72

Applications of Shortest Path Algorithms to Matrix Scalings*

Dedicated to Professor F.L. Bauer on the occasion of his 60th birthday

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Summary. A symmetric scaling of a nonnegative, square matrix A is a matrix XAX^{-1}, where X is a nonsingular, nonnegative diagonal matrix. By associating a family of weighted directed graphs with the matrix A we are able to adapt the shortest path algorithms to compute an optimal scaling of A, where we call a symmetric scaling A' of A optimal if it minimizes the maximum of the ratio of non-zero elements.

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1. Introduction

The scaling of matrices has been considered from several points of view. At one extreme some papers concentrate on theoretical characterizations, e.g. [12] or [18], and at the other papers report on the efficiency of algorithms, e.g. Tomlin [22]. This paper falls between these extremes for we are concerned with the development of algorithms which may however be stated as mathematical theorems.

We now briefly describe our paper. After some preliminaries in Sect. 2 we turn in Sect. 3 to the construction of an algorithm for a problem solved theoretically in [12]: Given nonnegative square matrices A, B, C when does there exist a symmetric scaling $A' = XAX^{-1}$ of A (where X is diagonal) for which $C \leq A' \leq B$? Our algorithm is related to Karp's algorithm for finding the minimum cycle mean in a weighted directed graph, see [14], and its first version was developed by one of us in [10, 11] and is there applied in the consideration of asymmetric scalings: $A' = XAY$. In [11] the present form of the algorithm is given and is there used for the approximation of bivariate

functions by univariate ones. Here we apply the algorithm to give an algorith-
mic answer to the problem described above.

Our main results may be found in Sects. 4 and 5. We again turn to a
problem previously solved theoretically, see [18, Theorem 6]: Given a non-
negative matrix A characterize the symmetric scalings A' which minimize the
maximum of the ratio of non-zero elements. By associating a family of weight-
ed directed graphs with the matrix A we are able to adapt our algorithm to
compute an optimal scaling A'. In Sect. 6 we then give a new proof of the
theorem in [18] based on this algorithm.

By means of the bipartite expansion of a directed graph it is possible to
obtain from any result on symmetric scalings a corresponding result on asym-
metric scalings, see [19, Sect. 3]. The asymmetric formulation of some of our
results is given in Sect. 7.

2. Notations

In this paper, n will be a fixed positive integer,

$$V = \{1; 2; \ldots; n\} \quad \text{and} \quad E \subseteq V \times V, \quad E \neq \emptyset.$$

By $G = (V, E)$ we denote the directed graph with vertices V and arcs $e = (i, j) \in E$.

A sequence $\beta = (e_1, \ldots, e_k)$, $k \geq 1$, of arcs $e_s = (i_s, j_s) \in E$, $s = 1, \ldots, k$, is a path (of
length $|\beta| = k$ from i_1 to j_k) if $j_s = i_{s+1}$, $s = 1, \ldots, k-1$.

A path β is closed if $i_1 = j_k$, and a closed path β is a directed cycle if
$i_1, \ldots, i_k$ are pairwise distinct. We put

$$W_j^k := \{\beta : \beta \text{ is a path of length } k \text{ ending at } j\},$$
$$W^k := \{\beta : \beta \text{ is a path of length } k\},$$
$$W_j := \{\beta : \beta \text{ is a path ending at } j\},$$
$$Y := \{\beta : \beta \text{ is a closed path}\},$$
$$Z := \{\beta : \beta \text{ is a directed cycle}\}.$$

Finally, we say that (V, E, d) is a weighted directed graph with the weight
function $d : E \to \mathbb{R}$.

The weight $d(\beta)$ of a path $\beta = (e_1, \ldots, e_k)$ is defined by

$$d(\beta) := \sum_{s=1}^{k} d(e_s),$$

and

$$\lambda^* := \min_{\beta \in Y} \frac{d(\beta)}{|\beta|} = \min_{\beta \in Z} \frac{d(\beta)}{|\beta|}$$

is called the minimum cycle mean in the weighted directed graph (V, E, d).

Notice that loops $\beta = (i, i) \in E$ are elements of Z.

3. Fundamental Results

We now turn to the problem discussed in the introduction.

Problem 1. *Let $a_{ij} \in \mathbb{R}$, $b_{ij} \in \mathbb{R} \cup \{\infty\}$, $c_{ij} \in \mathbb{R} \cup \{-\infty\}$, $i, j = 1, 2, \ldots, n$. Do there exist $x_i \in \mathbb{R}$, $i = 1, \ldots, n$, for which*

$$c_{ij} \leqq x_i + a_{ij} - x_j \leqq b_{ij}, \quad i, j = 1, \ldots, n? \tag{3.1}$$

A special case of Problem 1 is the following.

Problem 2. *Let $E \subseteq V \times V$, $d_{ij} \in \mathbb{R}$ for $(i,j) \in E$. Do there exist $x_i \in \mathbb{R}$, $i = 1, \ldots, n$, for which*

$$0 \leqq x_i + d_{ij} - x_j, \quad (i,j) \in E? \tag{3.2}$$

Our next lemma shows that Problem 1 is equivalent to Problem 2, and can be solved with the same x_i, $i = 1, \ldots, n$, provided that E and the d_{ij} are properly chosen.

Lemma 1. *Let a_{ij}, b_{ij}, c_{ij} be as in Problem 1. Let*

$$d_{ij} := \min\{a_{ij} - c_{ij}; b_{ji} - a_{ji}\}, \quad i, j = 1, \ldots, n, \tag{3.3}$$

$$E := \{(i,j) : d_{ij} \neq \infty\}. \tag{3.4}$$

Then $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ solves (3.1) if and only if x solves (3.2).

Proof. The inequalities (3.1) are equivalent to the $2n^2$ inequalities

$$\begin{aligned}0 \leqq x_i + a_{ij} - c_{ij} - x_j, \quad i, j = 1, \ldots, n \\ 0 \leqq x_i + b_{ji} - a_{ji} - x_j, \quad i, j = 1, \ldots, n\end{aligned}$$

where the last relations are obtained by interchanging i and j. Thus we obtain the equivalent form

$$0 \leqq x_i + d_{ij} - x_j, \quad i, j = 1, \ldots, n.$$

Since the inequalities above are trivially satisfied if $d_{ij} = \infty$, they are equivalent to (3.2). □

Our algorithm for solving Problem 2 (and thus Problem 1) is based on the following theorem which states that Problem 2 is equivalent to the negative cycle problem in weighted directed graphs.

Theorem 1. *Let $E \subseteq V \times V$, $d(e) := d_{ij}$ for $e = (i,j) \in E$.*
  (a) *If there is a directed cycle β of negative weight $d(\beta)$ in the weighted directed graph (V, E, d), then Problem 2 has no solution.*
  (b) *Otherwise the vector $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ defined by*

$$x_j := \min\left\{0; d(\beta) : \beta \in \bigcup_{k=1}^{n} W_j^k\right\}, \quad j = 1, \ldots, n \tag{3.5}$$

is a solution of (3.2).

Proof. (a) Let $x_i \in \mathbb{R}$, $i = 1, \ldots, n$, and put $d'_{ij} := x_i + d_{ij} - x_j$, $(i, j) \in E$. Then for any directed cycle β in (V, E), $d(\beta) = d'(\beta)$. Hence Problem 2 has no solution if $d(\beta) < 0$ for some directed cycle β in (V, E).

(b) Let x_j be defined by (3.5), $j = 1, \ldots, n$. Since there is no directed cycle of negative weight in (V, E, d), we have

$$x_j = \min\{0; d(\beta) : \beta \in W_j\}, \quad j = 1, \ldots, n.$$

Suppose that $x_i = d(\beta_i)$ where $\beta_i \in W_i$. If $(i, j) \in E$ and γ is the path β_i followed by the arc (i, j), then $d(\gamma) = x_i + d_{ij} \geq x_j$, since $\gamma \in W_j$. Hence $x_i + d_{ij} - x_j \geq 0$ if $(i, j) \in E$. $\square$

Corollary 1. *The minimum cycle mean λ^* of the weighted directed graph (V, E, d) is the largest of the numbers λ for which the inequalities*

$$x_i + d_{ij} - x_j \geq \lambda, \quad (i, j) \in E \tag{3.6}$$

have a solution $x \in \mathbb{R}^n$.

Proof. Applying Theorem 1 for the weights $d'_{ij} := d_{ij} - \lambda$, $(i, j) \in E$, we realize that (3.6) has a solution x iff

$$d'(\beta) = d(\beta) - \lambda |\beta| \geq 0 \quad \text{for all } \beta \in Z,$$

and hence iff $\lambda \leq \min\limits_{\beta \in Z} \dfrac{d(\beta)}{|\beta|} = \lambda^*$. $\square$

Remark 1. The above results are partly well-known and have already appeared in the literature in various guises and forms, see e.g. [11, Theorem 1.1] and the detailed theoretical investigations of Problem 1 in [12]. The equivalence of Problem 1 and Problem 2, but in particular our Theorem 1 play an essential role in algorithms which are concerned with the planning and scheduling in MPM networks (MPM = Metra-Potential-Method), see e.g. [23, Part III, § 1.31 and § 1.32].

Lemma 1 and Theorem 1 give the following information: Any algorithm which determines if a weighted directed graph (V, E, d) contains a directed cycle of negative weight can be applied to Problem 1 and Problem 2. Efficient algorithms of this kind have been constructed by Ford and Bellman, by Yen, by Dantzig, Blattner, Rao, and by others, see e.g. [15].

Our Algorithm 1 below is a modification of the famous Ford-Bellman algorithm and a first version of Karp's algorithm for finding the minimum cycle mean in a weighted directed graph. Algorithm 1 has been suggested in the present form in [11].

Algorithm 1 (The Minimum Cycle Mean Problem)

Step 0. We suppose that the forward and backward stars of each vertex $j \in V$ are non-empty. Otherwise, we may erase such vertices.

Set $x_j(0) := 0$ for $j \in V$, $J(0) := \{1; \ldots, n\}$, $k := 1$.

Step 1. For $j \in V$, compute

$$x_j(k) := \min\{x_j(k-1);\ x_i(k-1)+d_{ij}:\ i \in J(k-1),\ (i,j) \in E\};$$
$$J(k) := \{j \in V : x_j(k) < x_j(k-1)\}. \tag{3.7}$$

For any $j \in J(k)$, choose one of the indices $i \in J(k-1)$, for which the minimum in (3.7) is attained, and call it P_{jk}.

If $J(k) = \emptyset$, put

$$x_j := x_j(k),\quad j = 1, \ldots, n.\ \text{Stop.} \tag{3.8}$$

If $k = n$ and $J(n) \neq \emptyset$, Stop (Problem 2 has no solution).

If $k < n$ and $J(k) \neq \emptyset$, put $k := k+1$ and return to Step 1.

Step 2. Suppose that all sets $J(k)$, $k = 1, \ldots, n$, are non-empty.

(a) Compute the minimum cycle mean λ^* of (V, E, d) by

$$\lambda^* = \min_{j \in J(n)}\ \max_{0 \leq k \leq n-1}\ \frac{x_j(n) - x_j(k)}{n-k}. \tag{3.9}$$

(b) Choose any $j \in J(n)$ for which the minimum in (3.9) is attained. Construct the sequence of vertices by the recurrence

$$j_n := j,\quad j_{k-1} := P_{j_k, k}\quad \text{for } k = n, n-1, \ldots \tag{3.10}$$

until the first time equality $j_r = j_s$ occurs for some r and s, $0 \leq r < s \leq n$. Then

$$\beta^* := \{(j_r, j_{r+1}), \ldots, (j_{s-1}, j_s)\} \tag{3.11}$$

is a directed cycle in (V, E) with $d(\beta^*)/|\beta^*| = \lambda^*$.

Remark 2. The properties of Algorithm 1 have been studied in [11] under the assumption that the directed graph (V, E) has no loops. In our paper, loops are admitted and belong to the set Z of directed cycles in (V, E). But it turns out easily that these loops don't cause any additional difficulties in the proofs of [11].

In more detail, the following properties of Algorithm 1 have been proved in [11; Theorem 2.1, pp. 68 ff.].

Theorem 2. (a) *The minimum cycle mean* λ^* *of* (V, E, d) *is non-negative if and only if there exists an index* k, $1 \leq k \leq n$, *for which the set* $J(k)$ *is empty.*

(b) *If* $J(k)$ *is empty for some* k, *then the numbers* x_j, $j = 1, \ldots, n$, *defined by* (3.8) *and by* (3.5) *are the same, and therefore solve Problem 2.*

(c) *If all sets* $J(k)$, $k = 1, \ldots, n$, *are non-empty, then* λ^* *can be computed by* (3.9) *and the directed cycle* β^* *of form* (3.11) *constructed in Step 2 of Algorithm 1 has the property* $d(\beta^*)/|\beta^*| = \lambda^*$.

The computation of the minimum cycle mean λ^* and of an optimal directed cycle β^* in weighted directed graphs will play an important role in the Sects. 4–7. For the solution of Problem 2 we need less than Algorithm 1 achieves. If we want to solve Problem 2 (see also [11; Sect. 4]) then we can simplify Algorithm 1 to the following.

Algorithm 2 *(Problem 1 and Problem 2)*

0. Perform Step 0 of Algorithm 1.
1. Perform Step 1 of Algorithm 1 without noting the indices P_{jk}. If all sets $J(k)$, $k=1,\ldots,n$, are non-empty then Problem 2 has no solution, else the numbers x_j defined by (3.8) solve Problem 2.

4. Symmetric Scaling of Matrices

Let $A=(A_{ij})\neq 0$ be a non-negative $n\times n$-matrix. The matrices of the form XAX^{-1} with $X\in D_n^+$ are called *symmetric scalings of A* where D_n^+ is the set of non-negative, non-singular, diagonal $n\times n$-matrices. The measure of a symmetric scaling XAX^{-1} is given by

$$\alpha(X)=\frac{\max\limits_{(i,j)\in E_0}(XAX^{-1})_{ij}}{\min\limits_{(k,l)\in E_0}(XAX^{-1})_{kl}} \tag{4.1}$$

where $E_0:=\{(i,j):A_{ij}>0\}$.

It is the main purpose of our paper to present an efficient algorithm for the following *Symmetric Scaling Problem.*

Problem 3. *Find the infimum*

$$\alpha:=\inf_{X\in D_n^+}\alpha(X)$$

and a diagonal matrix $X^*\in D_n^+$ *with* $\alpha(X^*)=\alpha$.

We put

$$a_{ij}:=\log A_{ij},\qquad (i,j)\in E_0$$

and define the set $\Omega\in\mathbb{R}^2$ by

$$\Omega:=\begin{cases}(m,M)\text{: there exist }x_i\in\mathbb{R},\ i=1,\ldots,n,\text{ for which}\\ m\leq x_i+a_{ij}-x_j\leq M,\ (i,j)\in E_0.\end{cases} \tag{4.2}$$

Let $X=\text{diag}(X_1,\ldots,X_n)\in D_n^+$ and $x_i:=\log X_i$, $i=1,\ldots,n$. Taking the logarithm of (4.1) leads to

$$\log\alpha(X)=\max_{(i,j)\in E_0}(x_i+a_{ij}-x_j)-\min_{(i,j)\in E_0}(x_i+a_{ij}-x_j)$$

and thus to the following *additive form of the Problem 3.*

Problem 3′. *Find the infimum*

$$\log\alpha=\inf\{M-m:(m,M)\in\Omega\} \tag{4.3}$$

and numbers m^*, M^*, $x_1^*,\ldots,x_n^*$ *such that* $M^*-m^*=\log\alpha$ *and*

$$m^*\leq x_i^*+a_{ij}-x_j^*\leq M^*,\qquad (i,j)\in E_0. \tag{4.4}$$

The following lemma is obviously true.

Lemma 2. *The real numbers* $x_1^*, \ldots, x_n^*$ *solve* (4.4) *with* $M^* - m^* = \log \alpha$ *if and only if* $\alpha(X^*) = \alpha$ *for*

$$X^* = \mathrm{diag}(e^{x_1^*}, \ldots, e^{x_n^*}).$$

After associating with the matrix $A \neq 0$ a family of weighted directed graphs G^s, $s \in \mathbb{R}$, it will turn out that the Problem 3 and the Problem 3' are also equivalent to the Problem 3'' below which will be studied and solved by graph theoretical methods. But first, we need some notations.

We put

$$V := \{1, \ldots, n\}; \quad E := E_0 \cup \{(i,j):(j,i) \in E_0\}$$

and denote by Z the set of all directed cycles in the directed graph $G = (V, E)$. For any real number s, we define the weight function $d^s \colon E \to \mathbb{R}$ by

$$d_{ij}^s := \min{}^* \{a_{ij} - s; \; s - a_{ji}\}, \quad (i,j) \in E \qquad (4.5)$$

where in (4.5) the notation $\min^*$ indicates that we set in (4.5) $a_{ij} - s = +\infty$ if $(i,j) \notin E_0$ and $s - a_{ji} = +\infty$ if $(j,i) \notin E_0$. We put

$$E_1^s := \{(i,j) \in E : a_{ij} - s = d_{ij}^s\}$$
$$E_2^s := \{(i,j) \in E : s - a_{ji} = d_{ij}^s\} \qquad \cdot$$
$$E_3^s := E_1^s \cap E_2^s.$$

For any directed cycle $\beta \in Z$ we denote its number of arcs in E_r^s, $r = 1, 2, 3$, by $N_r^s(\beta) := |\beta \cap E_r^s|$ and put

$$n_{1s}(\beta) := N_1^s(\beta) + N_3^s(\beta) - N_2^s(\beta),$$
$$n_{2s}(\beta) := N_1^s(\beta) - N_3^s(\beta) - N_2^s(\beta).$$

Finally, with each $s \in \mathbb{R}$ we associate the weighted directed graph $G^s = (V, E, d^s)$ and denote the minimum cycle mean in G^s by $\varphi(s)$, i.e.

$$\varphi(s) := \min_{\beta \in Z} d^s(\beta)/|\beta|, \qquad (4.6)$$

and by $\beta_s \in Z$ one of the directed cycles in (V, E) for which the minimum in (4.6) is attained, i.e.

$$d^s(\beta_s)/|\beta_s| = \varphi(s). \qquad (4.7)$$

Since $d_{ij}^s \leq -d_{ji}^s$, $(i,j) \in E$, the minimum cycle mean $\varphi(s)$ of G^s is non-positive for each s.

Problem 3''. *Find the supremum*

$$\sup_{s \in R} \varphi(s) \qquad (4.8)$$

and a number s^* for which the supremum is attained.

The next theorem will show that the Problems 3, 3', 3'' are equivalent, that the supremum in (4.8) and thus the infima in the Problems 3 and 3' are attained, and how the optimal numbers $x_1^*, \ldots, x_n^*$ of the Problem 3' are connected with the minimum cycle mean problem.

Theorem 3. 1. *There exists an $s^* \in \mathbb{R}$ for which $\varphi(s^*) = \sup \varphi(s)$.*

2. *The supremum (4.8) is attained at a number s^* if and only if the infimum (4.3) of the Problem 3' is attained by*

$$m^* := s^* + \varphi(s^*), \qquad M^* := s^* - \varphi(s^*). \tag{4.9}$$

Additionally, any solution $x_1^*, \ldots, x_n^*$ of

$$x_i^* + d_{ij}^{s^*} - x_j^* \geqq \varphi(s^*), \qquad (i,j) \in E \tag{4.10}$$

is also a solution of (4.4), and conversely. And

$$\log \alpha = M^* - m^* = -2\varphi(s^*). \tag{4.11}$$

Proof. 1. The set $E_3^s = \{(i,j) \in E : a_{ij} - s = s - a_{ji}\}$ is empty except for finitely many numbers s, say except for $s_1 < s_2 < \ldots < s_q$. For any $\beta \in Z$, $d^s(\beta)$ is a continuous function in s and linear in the intervals $(-\infty, s_1), (s_1, s_2), \ldots, (s_q, \infty)$. Since the number of directed cycles $\beta \in Z$ is finite, φ is a piecewise linear continuous function consisting of only finitely many linear pieces. But as $\varphi(s) \leqq 0$ for all s, φ attains its maximal value.

2. We need the following lemma.

Lemma 3. *For any s, $|\varphi(s)| = -\varphi(s)$ is the smallest of all non-negative numbers ρ for which*

$$s - \rho \leqq x_i + a_{ij} - x_j \leqq s + \rho, \qquad (i,j) \in E_0 \tag{4.12}$$

has a solution $(x_1, \ldots, x_n)$.

Proof of Lemma 3. It follows from Corollary 1 that $\varphi(s)$ is the largest of all numbers λ for which

$$x_i + d_{ij}^s - x_j \geqq \lambda, \qquad (i,j) \in E \tag{4.13}$$

has a solution $(x_1, \ldots, x_n)$. But by Lemma 1, (4.12) and (4.13) are equivalent if $\rho = -\lambda$. □

Since $|\varphi(s^*)| = \min |\varphi(s)|$, the second part of Theorem 3 follows directly from Lemma 3 and the equivalence of (4.12) and (4.13) if $\rho = -\lambda$. □

Since the Problems 3, 3', 3″ are equivalent we shall solve the Symmetric Scaling Problem 3 by characterizing and computing the maximal points s^* of the function φ. For this purpose, we first have to study the essential properties of φ in our next theorem.

Theorem 4. 1. *φ is continuous, non-positive, and piecewise linear consisting of finitely many linear pieces.*

2. *$\varphi \in \mathrm{Lip}_1 1$, i.e. $|\varphi(s) - \varphi(t)| \leqq |s - t|$ for all s, t.*

3. *φ is concave.*

4. *φ satisfies the inequality*

$$|\beta_s|(\varphi(t) - \varphi(s)) \leqq \min_{r=1,2} (s-t) n_{rs}(\beta_s) \tag{4.14}$$

for any numbers s and t and any $\beta_s \in Z$ satisfying (4.7).

Proof. 1) has already been proved in the proof of Theorem 3.

2) Since $-|\beta_s| \leq n_{2s}(\beta_s) \leq n_{1s}(\beta_s) \leq |\beta_s|$, it follows from (4.14) that $\varphi(t) - \varphi(s) \leq |t - s|$ for all s and t. Hence, $\varphi \in \text{Lip}_1 1$.

3) Suppose that $s_0 < s_1$ and $s := (1 - u)s_0 + us_1$ for a fixed number u, $0 < u < 1$. It follows from (4.14) with $t = s_0$ that

$$\varphi(s_0) \leq \varphi(s) - (s_0 - s)n_{2s}(\beta_s)/|\beta_s|$$

and from (4.14) with $t = s_1$ that

$$\varphi(s_1) \leq \varphi(s) - (s_1 - s)n_{1s}(\beta_s)/|\beta_s|.$$

Hence we have

$$\varphi(s) - (1 - u)\varphi(s_0) - u\varphi(s_1) \geq u|\beta_s|^{-1}(s_1 - s)[n_{1s}(\beta_s) - n_{2s}(\beta_s)] \geq 0$$

which shows that φ is concave.

4) Let s and t be fixed. Then, for any $\beta \in Z$,

$$|\beta|\varphi(t) \leq d^t(\beta) \leq d^s(\beta) + (s - t)(N_1^s(\beta) \pm N_3^s(\beta) - N_2^s(\beta))$$

which already proves (4.14) if we take $\beta = \beta_s$. $\quad\square$

The next theorem gives an important characterization óf the maximal points s^* of the Problem 3″ and thus (by Theorem 3 and the relation (4.3)) of the minimal solutions (m^*, M^*) of the Problem 3′ and the minimum α of the Scaling Problem 3.

Theorem 5. *The following three statements are equivalent.*

(1) *φ attains its maximal value at s^*.*

(2) *There exist two (not necessarily distinct) directed cycles β and β' in Z for which the relations*

$$\varphi(s^*) = d^{s^*}(\beta)/|\beta| = d^{s^*}(\beta')/|\beta'| \tag{4.15}$$

and

$$n_{2s^*}(\beta') \leq 0 \leq n_{1s^*}(\beta) \tag{4.16}$$

are satisfied.

(3) *For any number s, $s \neq s^*$, and any directed cycle $\beta_s \in Z$ satisfying (4.7), the integers $n_{1s}(\beta_s)$ and $n_{2s}(\beta_s)$ have the properties*

$$n_{1s}(\beta_s) \geq n_{2s}(\beta_s) \geq 0 \quad \text{if } s > s^*$$
$$n_{2s}(\beta_s) \leq n_{1s}(\beta_s) \leq 0 \quad \text{if } s < s^*. \tag{4.17}$$

Proof. $(1) \Rightarrow (3)$: It follows from (4.14) that

$$0 \leq |\beta_s|(\varphi(s^*) - \varphi(s)) \leq \min_{r=1,2} (s - s^*)n_{rs}(\beta_s)$$

which leads to (4.17).

$(3) \Rightarrow (2)$: Suppose that the Statement (3) is valid. Since the number $|Z|$ of directed cycles is finite, we find sequences $s_k \to s^*$, $s_k > s^*$, and $s'_k \to s^*$, $s'_k < s^*$, and two (not necessarily distinct) directed cycles β and β' such that

$$\varphi(s_k) = d^{s_k}(\beta)/|\beta|, \quad \varphi(s'_k) = d^{s'_k}(\beta')/|\beta'| \tag{4.18}$$

holds for all k. Because of (4.17) and (4.18),

$$n_{1s_k}(\beta) \geq n_{2s_k}(\beta) \geq 0, \qquad n_{2s_k}(\beta') \leq n_{1s_k}(\beta') \leq 0.$$

for all k. Hence, for $k \to \infty$, the continuity of φ and d_{ij}^s guarantees that (4.15) and (4.16) are valid.

(2) $\Rightarrow$ (1): It follows from (4.14)–(4.16) that

$$|\beta|(\varphi(t) - \varphi(s^*)) \leq (s^* - t)n_{1s^*}(\beta) \leq 0 \qquad \text{if } t > s^*,$$
$$|\beta'|(\varphi(t) - \varphi(s^*)) \leq (s^* - t)n_{2s^*}(\beta') \leq 0 \qquad \text{if } t < s^*.$$

Hence, $\varphi(t) \leq \varphi(s^*)$ for each t. $\quad\square$

5. An Algorithm for the Problems 3, 3′, 3″

We have seen in Lemma 2 and Theorem 3 that any solution $s^*, x_1^*, \ldots, x_n^*$ of Problem 3″ and (4.10) leads by (4.9) to a solution $M^* - m^* = -2\varphi(s^*)$ of Problem 3′ (where (4.4) holds) and to a minimal symmetric scaling

$$X^* A X^{*-1}, \qquad X^* := \operatorname{diag}(e^{x_1^*}, \ldots, e^{x_n^*}) \tag{5.1}$$

with

$$\alpha = \alpha(X^*) = e^{-2\varphi(s^*)}. \tag{5.2}$$

Hence it suffices to find a maximal point s^* of φ and numbers $x_1^*, \ldots, x_n^*$ satisfying (4.10). For this purpose, we suggest the algorithm below which is based on Theorem 5, Inequality (4.14) and on Algorithm 1 which will compute the minimum cycle mean $\varphi(s)$ of the weighted directed graph $G^s = (V, E, d^s)$ for finitely many real numbers s.

Algorithm 3

Step 0. Put $L := -\infty$, $U := +\infty$. (L and U will serve as lower and upper bounds for a maximal point s^* of φ.) Choose a real number s.

Step 1. Apply the Algorithm 1 to the weighted directed graph $G^s = (V, E, d^s)$ and compute $\varphi(s)$, one directed cycle β_s satisfying (4.7), and the integers $n_{rs} := n_{rs}(\beta_s)$, $r = 1, 2$.
   a) If $n_{2s} \leq 0 \leq n_{1s}$, put $s^* := s$ and goto Step 2.
   b) If $n_{1s} < 0$, put $L := s$. If $n_{2s} > 0$, put $U := s$.
   c) Choose the new number s as follows.

$$s := s_{LU} := L + \frac{|\beta_L|(U - L)n_{2U} + |\beta_L||\beta_U|(\varphi(U) - \varphi(L))}{|\beta_L|n_{2U} - |\beta_U|n_{1L}}$$

if $-\infty < L \leq U < +\infty$, and

$$s := s_{LU} := \begin{cases} L + |\beta_L|\varphi(L)/n_{1L} & \text{if } U = +\infty \\ U + |\beta_U|\varphi(U)/n_{2U} & \text{if } L = -\infty. \end{cases}$$

d) If $s_{LU}=L$ or $s_{LU}=U$, put $s^*:=s_{LU}$ and go to Step 2. Else return to Step 1.

Step 2. s^* is a maximal point of φ.

Apply the Algorithm 1 to the weighted directed graph (V,E,d^*) where the weight function d^* is defined by

$$d_{ij}^*:=d_{ij}^{s^*}-\varphi(s^*)=\min^*\{a_{ij}-s^*;\ s^*-a_{ji}\}-\varphi(s^*),$$

$(i,j)\in E$. Since the minimum cycle mean of (V,E,d^*) is equal to zero, the event (3.8) must occur for some index k, $1\leq k\leq n$, and the numbers $x_j^*:=x_j(k)$, $j=1,\ldots,n$, which we obtain from (3.8), are the desired solution of the inequalities (4.10).

Remark 3. 1. If the events (a) or (d) of Step 1 occur, then the inequalities (4.14) imply that s is a maximal point of φ.

2. The number s_{LU} in Step 1 is the unique point where the straight lines

$$y_1(t):=\begin{cases}\varphi(L)-(t-L)n_{1L}/|\beta_L|, & \text{if } L>-\infty\\ 0, & \text{if } L=-\infty\end{cases}\ t\in\mathbb{R}$$

$$y_2(t):=\begin{cases}\varphi(U)-(t-U)n_{2U}/|\beta_U|, & \text{if } U<+\infty\\ 0, & \text{if } U=+\infty\end{cases}\ t\in\mathbb{R}$$

intersect. Hence, $L\leq s_{LU}\leq U$.

3. The above Algorithm 3 finds a maximal point s^* of φ in finitely many iterations. That follows at once from the inequalities (4.14).

6. An Explicit Characterization of the Minimum α

In [18; Theorem 6], Rothblum and Schneider haved proved a characterization of the minimal value α of the Problem 3. We shall now show how this result follows from our Theorem 5.

Let the notation be the same as in Sect. 4. More exactly, let $A=(A_{ij})\neq 0$ be a non-negative $n\times n$-matrix, $V:=\{1,\ldots,n\}$, $E_0:=\{(i,j):A_{ij}>0\}$, and

$$a_{ij}:=\log A_{ij}\quad\text{for }(i,j)\in E_0.$$

We now consider another directed graph, $G_0:=(V,E_0)$. In this section, G_0 and the (oriented) cycles c in G_0 play the essential role.

Definition. $c=c^+\cup c^-=\{e_1,\ldots,e_k\}\subseteq E_0$, $k\geq1$, is called a *cycle* in G_0 if there exist k distinct vertices in V, say $i_1,\ldots,i_k$, such that

$$e_r=\begin{cases}(i_r,i_{r+1}), & \text{if } e_r\in c^+\\ (i_{r+1},i_r), & \text{if } e_r\in c^-\end{cases}\ r=1,\ldots,k$$

where we set $i_{k+1}:=i_1$. By Z_0 we denote the set of all cycles $c=c^+\cup c^-$ in G_0.

We associate with any arc $e=(i,j)\in E_0$ its weight $D(e):=a_{ij}$ and with any cycle $c=c^+\cup c^-\in Z_0$ its weight

$$D(c):=\sum_{e\in c^+}D(e)-\sum_{e\in c^-}D(e).$$

We put $N(c):=|c^+|-|c^-|$ and

$$C^0(G_0):=\{c\in Z_0:N(c)=0\},$$
$$C^2(G_0):=\{(c_1,c_2)\in Z_0\times Z_0:N(c_2)<0<N(c_1)\}.$$

Finally, with any cycle $c\in Z_0$ we associate the directed cycle $\bar{c}\in Z$ in $G=(V,E)$ defined by

$$\bar{c}:=\{(i,j)\in E:(i,j)\in c^+\}\cup\{(j,i)\in E:(i,j)\in c^-\}.$$

Here, E and Z are defined as in Sect. 4.

We then have for each cycle $c=c^+\cup c^-\in Z_0$ and each s

$$d^s(\bar{c})=\sum_{(i,j)\in c^+}\min^*\{a_{ij}-s;\;s-a_{ji}\}+\sum_{(i,j)\in c^-}\min^*\{a_{ji}-s;\;s-a_{ij}\} \qquad (6.1)$$

and hence

$$|\bar{c}|\,\varphi(s)=|c|\,\varphi(s)\leqq d^s(\bar{c})\leqq D(c)-sN(c) \qquad (6.2)$$

and

$$\varphi(s)\leqq D(c)/|c| \qquad \text{if } c\in C^0(G_0), \qquad (6.3)$$

$$\varphi(s)\leqq\frac{D(c_2)N(c_1)-D(c_1)N(c_2)}{|c_2|N(c_1)-|c_1|N(c_2)}=:I(c_1,c_2) \qquad (6.4)$$

if $(c_1,c_2)\in C^2(G_0)$. Here, the number $I(c_1,c_2)$ is defined by the equation of (6.4). The inequality in (6.4) follows from (6.2) since the straight lines

$$\tilde{y}_r(s)=(D(c_r)-sN(c_r))/|c_r|,\qquad s\in\mathbb{R},\;r=1,2$$

intersect at $(\tilde{s},I(c_1,c_2))\in\mathbb{R}^2$ where

$$\tilde{s}=\frac{|c_2|D(c_1)-|c_1|D(c_2)}{|c_2|N(c_1)-|c_1|N(c_2)}.$$

On the other hand, using our Theorem 5, there exist directed cycles β and β' in Z which satisfy (4.15)-(4.16) for a maximal point s^* of φ. Hence we can choose the cycles $z_r=z_r^+\cup z_r^-\in Z_0$, $r=1,2$, such that

$$\bar{z}_1=\beta \quad \text{and} \quad z_1^+=\beta\cap E_1^{s^*} \qquad \text{which implies } N(z_1)=n_{1s^*}(\beta)$$
$$\bar{z}_2=\beta' \quad \text{and} \quad z_2^+=\beta'\cap E_1^{s^*}\smallsetminus E_3^{s^*} \qquad \text{which implies } N(z_2)=n_{2s^*}(\beta').$$

Applying (4.15) and (6.1) leads to

$$|z_r|\,\varphi(s^*)=D(z_r)-s^*N(z_r),\qquad r=1,2 \qquad (6.5)$$

and thus to

$$\varphi(s^*)=I(z_1,z_2) \qquad \text{unless } N(z_1)=N(z_2)=0. \qquad (6.6)$$

Combining (6.3)–(6.6) completes the proof of the following explicit characterization of the infimum α of the Symmetric Scaling Problem 3 which is due to Rothblum and Schneider [18; Theorem 6].

Theorem 6. *Under the above setting and notations,*

$$\log \alpha = -2 \min \left\{ \min_{c \in C^0(G_0)} \frac{D(c)}{|c|}; \quad \min_{(c_1, c_2) \in C^2(G_0)} I(c_1, c_2) \right\}.$$

Example. Let $E_0 := \{(1, 1), (1, 2), (1, 3), (2, 3), (3, 2)\}$,

$$\log A := (a_{ij}) := \begin{pmatrix} 1 & 2 & 4 \\ * & * & 1 \\ * & 2 & * \end{pmatrix}.$$

We apply Algorithm 3 starting with $s := 0$. Then we obtain the following values $s = s_{LU}$ and matrices d^s:

$$s = 0: \ d^s = \begin{pmatrix} -1 & 2 & 4 \\ -2 & * & -2 \\ -4 & -1 & * \end{pmatrix},$$

$$\varphi(s) = -3/2; \ \beta_s = \{(2, 3), (3, 2)\},$$

$n_{1s} = n_{2s} = -2$, $L := 0$, $s_{LU} = 3/2$

$$s = 3/2: \ d^s = \begin{pmatrix} -1/2 & 1/2 & 5/2 \\ -1/2 & * & -1/2 \\ -5/2 & 1/2 & * \end{pmatrix}$$

$$\varphi(s) = -5/6; \ \beta_s = \{(1, 2), (2, 3), (3, 1)\},$$

$n_{1s} = 1$, $n_{2s} = -1$. Hence, $s^* = 3/2$ is a maximal point of φ, and

$$\varphi(s^*) = -5/6, \quad m^* = 2/3, \quad M^* = 7/3, \quad \log \alpha = 5/3.$$

Step 2 of Algorithm 3. $s^* = 3/2$

$$d^* = (d_{ij}^{s^*} - \varphi(s^*)) = \begin{pmatrix} 1/3 & 4/3 & 10/3 \\ 1/3 & * & 1/3 \\ -5/3 & 4/3 & * \end{pmatrix}$$

$$x_1^* = -5/3, \quad x_2^* = -1/3, \quad x_3^* = 0,$$

$$\log(X^* A X^{*-1}) = (x_i^* + a_{ij} - x_j^*) = \begin{pmatrix} 1 & 2/3 & 7/3 \\ * & * & 2/3 \\ * & 7/3 & * \end{pmatrix}.$$

The directed cycles of Theorem 5 are

$$\beta = \beta' = \{(1, 2), (2, 3), (3, 1)\},$$

and the minimum in Theorem 6 is attained by $(z_1, z_2) \in C^2(G_0)$ where

$$z_1^+ = \{(1,2), (2,3)\}, \quad z_1^- = \{(1,3)\}, \qquad D(z_1) = -1,$$
$$z_2^+ = \{(1,2)\}, \qquad z_2^- = \{(3,2), (1,3)\}, \quad D(z_2) = -4$$

and $N(z_1) = 1$, $N(z_2) = -1$.

7. Asymmetric Scaling

Let $A = (A_{ij}) \neq 0$ be a non-negative $m \times n$-matrix. The matrices of the form XAY with $X \in D_m^+$ and $Y \in D_n^+$ are called *asymmetric scalings of* A. The measure of a asymmetric scaling is given by

$$\gamma(X, Y) = \frac{\max\limits_{(i,j) \in E_0} (XAY)_{ij}}{\min\limits_{(k,l) \in E_0} (XAY)_{kl}} \tag{7.1}$$

where $E_0 := \{(i,j) : A_{ij} > 0\}$.

Problem 4. *Find the infimum*

$$\gamma := \inf\{\gamma(X, Y) : X \in D_m^+, Y \in D_n^+\}$$

and diagonal matrices $X^* \in D_m^+$, $Y^* \in D_n^+$ *with* $\gamma(X^*, Y^*) = \gamma$.

We put $a_{ij} := \log A_{ij}$, $(i,j) \in E_0$, and

$$x_i := -\log X_i, \quad i = 1, \ldots, m, \quad \text{if } X = \text{diag}(X_1, \ldots, X_m) \in D_m^+,$$
$$y_j := -\log Y_j, \quad j = 1, \ldots, n, \quad \text{if } Y = \text{diag}(Y_1, \ldots, Y_n) \in D_n^+.$$

Taking the logarithm of (7.1) leads to

$$\log \gamma(X, Y) = \max_{(i,j) \in E_0} (a_{ij} - x_i - y_j) - \min_{(i,j) \in E_0} (a_{ij} - x_i - y_j).$$

Hence, the Scaling Problem 4 is equivalent to

Problem 4'. *Find the infimum*

$$\tfrac{1}{2} \log \gamma = \inf_{x_i, y_j} \max_{(i,j) \in E_0} |a_{ij} - x_i - y_j|$$

and numbers x_i^*, $i = 1, \ldots, m$, *and* y_j^*, $j = 1, \ldots, n$, *for which the infimum is achieved.*

Meanwhile, many people have studied the Problems 4 and 4' or related topics. We do not intend to survey these results, but have listed a few of these articles in the References below. We only want to point out that the Problems 4 and 4' are equivalent to the minimum cycle mean problem in a properly chosen weighted directed graph $\tilde{G}$. More detailed, we shall prove Theorem 7 below by introducing $\tilde{G}$ as follows. We put

$$\tilde{V} := \{1, 2, \ldots, m+n\},$$
$$\tilde{E} := \{(i, m+j); (m+j, i) : (i,j) \in E_0\} \subseteq V \times V,$$

and define the weight function $\tilde{d}\colon \tilde{E} \to \mathbb{R}$ by

$$\tilde{d}_{i,m+j} := a_{ij}, \qquad \tilde{d}_{m+j,i} := -a_{ij}, \quad (i,j) \in E_0.$$

Theorem 7. *Let λ^* denote the minimum cycle mean of the weighted directed graph $\tilde{G} = (\tilde{V}, \tilde{E}, \tilde{d})$. Then,*

$$\tfrac{1}{2} \log \gamma = -\lambda^*; \tag{7.2}$$

and the numbers $(x_1, \dots, x_{m+n}) \in \mathbb{R}^{m+n}$ are a solution of

$$x_k + \tilde{d}_{kl} - x_l \geq \lambda^*, \qquad (k,l) \in \tilde{E} \tag{7.3}$$

if and only if

$$\max_{(i,j) \in E_0} |a_{ij} + x_i - x_j| = \tfrac{1}{2} \log \gamma. \tag{7.4}$$

Proof. The inequalities (7.3) are equivalent to

$$\left. \begin{array}{c} x_i + a_{ij} - x_{m+j} \geq \lambda^* \\ x_{m+j} - a_{ij} - x_i \geq \lambda^* \end{array} \right\} \quad (i,j) \in E_0 \tag{7.5}$$

and hence to

$$|a_{ij} + x_i - x_{m+j}| \leq -\lambda^*, \quad (i,j) \in E_0. \tag{7.6}$$

That already concludes the proof of Theorem 7 since, by Corollary 1, λ^* is the largest number for which (7.3) has a solution $x \in \mathbb{R}^{m+n}$. $\qquad \square$

Theorem 7 shows that any algorithm which solves the minimum cycle mean problem is also applicable to the Problems 4 and 4'. For instance, if we apply our Algorithm 1 of Sect. 3, we obtain the recurrences

$$x_i(k) := \max\{x_i(k-1); a_{ij} - y_j(k-1) \colon j = 1, \dots, n\}, \qquad i = 1, \dots, m, \tag{7.7a}$$

and

$$y_j(k) := \min\{y_j(k-1); a_{ij} - x_i(k-1) \colon i = 1, \dots, m\}, \qquad j = 1, \dots, n, \tag{7.7b}$$

which we have to compute for $k = 1, 2, \dots, m+n$ until we can apply (3.9) for the computation of λ^*.

In [10; Algorithm 1] however, the recurrence (7.7a) is the same, but (7.7b) is replaced by

$$y_j(k) := \min\{y_j(k-1); a_{ij} - x_i(k) \colon i = 1, \dots, m\}, \qquad j = 1, \dots, n, \tag{7.7b'}$$

using the latest value of the x_i.

But then, we only need to compute (7.7a), (7.7b)' for $k = 1, \dots, n$, which is an essential improvement.

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