation we define a new set $\hat{s}(A, \alpha)$ by the following algorithm.

Algorithm 6.15

Initial step: Set $\beta = \alpha$.

Step $k, k = 2, 3, \dots$: If $\beta = s(A, \beta)$ then go to the final step. Otherwise set $\beta := s(A, \beta)$.

Final step: Set $\hat{s}(A, \alpha) = \beta$.

Observe that the algorithm terminates after a finite number of steps, since all the sets involved in the process are subsets of $\langle n \rangle$.

The following Lemma is now an immediate consequence of Lemma 6.14.

LEMMA 6.16 *Let $A \in \mathbb{R}^{n,n}$ be such that $A + A^T$ is positive semidefinite, and let α be a nonempty subset of $\langle n \rangle$ such that the identity matrix is the unique Lyapunov scaling factor of $A[\alpha]$. Then the identity matrix is the unique Lyapunov scaling factor of $A[\hat{s}(A, \alpha)]$.*

Example 6.17 Let

$$A = \begin{bmatrix} 1 & 1 & 2 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 2 & 1 & 2 \\ 2 & 2 & 0 & 1 \end{bmatrix}$$

Clearly,

$$A + A^T = \begin{bmatrix} 2 & 2 & 2 & 2 \\ 2 & 2 & 2 & 2 \\ 2 & 2 & 2 & 2 \\ 2 & 2 & 2 & 2 \end{bmatrix} \quad \text{is positive semidefinite.}$$

The only singular principal submatrices of A are $A[1, 2]$ and $A[1, 2, 4]$. Observe that

$$E_1 = E(N(A[1, 2])) = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

and

$$E_2 = E(N(A[1, 2, 4])) = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}.$$

Obviously $H(E_1)$ is connected and thus, by Proposition 6.8, the identity matrix is the unique Lyapunov scaling factor of $A[1, 2]$. On the other $H(E_2)$ is not connected, and it is easy to verify that $D = \text{diag}\{1, 1, d\}$ is a Lyapunov scaling factor of $A[1, 2, 4]$ whenever $d \geq 1$. Thus, $\alpha = \{1, 2\}$ is the only subset of $\langle n \rangle$ that satisfies the conditions of Lemma 6.16. Observe that

$$\beta = s(A, \alpha) = \{1, 2, 3\},$$

and

$$\hat{s}(A, \alpha) = s(A, \beta) = \{1, 2, 3, 4\}.$$

Hence, by Lemma 6.16, the identity is the unique Lyapunov scaling factor of A .

In order to further generalize Proposition 6.8 we now define a non-directed graph $U(A)$ associated with an $n \times n$ matrix A . The set of vertices of $U(A)$ is $\langle n \rangle$. For any i and j , $i \neq j$, $1 \leq i, j \leq n$ there is an edge between i and j if i and j belong to a set $\hat{s}(A, \alpha)$ where $A[\alpha]$ is singular and $H(E(N(A[\alpha])))$ is connected.

THEOREM 6.18 *Let $A \in \mathbb{R}^{n,n}$ be such that $A + A^T$ is positive semidefinite. If the graph $U(A)$ is connected then the identity matrix is the unique Lyapunov scaling factor of A .*

Proof Let D be a Lyapunov scaling factor of A , and let i and j , $i \neq j$, be any two indices between 1 and n . Since $U(A)$ is connected there exist k indices $1 \leq i_1, \dots, i_k \leq n$ such that $i_1 = i$, $i_k = j$ and such that there is an edge between i_t and i_{t+1} , $t = 1, \dots, k-1$ in $U(A)$. Choose t , $1 \leq t \leq k-1$. By the definition of $U(A)$, i_t and i_{t+1} belong to a set $\hat{s}(A, \alpha)$ where $A[\alpha]$ is singular and $H(E(N(A[\alpha])))$ is connected. Since $A + A^T$ is positive semidefinite it follows from Proposition 6.8 and Lemma 6.16 that

$$d_{i_t} = d_{i_{t+1}}. \quad (6.19)$$

Since (6.19) hold for all t , $1 \leq t \leq k-1$, we have $d_i = d_j$ and the proof is completed. ■

An example of matrix to which Theorem 6.18 applies is the following.
Let

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}.$$

Clearly, $A + A^T$ is positive semidefinite. Observe that for all sets $\alpha \subseteq \{1, 2, 3, 4\}$ such that $|\alpha| = 2$, the matrix $A[\alpha]$ is singular and $H(E(N(A[\alpha])))$ is connected. Furthermore, for any such α , $\alpha = \hat{s}(A, \alpha)$. The graph $U(A)$, which is shown below in Figure 1 is connected and hence, by Theorem 6.18 the identity matrix is the unique Lyapunov scaling factor of A .

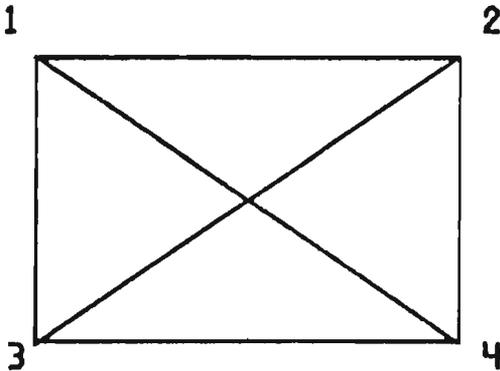


FIGURE 1.

We remark that the sufficient condition for uniqueness of Lyapunov scaling factor, introduced in Theorem 6.18 is not necessary as demonstrated by the matrix

$$A = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 2 \\ 2 & 0 & 1 \end{bmatrix}.$$

Here A is Lyapunov diagonally semistable since

$$A + A^T = \begin{bmatrix} 2 & 2 & 2 \\ 2 & 2 & 2 \\ 2 & 2 & 2 \end{bmatrix}.$$

Assume that for a positive diagonal matrix D

$$C = AD + DA^T = 2 \begin{bmatrix} d_1 & d_2 & d_1 \\ d_2 & d_2 & d_3 \\ d_1 & d_3 & d_3 \end{bmatrix}$$

is positive semidefinite. Then

$$\det C[1, 2] \geq 0 \Rightarrow d_1 \geq d_2,$$

$$\det C[1, 3] \geq 0 \Rightarrow d_3 \geq d_1,$$

$$\det C[2, 3] \geq 0 \Rightarrow d_2 \geq d_3.$$

These inequalities imply $d_1 = d_2 = d_3$. So, the identity matrix is the unique Lyapunov scaling factor of A , though A does not satisfy the conditions of Theorem 6.18 ($U(A)$ has no edges since A is a P -matrix).

We now prove a necessary condition for uniqueness of Lyapunov scaling factors.

THEOREM 6.20 *Let $A \in \mathbb{R}^{n \times n}$ be Lyapunov diagonally semistable. If the identity matrix is the unique Lyapunov scaling factor of A then A is irreducible and either the graph $U(A)$ is connected or for every connected component μ of $U(A)$ there exists a set $v \subseteq \langle n \rangle$ such that $v \cap \mu \neq \emptyset$, $v \setminus \mu \neq \emptyset$, $A[v]$ is a P -matrix and $A[v] + A[v]^T$ is singular.*

Proof Assume that A is reducible. Then A may be written in the following block form,

$$A = \begin{bmatrix} A_{11} & 0 \\ A_{21} & A_{22} \end{bmatrix},$$

where A_{11} is a square matrix of order k , $k < n$. Also, the matrices $A + A^T$ and $A_{22} + A_{22}^T$ are positive semidefinite. Define the diagonal D by

$$d_{ii} = \begin{cases} 1, & i \leq k \\ 2, & k < i. \end{cases}$$

Observe that the matrix

$$AD + DA^T = A + A^T + \begin{bmatrix} 0 & 0 \\ 0 & A_{22} + A_{22}^T \end{bmatrix}$$

is a sum of two positive semidefinite matrices and as such $AD + DA^T$ is also positive semidefinite. Hence, D is a Lyapunov scaling factor of A , a contradiction to uniqueness. Therefore A is irreducible.

Assume that $U(A)$ is not connected and let μ be a connected component of $U(A)$. Assume also that for every set $v \subseteq \langle n \rangle$ such that $v \cap \mu \neq \phi, v \setminus \mu \neq \phi$ and $A[v]$ is a P -matrix, the matrix $A[v] + A[v]^T$ is positive definite. Define a positive diagonal matrix $F = \text{diag}\{f_1, \dots, f_n\}$ by

$$f_i = \begin{cases} 1 + \varepsilon, & i \in \mu \\ 1, & i \in \langle n \rangle \setminus \mu, \end{cases} \tag{6.21}$$

where ε is a positive number. Observe that since $U(A)$ is not connected F is not a scalar matrix. Let $B = A + A^T, C = AF + FA^T$, and let $\alpha \subseteq \langle n \rangle$. To prove that $\det(C[\alpha]) \geq 0$ we distinguish between three possible cases.

Case 1 $\alpha \setminus \mu = \phi$. In this case

$$\det(C[\alpha]) = (1 + \varepsilon)^{|\alpha|} \det(B[\alpha]) \geq 0.$$

Case 2 $\alpha \cap \mu = \phi$. Here

$$\det(C[\alpha]) = \det(B[\alpha]) \geq 0.$$

Case 3 $\beta = \alpha \setminus \mu \neq \phi$ and $\gamma = \alpha \cap \mu \neq \phi$. If $A[\alpha]$ is a P -matrix then by our assumption $\det(B[\alpha]) > 0$, and using continuity arguments it follows that for ε small enough $\det(C[\alpha]) > 0$. If $A[\alpha]$ is not a P -matrix then, since a Lyapunov diagonally semistable matrix is a P_0 -matrix, e.g. [6], $A[\alpha]$ has a singular principal submatrix. Let $A[\delta]$ be a singular principal submatrix of $A[\alpha]$ of minimal order, i.e. for all $\eta \subset \delta$ the matrix $A[\eta]$ is nonsingular (where $\det(A[\phi]) = 1$). As observed in Corollary 5.5, $H(E(N(A[\delta])))$ is connected and, hence, by the definition of $U(A)$, $\hat{s}(A, \delta)$ is connected in $U(A)$. Therefore, since μ is a connected component of $U(A)$, either

$$\hat{S}(A, \delta) \subseteq \beta, \tag{6.23}$$

or

$$\hat{S}(A, \delta) \subseteq \gamma. \tag{6.24}$$

Assume (6.23) holds. Since $A[\delta]$ is singular it follows from Lemma 6.3 that $B[\delta]$ is a singular principal submatrix of the positive semidefinite matrix B . Hence, as mentioned above

$$A[\delta|j] + A[j|\delta]^T = B[\delta|j] \in \text{span}(B[\delta]), \quad j = 1, \dots, n. \quad (6.25)$$

Let $k \in \mu$. By (6.23) $k \notin \hat{s}(A, \delta)$ and hence

$$A[\delta|k] \in \text{span}(B[\delta]), \quad \text{for all } k \in \mu. \quad (6.26)$$

By (6.25) and (6.26)

$$A[k|\delta]^T \in \text{span}(B[\delta]), \quad \text{for all } k \in \mu. \quad (6.27)$$

Since, by (6.23) $C[\delta] = B[\delta]$ and

$$C[\delta|j] = \begin{cases} (1 + \varepsilon)A[\delta|j] + A[j|\delta]^T, & \text{for all } j \in \mu \\ A[\delta|j] + A[j|\delta]^T, & \text{for all } j \in \langle n \rangle \setminus \mu, \end{cases}$$

it follows from (6.25), (6.26) and (6.27) that

$$C[\delta|j] \in \text{span}(C[\delta]), \quad j = 1, \dots, n. \quad (6.28)$$

Since $\det(C[\delta]) = \det(B[\delta]) = 0$, (6.28) implies that

$$r(C[\delta|\langle n \rangle]) = r(C[\delta]) < |\delta|. \quad (6.29)$$

Since $\delta \subseteq \alpha$ it follows from (6.29) that

$$\det(C[\alpha]) = 0. \quad (6.30)$$

In case that (6.24) holds we obtain (6.30) in a very similar way.

We have proven that C is positive semidefinite. Thus F is a Lyapunov scaling factor of A which is not a scalar matrix. This is a contradiction to the hypothesis of the theorem. Hence, in case that $U(A)$ is not connected, for every connected component μ of $U(A)$ there exists a set $v \subseteq \langle n \rangle$ such that $v \cap \mu \neq \emptyset$, $v \setminus \mu \neq \emptyset$, $A[v]$ is a P -matrix and $A[v] + A[v]^T$ is singular. \blacksquare

The necessary condition for uniqueness of Lyapunov scaling factors, introduced in Theorem 6.20 is not sufficient as demonstrated by the example discussed in the beginning of the section. It is easy to verify that

for

$$A = \begin{bmatrix} 2 & 2 & 3 \\ 2 & 2 & 3 \\ 1 & 1 & 2 \end{bmatrix}$$

the graph $U(A)$ has two components: $\{1, 2\}$ and $\{3\}$. Observe that if $v = \{1, 3\}$ (or $\{2, 3\}$), then $A[v]$ is a P -matrix and $A[v] + A[v]^T$ is singular. Hence the necessary conditions of Theorem 6.20 is satisfied. However, as shown above, the Lyapunov scaling factor of A is not unique.

Remark We would like to emphasize that although Theorems 6.18 and 6.20 are stated for matrices A such that $A + A^T$ is positive semidefinite, they provide a sufficient condition and a necessary condition for the uniqueness of the Lyapunov scaling factor for any Lyapunov diagonally semistable matrix. If D is a Lyapunov scaling factor of the matrix A then we simply apply Theorems 6.18 and 6.20 to AD .

We conclude the paper with a question and two conjectures that seem to be related.

QUESTION 6.31 *Can one strengthen Theorem 6.20 by replacing the phrase " $A[v] + A[v]^T$ is singular" (at the end of the theorem) by " $A[v]$ is Lyapunov diagonally near stable"?*

CONJECTURE 6.32 *Let $A \in \mathbb{R}^{n,n}$ be a Lyapunov diagonally near stable irreducible P -matrix. Then A has a unique Lyapunov scaling factor.*

If Conjecture 6.32 is true then we may define a graph $U'(A)$ which is the graph $U(A)$ with the following additional edges: We add an edge between i and j if i and j belong to a set $S(A, \alpha)$ where $A[\alpha]$ is a Lyapunov diagonally near stable P -matrix. We may then replace $U(A)$ by $U'(A)$ in Theorems 6.18 and 6.20. Furthermore, if the answer to Question 6.31 is positive, then we have a proof of the following conjecture.

CONJECTURE 6.33 *Let $A \in \mathbb{R}^{n,n}$ be such that $A + A^T$ is positive semidefinite. Then the identity matrix is the unique Lyapunov scaling factor of A if and only if the graph $U'(A)$ is connected.*

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