Mathematical Programming 27 (1983) 291–306
North-Holland

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A CONFORMING DECOMPOSITION THEOREM, A PIECEWISE LINEAR THEOREM OF THE ALTERNATIVE, AND SCALINGS OF MATRICES SATISFYING LOWER AND UPPER BOUNDS*

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Received 15 March 1982
Revised manuscript received 23 November 1982

A scaling of a nonnegative matrix A is a matrix XAY^{-1}, where X and Y are nonsingular, nonnegative diagonal matrices. Some condition may be imposed on the scaling, for example, when A is square, $X = Y$ or $\det X = \det Y$. We characterize matrices for which there exists a scaling that satisfies predetermined upper and lower bound. Our principal tools are a piecewise linear theorem of the alternative and a theorem decomposing a solution of a system of equations as a sum of minimal support solutions which conform with the given solutions.

Key words: Conforming Decompositions, Theorem of the Alternative, Scalings, Matrices.

1. Introduction

A *symmetric scaling* of a nonnegative, square matrix $A \neq 0$ is a matrix of the form XAX^{-1} where X is a nonnegative, nonsingular, diagonal matrix having the same dimension as A. An *asymmetric scaling* of a nonnegative rectangular matrix $A \neq 0$ is a matrix of the form XAY^{-1} where X and Y are nonnegative, nonsingular, diagonal matrices having appropriate dimensions. Such an asymmetric scaling is called *constrained*, if the corresponding diagonal matrices X and Y can be chosen such that $\det X = \det Y$. There are many models in which a matrix can be replaced by any one of its scalings without changing the character of the problem (e.g. linear programming). A number of recent investigations studied the problem of finding and characterizing scalings of a matrix which are efficient in some way, e.g. Bank (1979), Bauer (1959, 1963), Curtis and Reid (1972), Fulkerson and Wolfe (1962), Hamming (1971) and Tomlin (1975). Scaling problems occur in many applications,

* This research was partially supported by National Science Foundation Grants ENG-78-25182 and MCS-80-26132.

e.g. input–output models in economics, e.g. Bacharach (1970) and telecommunication, e.g. Orchard–Hays (1968). They are also related to problems in approximation theory, e.g. Golitschek (1982).

The systematic theoretical study of diagonal scaling problems has a history of 15–20 years. We now give a condensed (and incomplete) account of this history as it affects the present paper. An early paper, Fielder and Plak (1969), examined an additive scaling problem whose multiplicative analog requires the elements of the matrix to be positive. This condition was relaxed in Engel and Schneider (1973), where diagonal scaling theorems are proved for irreducible nonnegative matrices, with some extensions to reducible nonnegative matrices by use of the Frobenius normal forms. The papers mentioned so far use as their principal tool circuit products of elements of the matrix (such as occur in the evaluation of determinants). In Saunders and Schneider (1978) the simple but crucial observation was made that restrictions such as irreducibility could be removed by the consideration of general cyclic products which allow reverse arcs. Diagonal scaling problems could now be reformulated as fairly standard problems in graph theory or the duality of cones, see Saunders and Schneider (1979), or as linear programming problems, see Rothblum and Schneider (1980). It is the purpose of the present paper to continue the unification of diagonal scaling problems by the methods of linear programming and to obtain additional results. A new feature of our paper is the introduction of cyclic products which draw their elements from more than one matrix. Another consequence of the reformulation referred to above has been the application of specially developed variants of algorithms employed in combinatorial optimization to diagonal scaling problems, e.g. v. Golitschek (1980) and Engel and Schneider (1982). It is planned to implement the results of this paper, which are theoretical, for publication elsewhere.

In greater detail, it is the aim of our paper to characterize matrices which have scalings that satisfy predetermined upper and lower bounds, see Sections 4 and 5. As corollaries we obtain known results where only the upper (or lower) bound is given, or where the two bounds coincide. A sample question concerning constrained asymmetric scalings is:

> Given nonnegative matrices B, A, C, are there nonnegative diagonal matrices X and Y with $\det X = \det Y$ for which $B \leq XAY^{-1} \leq C$? (1.1)

Using techniques related to those found in Saunders and Schneider (1978, 1979) and Rothblum and Schneider (1980) we derive an equivalent formualtion of the scaling problem:

> Given a real rectangular matrix Δ and vectors a, b, c, f of appropriate size, is there a vector x for which $b \leq a + \Delta x \leq c$ and $f^{\mathrm{T}}x = 0$? (1.2)

We answer this question by means of an equivalent condition (see (1.3) below) given by a theorem of the alternative, which is piecewise linear as it involves the positive and negative parts of a real vector w, respectively, defined by $w^+ = \max(w, 0)$ and $w^- = \max(-w, 0)$, see Section 2. Specifically, we show that (1.2) is equivalent to:

$$\text{If } \Delta^{\mathrm{T}} w = 0 \text{ or } \Delta^{\mathrm{T}} w = \pm f, \quad \text{then } b^{\mathrm{T}} w^+ - c^{\mathrm{T}} w^- \leqslant a^{\mathrm{T}} w \leqslant c^{\mathrm{T}} w^+ - b^{\mathrm{T}} w^-. \tag{1.3}$$

The matrix Δ which occurs in (1.3) is the incidence matrix of a graph and f is a vector whose entries are ± 1. We strengthen our theorem of the alternative by showing that (1.2) is equivalent to a weaker form of (1.3) where one considers only minimal support solutions of $\Delta^{\mathrm{T}} w = 0$ or $\Delta^{\mathrm{T}} w = \pm f$, which are called, respectively, *cycles* and *pseudodiagonals* of the corresponding graph. This is done by using a conforming decompositions theorem of solutions to linear systems proved in Section 2. The homogenous case of this theorem is well known (see Rockafellar (1968, p. 109)).

As we have just explained, our considerations of scalings of matrices have led to a theorem of the alternative which involves conforming decompositions of vectors. We attach equal importance to each of the three types of results mentioned in the title of our paper and presented here.

One can view upper and lower bounds on a scaling as a requirement that a (multiattribute) efficiency level must be satisfied. We hope that our theoretical analysis will eventually lead to a better understanding of the relationship between theoretical properties and numerical efficiency of scalings.

2. Conforming decompositions of solutions to linear systems

The purpose of this section is to establish a conforming decomposition of solutions to a system of linear equations. The result concerning homogenous systems generalizes a result of Tutte (1956) and is well known (e.g. Rockafellar (1968, p. 109)). We first need a few definitions.

Let $\mathbb{R}$ be the real field and let $\mathbb{R}_+$ be the set of nonnegative reals. By $\mathbb{R}^{mn}$ (resp. $\mathbb{R}_+^{mn}$) we denote the set of all $m \times n$ matrices with elements in $\mathbb{R}$ (resp. $\mathbb{R}_+$). As usual, $\mathbb{R}^n$ (resp. $\mathbb{R}_+^n$) will stand for $\mathbb{R}^{n1}$ (resp. $\mathbb{R}_+^{n1}$). By convention, the empty sum of vectors in $\mathbb{R}^n$ is defined to be zero.

Let $x \in \mathbb{R}^m$. The *support* of x, denoted $S(x)$, is defined to be the index set $\{i: x_i \neq 0\}$. Also, we define x^+ and x^- to be, respectively, $\max\{x, 0\}$ and $\max\{-x, 0\}$ where these maxima are taken elementwise. We also let $|x| = x^+ + x^-$. Next, if $x, y \in \mathbb{R}^m$, we say that *x conforms to y* if $S(x) \subseteq S(y)$ and for $i = 1, \ldots, m$, $x_i y_i \geqslant 0$. (This definition is asymmetric in x and y and follows Tutte (1956, p. 22)). We say that $x^1, \ldots, x^q \in \mathbb{R}^m$ are *symmetrically conforming* if for every $r, s = 1, \ldots, q$ and $i = 1, \ldots, m$, $x_i^r x_i^s \geqslant 0$.

We consider solutions to linear systems of the form

$$D\xi = d, \qquad E\xi \leqslant e$$

where D and E (resp. d and e) are matrices (resp. vectors) of appropriate sizes. A solution x to such a system is called a *minimal-support solution* if x is nonzero and there exists no solution $y \neq 0$ to the above system with $S(y) \subset S(x)$ (where we use $\subset$ to denote proper inclusion). We remark that $x = 0$ has to be excluded in this definition only when $d = 0 \leqslant e$. In this case, if x is a (minimal-support) solution of the above system, we call $\{\alpha x : \alpha \in \mathbb{R}_+\}$ a *minimal-support solution ray*.

Theorem 2.1 (Rockafellar (1967, p. 109)). *Let $A \in \mathbb{R}^{mn}$. Consider the system*

$$A\xi = 0. \tag{H}$$

Then

(a) *The system* (H) *has a finite number of minimal-support solution rays.*

(b) *For every solution x of* (H) *there exists minimal-support solutions of* (H), *$z^1, \ldots, z^p$, such that $x = \sum_{j=1}^{p} z^r$ and $z^1, \ldots, z^p$ all conform to x.* □

Extensions of Theorem 2.1 in the setting of oriented matroids appear in Bland (1974) and Bland and Las Vergnas (1979). Using a familar technique, e.g. Bachem and Grötschel (1982), we now derive a theorem on inhomogenous systems.

Theorem 2.2. *Let $A \in \mathbb{R}^{mn}$ and $b \in \mathbb{R}^m$. Consider the system*

$$A\xi = b. \tag{G}$$

Then

(a) *If $b \neq 0$, then the system* (G) *has a finite number of minimal-support solutions.*

(b) *For every solution x of* (G) *there exist minimal-support solutions of* (H), *$x^1, \ldots, x^q$, minimal-support solutions of* (G), *$y^1, \ldots, y^t$ and positive numbers $\beta_1, \ldots, \beta_t$ such that $\sum_{s=1}^{t} \beta_s = 1$, $x = \sum_{r=1}^{q} x^r + \sum_{s=1}^{t} \beta_s y^s$ and $x^1, \ldots, x^q, y^1, \ldots, y^t$ all conform to x.*

Proof.[1] For $x \in \mathbb{R}^n$ let $\tilde{x} = (x^T, 1) \in \mathbb{R}^{n+1}$. Then x is a (minimal-support) solution of (G) if and only if $\tilde{x}$ is a (minimal-support) solution of

$$(A, -b)\xi = 0. \tag{$\tilde{\mathrm{H}}$}$$

Hence, by the first conclusion in Theorem 2.1, the number of minimal-support solutions of (G) is finite, proving (a).

Next observe that by Theorem 2.1 there exist minimal-support solutions $\tilde{z}^1, \ldots, \tilde{z}^p$ of ($\tilde{\mathrm{H}}$), all conforming to $\tilde{x}$ such that $\tilde{x} = \sum_{j=1}^{p} \tilde{z}^j$. By the conformality condition, $\tilde{z}^j_{n+1} \geqslant 0$ for $j = 1, \ldots, p$. For $j = 1, \ldots, p$, let $z^j \in \mathbb{R}^n$ consist of the first n coordinates of $\tilde{z}^j$ and let $I = \{j : \tilde{z}^j_{n+1} > 0\}$ and $K = \{j : \tilde{z}^j_{n+1} = 0\} = \{1, \ldots, p\} \backslash I$. Then $x = \sum_{j=1}^{p} z^j = \sum_{j \in I} \tilde{z}^j_{n+1}[(\tilde{z}^j_{n+1})^{-1} z^j] + \sum_{j \in K} z^j$. Moreover, $\sum_{j \in I} \tilde{z}^j_{n+1} = 1$, and $(\tilde{z}^j_{n+1})^{-1} z^j$ for $j \in I$ are minimal-support solutions of (H) all conforming to x and z^j for $j \in K$ are minimal-support solutions of (G) all conforming to x. □

The following two examples illustrate conforming decompositions of the type established in Theorem 2.2. In the first example each solution of the given system

[1] The current proof is due to Alan Hoffman.

has a unique conforming decomposition. This example also illustrates that the following are possible: $t>1$, $q>0$ or $q=0$. The second example illustrates that, in general, nonuniqueness of these decompositions is possible.

Example 2.3. Let $A=(1,-1)$ and $b=(1)$. Then $\{x : Ax=b\}=\{(\alpha,\alpha-1)^{\mathrm{T}} : \alpha\in\mathbb{R}\}$. Let $x=(\alpha,\alpha-1)^{\mathrm{T}}$ be a solution of $A\xi=b$. Then the (unique) conforming decomposition of $x=(\alpha,\alpha-1)$ is given by: $x=(1,0)^{\mathrm{T}}+(\alpha-1,\alpha-1)^{\mathrm{T}}$ when $\alpha\geqslant 1$, $x=\alpha(1,0)^{\mathrm{T}}+(1-\alpha)(0,-1)^{\mathrm{T}}$ when $0\leqslant\alpha\leqslant 1$, and $x=(0,-1)^{\mathrm{T}}+(\alpha,\alpha)^{\mathrm{T}}$ when $\alpha\leqslant 0$. When $1\leqslant\alpha$ and $\alpha\leqslant 0$ there clearly exist solutions y of the homogeneous equation which conforms to x, but no such y exists when $0<\alpha<1$.

Example 2.4. Let

$$A=\begin{pmatrix}1 & -1 & 0 & 0\\ 0 & 0 & 1 & -1\end{pmatrix} \quad\text{and}\quad b=\begin{pmatrix}2\\2\end{pmatrix}.$$

Then $x=(1,-1,1,-1)^{\mathrm{T}}$ has two conforming decompositions given by

$$x=0.5(2,0,2,0)^{\mathrm{T}}+0.5(0,-2,0,-2)^{\mathrm{T}}=0.5(2,0,0,-2)^{\mathrm{T}}+0.5(0,-2,2,0)^{\mathrm{T}}.$$

We now compare and contrast Theorem 2.2 with the following well-known ‘Resolution Theorem’.

Theorem 2.5 (Motzkin (1936, p. 40), cf. Goldman (1956)). *Consider the inequalities*

$$A\xi\leqslant b, \tag{G'}$$

and the corresponding homogeneous inequalities

$$A\xi\leqslant 0. \tag{H'}$$

Then there exist solutions $x^1,\ldots,x^q$ of (H′) *and solutions $y^1,\ldots,y^t$ of* (G′) *such that every solution x of* (G) *is a nonnegative combination of $x^1,\ldots,x^q$ plus a convex combination of $y^1,\ldots,y^t$.* □

In the case of equalities, Theorem 2.5 is an immediate consequence of our Theorem 2.2. The general case of Motzkin’s theorem is also obtained from Theorem 2.2 by a standard transformation of an inequality system to an equality system. Conversely, it is possible to obtain Theorem 2.2 from Theorem 2.5. When the solution set S of (G′) is pointed, (i.e. contains no lines) one can require that $y^1,\ldots,y^t$ (resp. $x^1,\ldots,x^q$) in Theorem 2.5 to be extreme points (resp. extreme rays) of the polyhedron S. In particular, this holds for the set of solutions of (G) augmented by nonnegativity and/or nonpositivity constraints on each of the coordinates. We remark, without proof, that for such systems, the minimal support solutions are precisely the extreme points of the corresponding polytope and the extreme rays of that polytope consist of minimal-support solutions of the corresponding homogeneous system. Thus our Theorem 2.2 follows from Theorem 2.5 by considering separately each of the 2^n systems obtained by adjoining to (G) a set of nonnegativity and/or nonpositivity constraints.

3. A piecewise linear Theorem of the Alternative

The purpose of this section is to obtain a Theorem of the Alternative that gives necessary and sufficient conditions that a linear system having upper and lower bounds inequalities has a solution.

Theorem 3.1. *Let $A \in \mathbb{R}^{mn}$, $d \in \mathbb{R}^n$ and b, $c \in \mathbb{R}^m$ where $b \leqslant c$. Then the following three conditions are equivalent:*

(a) *The system*

$$b \leqslant A\xi \leqslant c, \qquad d^{\mathrm{T}}\xi = 0 \tag{3.1}$$

has a solution.

(b) *Let $w \in \mathbb{R}^m$. If either $A^{\mathrm{T}}w = 0$ or $A^{\mathrm{T}}w = d$ or $A^{\mathrm{T}}w = -d$, then*

$$c^{\mathrm{T}}w^{+} - b^{\mathrm{T}}w^{-} \geqslant 0. \tag{3.2}$$

(c) *Let $w \in \mathbb{R}^m$. If w is a minimal-support solution of one of the following three systems: $A^{\mathrm{T}}\eta = 0$, $A^{\mathrm{T}}\eta = d$, $A^{\mathrm{T}}\eta = -d$, then* (3.2) *holds.*

Moreover, the above equivalence holds when (a) *is weakened by replacing $d^{\mathrm{T}}\xi = 0$ in* (3.1) *by $d^{\mathrm{T}}\xi \geqslant 0$ (resp. $d^{\mathrm{T}}\xi \leqslant 0$),* (b) *is weakened by not requiring that $A^{\mathrm{T}}w = -d$ (resp. $A^{\mathrm{T}}w = d$) implies* (3.2) *and* (c) *is weakened by not requiring that a minimal-support solution w of $A^{\mathrm{T}}\eta = -d$ (resp. $A^{\mathrm{T}}\eta = d$) satisfies* (3.2).

Proof. (a) $\Leftrightarrow$ (b): First, observe that by a standard application of Farkas Lemma (e.g. Gale (1960, Theorem 2.7, p. 46)) (a) is equivalent to the implication

$$u, v \in \mathbb{R}^m_+, \ \gamma \in \mathbb{R}, \ A^{\mathrm{T}}(u - v) = \gamma d \ \Rightarrow \ c^{\mathrm{T}}u - b^{\mathrm{T}}v \geqslant 0. \tag{3.3}$$

The fact that (3.3) implies (b) is straightforward. To see the reverse implication, assume that (b) holds and u, $v \in \mathbb{R}^m_+$ and $\gamma \in \mathbb{R}$ satisfy $A^{\mathrm{T}}(u - v) = \gamma d$. We will show that $c^{\mathrm{T}}u - b^{\mathrm{T}}v \geqslant 0$. First consider the case where $\gamma \neq 0$. Let $w = |\gamma|^{-1}(u - v)$ and $z = |\gamma|^{-1}u - w^{+}$. As $|\gamma|^{-1}u \geqslant 0$ and $|\gamma|^{-1}u = w + |\gamma|^{-1}v \geqslant w$ we have that $|\gamma|^{-1}u \geqslant \max\{0, w\} = w^{+}$, assuring that $z \geqslant 0$. Hence, as $c \geqslant b$, we have that $c^{\mathrm{T}}z \geqslant b^{\mathrm{T}}z$. Also, as $w^{+} - w^{-} = w = |\gamma|^{-1}(u - v)$, we have that $|\gamma|^{-1}v - w^{-} = |\gamma|^{-1}u - w^{+} = z$. Now, if $\gamma > 0$ then $A^{\mathrm{T}}w = d$ and if $\gamma < 0$ then $A^{\mathrm{T}}w = -d$. In either case (b) implies that (3.2) holds. Consequently,

$$\begin{aligned} c^{\mathrm{T}}u - c^{\mathrm{T}}v &= |\gamma|[c^{\mathrm{T}}(|\gamma|^{-1}u) - b^{\mathrm{T}}(|\gamma|^{-1}v)] \\ &= |\gamma|[c^{\mathrm{T}}(w^{+} + z) - b^{\mathrm{T}}(w^{-} + z)] \\ &= |\gamma|[c^{\mathrm{T}}w^{+} - b^{\mathrm{T}}w^{-}) + (c^{\mathrm{T}}z - b^{\mathrm{T}}z)] \geqslant 0. \end{aligned}$$

The case where $\gamma = 0$ follows similarly by setting $w = u - v$. Thus the proof that (b) implies (3.3) is complete, establishing the equivalence of (a) and (b).

(b) $\Leftrightarrow$ (c): The implication (b) $\Rightarrow$ (c) is trivial. Next assume (c) holds and assume that either $A^{\mathrm{T}}w = 0$ of $A^{\mathrm{T}}w = d$ or $A^{\mathrm{T}}w = -d$. It follows from Theorem 2.2 that there exist minimal-support solutions $w^1, \ldots, w^t$ of the system $A^{\mathrm{T}}\eta = A^{\mathrm{T}}w$, minimal-support solutions $w^{t+1}, \ldots, w^{t+q}$ of the system $A^{\mathrm{T}}\eta = 0$ and positive numbers $\beta_1, \ldots, \beta_t$ such that $\sum_{s=1}^{t} \beta_s = 1$, $w = \sum_{s=1}^{t} \beta_s w^s + \sum_{s=t+1}^{t+q} w^s$ and $w^1, \ldots, w^{t+q}$ all conform to w. In particular, $w^+ = \sum_{s=1}^{t} \beta_s (w^s)^+$ and $w^- = \sum_{s=1}^{t} \beta_s (w^s)^- + \sum_{s=t+1}^{t+q} (w^s)^-$. It follows from (c) that for $s = 1, \ldots, t+q$ $c^{\mathrm{T}}(w^s)^+ - b^{\mathrm{T}}(w^s)^- \geqslant 0$. Thus,

$$c^{\mathrm{T}}w^+ - b^{\mathrm{T}}w^- = \sum_{s=1}^{t} \beta_s[c^{\mathrm{T}}(w^s)^+ - b^{\mathrm{T}}(w^s)^-] + \sum_{s=t+1}^{t+q} [c^{\mathrm{T}}(w^s)^+ - b^{\mathrm{T}}(w^s)^-] \geqslant 0,$$

completing the proof that (c) implies (b).

Finally, the equivalence of the weakened forms of (a), (b) and (c) follows from analogous arguments. $\square$

Observe that w is a solution of the system $A^{\mathrm{T}}\eta = d$ if and only if $-w$ is a solution of the system $A^{\mathrm{T}}\eta = -d$. Also, recall that $(-w)^+ = w^-$ and $(-w)^- = w^+$. These observations allow one to replace the requirement (3.2) by

$$b^{\mathrm{T}}w^+ - c^{\mathrm{T}}w^- \leqslant 0 \tag{3.4}$$

in parts (b) and (c) of Theorem 3.1. Of course, one can combine (3.6) and (3.2) to a joint inequality

$$b^{\mathrm{T}}w^+ - c^{\mathrm{T}}w^- \leqslant 0 \leqslant c^{\mathrm{T}}w^+ - b^{\mathrm{T}}w^-. \tag{3.5}$$

We remark that (a) implies that $b \leqslant c$, but as the following example illustrates (b) and (c) can hold even when $b \nleqslant c$. We remark that our proof shows that (b) and (c) are equivalent independently of the condition $b \leqslant c$.

Example 3.2. Let

$$A = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}, \quad b = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \text{and} \quad c = \begin{pmatrix} 2 \\ -2 \end{pmatrix}.$$

Then the only solutions to $A^{\mathrm{T}}\eta = 0$ are of the form $w = \alpha(1, 1)^{\mathrm{T}}$, where $\alpha \in \mathbb{R}$, and for these vectors either $w^+ = w$ or $w^- = w$; in either case (3.2) is satisfied.

4. Symmetric scalings of square matrices

In this section we consider symmetric scalings of square matrices. After appropriate definitions, we derive our main result (Theorem 4.1). From it we derive a number of corollaries most of which are known.

For a positive integer s let $\langle s\rangle \equiv \{1, \ldots, s\}$. A (directed) *graph* G is an ordered pair (G_v, G_a) where $G_a \subseteq G_v \times G_v$. In this paper we have $G_v = \langle n\rangle$ for some positive integer n. We order G_a lexicographically, viz. $G_a = (g_1, \ldots, g_m)$ where m denotes the number of elements of G_a. If G is a graph, the *incidence matrix* of G, denoted $\Gamma(G)$, is the $n \times m$ matrix defined in the following way: if $g_r = (i, j)$ then

$$\Gamma(G)_{kr} = \begin{cases} 1 & \text{if } k = i \neq j, \\ -1 & \text{if } k = j \neq i, \\ 0 & \text{otherwise.} \end{cases}$$

(Observe that $\Gamma(G)_{ir} = 0$ if $g_r = (i, i)$ for some $i \in \langle n\rangle$.) A *circulation* of G is a vector $w \in \mathbb{R}^m$ such that $\Gamma(G)w = 0$. A *cycle* of G is a vector $w \neq 0$ with $w_r \in \{-1, 0, 1\}$ for $r = 1, \ldots, m$ where w is a minimal-support solution to the system $\Gamma(G)w = 0$. A *circuit* (or directed cycle) is a cycle w of G with $w \geqslant 0$.

Let $A \in \mathbb{R}^{nn}$. The (directed) *graph associated with* A, written $G(A)$, has $G(A)_v = \langle n\rangle$ and $G(A)_a = \{(i, j) \in \langle n\rangle \times \langle n\rangle \mid A_{ij} \neq 0\}$. Also, the *incidence matrix associated* with A, written $\Gamma(A)$, is the matrix $\Gamma[G(A)]$.

If $A \in \mathbb{R}^{nn}_+$ and $y \in \mathbb{R}^m$, with $g_1, \ldots, g_m$ being the elements of $G(A)_a$ when ordered lexicographically, we define

$$\Pi_y(A) = \prod_{r=1}^{m} A_{ij}^{y_r},$$

where for each r in the above product, (i, j) is chosen as the (unique) pair with $g_r = (i, j)$. Observe that $\Pi_y(A) = \Pi_{y^+}(A)/\Pi_{y^-}(A)$. Notice that if B is any matrix in $\mathbb{R}^{mm}_+$ we may use the notation $\Pi_y(B) = \prod_{r=1}^{m} B_{ij}^{y_r}$ without difficulty, provided that $y_r \geqslant 0$ whenever $B_{ij} = 0$. (We use the convention $0^0 = 1$.) The matrix A involved will not appear explicitly in the notation and will be clear from the context. If $\Pi_{w^-}(B) = 0$ in (4.1) below and elsewhere the inequality of the type $\Pi_w(A) \leqslant \Pi_{w^+}(C)/\Pi_{w^-}(B)$ is to be interpreted as $\Pi_{w^-}(B)\Pi_w(A) \leqslant \Pi_{w^+}(C)$.

Theorem 4.1. *Let B, A, $C \in \mathbb{R}^{nn}_+$ with $B \leqslant C$ and $G(B)_a \subseteq G(A)_a \subseteq G(C)_a$. Then the following are equivalent:*

(a) *There exists a nonsingular diagonal matrix $X \in \mathbb{R}^{nn}_+$ for which $B \leqslant XAX^{-1} \leqslant C$.*

(b) *For each circulation w of $G(A)$,*

$$\Pi_w(A) \leqslant \Pi_{w^+}(C)/\Pi_{w^-}(B). \tag{4.1}$$

(c) *For each cycle w of $G(A)$ (4.1) holds.*

Proof. First assume that $G(A)_a = G(B)_a = G(C)_a$. Let $|G(C)|_a = m$ and let $a, b, c \in \mathbb{R}^m$ be defined by

$$a_r = \log A_{ij}, \quad b_r = \log B_{ij}, \quad c_r = \log C_{ij}, \tag{4.2}$$

where $g_r = (i, j)$ (i.e. (i, j) is the rth element of $G(A)_a$ when ordered lexicographically). It is immediate to see that, with $\Gamma = \Gamma(A)$, (a) is equivalent to the existence of a solution to the system

$$b \leqslant a + \Gamma^{\mathrm{T}}\xi \leqslant c \tag{4.3}$$

and (b) and (c) are, respectively, equivalent to the implications

$$w \in \mathbb{R}^m \text{ is a circulation } \Rightarrow c^{\mathrm{T}}w^+ - b^{\mathrm{T}}w^- \geqslant a^{\mathrm{T}}w, \tag{4.4}$$

$$w \in \mathbb{R}^m \text{ is a cycle } \Rightarrow c^{\mathrm{T}}w^+ - b^{\mathrm{T}}w^- \geqslant a^{\mathrm{T}}w, \tag{4.5}$$

where circulations and cycles are with respect to $G(A)$. The fact that the existence of a solution to (4.3) is equivalent to both (4.4) and to (4.5) follows immediately from Theorem 3.1 (with $d = 0$), and the definitions of circulations and cycles.

We now consider the general case where $G(A)_a \subseteq G(B)_a \subseteq G(C)_a$. Consider the matrix C' obtained from C by replacing C_{ij} by 0 for all i, j with $A_{ij} = 0 < C_{ij}$. To see that (a) $\Rightarrow$ (b), assume that (a) holds with the diagonal matrix X. Evidently, $B' \leqslant X^{-1}AX$ where B' is obtained from B by replacing $B_{ij} = 0$ for which $A_{ij} > 0$ by $(X^{-1}AX)_{ij}$. As B', A and C' satisfy (b). Since $\Pi_{w^-}(B') \geqslant \Pi_{w^-}(B)$ and $\Pi_{w^+}(C) \geqslant \Pi_{w^+}(C')$ for each circulation of A, it easily follows that B, A and C satisfy (b). The implication (b) $\Rightarrow$ (c) is trivial. It remains to see that (c) $\Rightarrow$ (a). Assume that (c) holds. Since the number of cycles of $G(A)$ is finite, it follows that for some $\alpha > 0$ and matrix B' obtained from B by replacing $B_{ij} = 0$ for which $A_{ij} > 0$ by α, satisfies (4.1) together with the matrices A and C' for every cycle w of $G(A)$. As B', A and C' satisfy (c) and it follows that B', A and C' also satisfy (a), which immediately implies that B, A and C satisfy (a). □

Inequality (4.1) in conditions (b) and (c), respectively, of Theorem 4.1 may be given in an alternative form (cf. the remark following Theorem 2.2). Evidently, w is a circulation (resp. a cycle) of $G(A)$ if and only if $-w$ is. Further, for every $y \in \mathbb{R}^m$, $(-y)^+ = y^-$, $(-y)^- = y^+$ and $\pi_{-y}(A) = \pi_y(A)^{-1}$. Hence, we can replace (4.1) by

$$\Pi_w(A) \leqslant \Pi_{w^+}(B)/\Pi_{w^-}(C). \tag{4.6}$$

Of course, we could also augment (4.6) by (4.1) thereby replacing (4.1) in Theorem 4.1 by

$$\Pi_{w^+}(B)/\Pi_{w^-}(C) \leqslant \Pi_w(A) \leqslant \Pi_{w^+}(C)/\Pi_{w^-}(B). \tag{4.7}$$

The following example illustrates that the condition $B \leqslant C$ cannot be dropped from Theorem 4.1. Of course, (a) implies that $B \leqslant C$ and as our proof establishes (b) and (c) are equivalent, independently of the condition $B \leqslant C$.

Example 4.2. Let

$$A = B = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \qquad C = \begin{pmatrix} 0 & 2 & 0 \\ 0 & 0 & 0.5 \\ 1 & 0 & 0 \end{pmatrix}.$$

It is easy to see that the two cycles of $G(A)$ are $(1, 1, 1)$ and $(-1, -1, -1)$. If w is either of these, then $\Pi_w(A) = \Pi_w(B) = \Pi_w(C)$ and either $w = w^+$ or $w = w^-$. Consequently, (b) and (c) are satisfied, though $B_{23} > C_{23}$. One can modify the above example and obtain an alternative (trivial) example by setting $A_{31} = B_{31} = C_{31} = 0$. Then $G(A)$ has no cycles or circulations and (b) and (c) are trivially satisfied, though (still) $B_{23} > C_{23}$.

It is easily seen that if part (a) of Theorem 4.1 holds then, necessarily, $G(B)_a \subseteq G(A)_a \subseteq G(C)_a$. However, parts (b) and (c) of Theorem 4.1 can hold when either $G(B)_a \not\subseteq G(A)_a$ or $G(A)_a \not\subseteq G(C)_a$ (though part (a) will necessarily be violated).

Example 4.3. Let

$$B = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad A = C = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

Part (b) of Theorem 4.1 trivially holds (as $G(A)$ has no circulations), but part (a) does not hold as $G(B)_a \not\subseteq G(A)_a$.

Corollary 4.4 (Saunders and Schneider (1978, Theorem 2.1 and Corollary 2.4)). *Let $A, C \in \mathbb{R}_+^{nn}$ with $G(A)_a = G(C)_a$. Then the following are equivalent:*

(*a*) *There exists a nonsingular diagonal matrix $X \in \mathbb{R}_+^{nn}$ such that $XAX^{-1} = C$.*

(*b*) *For each circulation w of $G(A)$*

$$\Pi_w(A) = \Pi_w(C). \tag{4.8}$$

(*c*) *For each cycle w of $G(A)$,* (4.8) *holds.*

Proof. Let $B = C$. Since $\Pi_{w^+}(B)/\Pi_{w^-}(B) = \Pi_w(B)$ we have that (4.7) is equivalent to (4.8). Consequently, the corollary is immediate from Theorem 4.1 and the following remark. □

Corollary 4.5 (Saunders and Schneider (1979, Theorem 5.2 and Remark 5.6)). *Let $A, C \in \mathbb{R}_+^{nn}$ with $G(A)_a \subseteq G(C)_a$. Then the following are equivalent:*

(a) *There exists a nonsingular diagonal matrix $X \in \mathbb{R}_+^{nn}$ such that $XAX^{-1} \leqslant C$.*

(b) *For each nonnegative circulation w of $G(A)$*

$$\Pi_w(A) \leqslant \Pi_w(C). \tag{4.9}$$

(c) *For each circuit w of $G(A)$,* (4.9) *holds.*

Proof. The corollary is immediate from Theorem 4.1 and the observation that (4.1) is immediate when $w \geqslant 0$. □

5. Asymmetric scalings of rectangular matrices

In this section we consider asymmetric scalings of rectangualr matrices. We start this section with some additional definitions. Our first result and its corollary are then obtained by applying results of the previous section to the bipartitie expansion (definition to follow) of the matrix under consideration. We then derive our main result concerning asymmetric scalings (Theorem 5.5). From it we derive a number of corollaries most of which are known.

A directed graph $H = (H_v, H_a)$ is called *(p, n)-bipartite*, where p and n are positive integers, if $H_v = \langle p+n \rangle$ and $H_a \subseteq \{1, \ldots, p\} \times \{p+1, \ldots, p+n\}$. Cycles of such a graph are called *polygons*. We typically use Δ to denote the incidence matrix of such a graph. Let H be a (p, n)-bipartite graph with incidence matrix $\Delta \in \mathbb{R}^{p+n,m}$. A *pseudodiagonal* of H is a vector $x \in \mathbb{R}^m$ with $\Delta x = f$ where $f \in \mathbb{R}^{p+n}$ is defined by $f_r = 1$ for $r = 1, \ldots, p$ and $f_r = -1$ for $r = p+1, \ldots, p+n$. We call a pseudodiagonal *integral* if in addition x_r is an integer for $r = 1, \ldots, p+n$. (Our definitions differ from those introduced in Saunders and Schneider (1978) where integrality is required in the definition of a pseudodiagonal.) A *minimal-support pseudodiagonal* of H is a minimal-support solution to the system $\Delta\xi = f$. A *diagonal* of H is a pseudodiagonal w with $w_i \in \{0, 1\}$ for $i = 1, \ldots, m$. Some relationships between these definitions are summarized in the Appendix. In particular, H has no pseudodiagonals when $p \neq n$. Other properties of pseudodiagonals can be found in Saunders and Schneider (1978).

Let $A \in \mathbb{R}^{pn}$. The *bipartite expansion* of A is defined to be the matrix A^* given by

$$A^* = \begin{pmatrix} 0 & A \\ 0 & 0 \end{pmatrix},$$

(where the zero matrices are of apporpriate dimensions). The (directed) *(p, n)-bipartite graph associated with A*, written $H(A)$, is defined to be the graph $G(A^*)$. Also, the *(p, n)-bipartite incidence matrix associated* with A, written $\Delta(A)$, is the matrix $\Gamma(A^*)$. If $A \in \mathbb{R}^{pn}$ and $y \in \mathbb{R}^m$ where the number of elements in $H(A)_a$ is m, we define $\Pi_y(A) \doteq \Pi_y(A^*)$, where $\Pi_y(A^*)$ is defined as in Section 4. Of course, when A is square, $\Pi_y(A) = \Pi_y(A^*)$ and therefore this definition is consistent with the one for square matrices.

Theorem 5.1. *Let $B, A, C \in \mathbb{R}^{pn}$ with $B \leq C$ and $H(B)_a \subseteq H(A)_a \subseteq H(C)_a$. Then the following are equivalent*:

(a) *There exist nonsingular diagonal matrices $X \in \mathbb{R}_+^{pp}$ and $Y \in \mathbb{R}_+^{nn}$ for which $B \leq XAY^{-1} \leq C$.*

(b) *For each circulation w of $H(A)$*

$$\Pi_w(A) \leq \Pi_{w^+}(C)/\Pi_{w^-}(B). \tag{5.1}$$

(c) *For each polygon w of $H(A)$ (5.1) holds.*

Proof. If $X \in \mathbb{R}_+^{pp}$ and $Y \in \mathbb{R}_+^{nn}$ are nonsingular diagonal matrices and $Z = X \oplus Y \in \mathbb{R}^{p+n,p+n}$, then $ZA^*Z^{-1} = C$ is equivalent to $XAY^{-1} = C$, where A^* is the bipartite expansion of A. The equivalence of (a), (b) and (c) now follows immediately from the fact that $\Pi_y(A) = \Pi_y(A^*)$, combined with the application of Theorem 4.1 to A^*. □

Corollary 5.2 (Saunders and Schneider (1979, Theorem 3.1 and Remark 5.3)). *Let $A, C \in \mathbb{R}^{pn}$ with $H(A)_a = H(C)_a$. Then the following are equivalent*:

(a) *There exist nonsingular diagonal matrices $X \in \mathbb{R}_+^{pp}$ and $Y \in \mathbb{R}_+^{nn}$ for which $XAY^{-1} = C$.*

(b) *For each circulation w of $H(A)$*

$$\Pi_w(A) = \Pi_w(C). \tag{5.2}$$

(c) *For each polygon w of $H(A)$, (5.2) holds.*

Proof. The proof follows the same lines as that of Corollary 4.4. □

The analog to Corollary 4.5 is trivial. For $A, C \in \mathbb{R}^{pn}$ with $H(A) = H(C)$, one can always find nonsingular diagonal matrices $X \in \mathbb{R}_+^{pp}$ and $Y \in \mathbb{R}_+^{nn}$ such that $XAY^{-1} \leqslant C$. Also, $H(A)$ has no nonnegative circulations. So, (a), (b) and (c) of Corollary 4.5 trivially apply to A^* and C^*. It therefore seems natural to impose some normalization condition on the corresponding diagonal matrices.

Theorem 5.3. *Let $B, A, C \in \mathbb{R}_+^{pn}$ with $B \leqslant C$ and $H(B)_a \subseteq H(A)_a \subseteq H(C)_a$. Then the following are equivalent*:

(a) *There exist nonsingular diagonal matrices $X \in \mathbb{R}_+^{pp}$ and $Y \in \mathbb{R}_+^{nn}$ with* $\det X = \det Y$ *such that $B \leqslant XAY^{-1} \leqslant C$.*

(b) *For each circulation w of $H(A)$ (5.2) holds and for each pseudodiagonal w of $H(A)$*

$$\Pi_{w^+}(B)/\Pi_{w^-}(C) \leqslant \Pi_w(A) \leqslant \Pi_{w^+}(C)/\Pi_{w^-}(B). \tag{5.3}$$

(c) *For each polygon w of $H(A)$, (5.1) holds and for each minimal-support pseudodiagonal w of $H(A)$, (5.3) holds.*

Proof. The proof follows the same lines as that of Theorem 4.1. □

The following example illustrates that, unlike Saunders and Schneider (1978, Theorem 4.6), it is not possible to weaken (b) in Theorem 5.3 by requiring that (just) one pseudodiagonal of $H(A)$ satisfies (5.3) (provided a pseudodiagonal of $H(A)$ exists).

Example 5.4. Let

$$B = \begin{pmatrix} 1 & 1 \\ 1 & e \end{pmatrix}, \quad A = \begin{pmatrix} 1 & 1 \\ e^2 & e^2 \end{pmatrix} \quad \text{and} \quad C = \begin{pmatrix} 1 & 1 \\ e & e^2 \end{pmatrix}.$$

Then $b = (0, 0, 0, 1)^T$, $a - (0, 0, 2, 2)^T$, $c = (0, 0, 1, 2)^T$, and

$$\Delta = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ -1 & 0 & -1 & 0 \\ 0 & -1 & 0 & -1 \end{pmatrix}.$$

The only two cycles of $H(A)$ are $w = (1, -1, -1, 1)^T$ and $-w$. Now, $w^+ = (1, 0, 0, 1)^T = (-w)^-$, $w^- = (0, 1, 1, 0)^T = (-w)^+$, $a^T w = 0$, $b^T w^+ = 1$, $b^T w^- = 0$, $c^T w^+ = 2$ and $c^T w^- = 1$. So, $a^T w = 0 \leq 2 = c^T w^+ - b^T w^-$ and $a^T(-w) = 0 \leq c^T(-w)^+ - b^T(-w)^-$. Next, observe that $u = w^-$ is a diagonal (with $u^+ = u = w^-$ and $u^- = 0$). So, $b^T u^+ - c^T u^- = 1 \leq 2 = a^T u = c^T u^+ - b^T u^-$. But $v = w^+$ is a diagonal (with $v^+ = v = w^+$ and $v^- = 0$) and $a^T v = 2 > 1 = c^T v^+ - b^T v^-$.

A characterization of (p, n)-bipartite graphs having no pseudodiagonals is given in the Appendix. In particular, (p, n)-bipartite graphs have no pseudodiagonals when $p \neq n$. When $H(A)$ has no pseudodiagonals, conditions (b) and (c) of Theorem 5.3 reduce, respectively, to the corresponding conditions of Theorem 5.1. Formally, we obtain the following immediate corollary.

Corollary 5.5. *Let $B, A, C \in \mathbb{R}^{pn}$ with $B \leq C$ and $H(B)_a \subseteq H(A)_a \subseteq H(C)_a$. If $H(A)$ has no pseudodiagonals then there exist nonsingular diagonal matrices $X \in \mathbb{R}_+^{pp}$ and $Y \in \mathbb{R}_+^{nn}$ such that $B \leq XAY^{-1} \leq C$ if and only if there exist nonsingular diagonal matrices $\tilde{X} \in \mathbb{R}_+^{pp}$ and $\tilde{Y} \in \mathbb{R}_+^{nn}$ with* $\det \tilde{X} = \det \tilde{Y}$ *such that $B \leq \tilde{X}A\tilde{Y}^{-1} \leq C$.* □

The following two corollaries follow from Theorem 5.3 by the same arguments used to deduce Corollaries 4.4 and 4.5, respectively, from Theorem 4.1.

Corollary 5.6. *Let $A, C \in \mathbb{R}_+^{pn}$ with $H(A)_a = H(C)_a$. Then the following are equivalent*:

(a) *There exist nonsingular diagonal matrices $X \in \mathbb{R}_+^{pp}$ and $Y \in \mathbb{R}_T^{nn}$ with* $\det X = \det Y$ *such that $XAY^{-1} = C$.*

(b) *For each circulation w of $H(A)$ and for each pseudodiagonal w of $H(A)$.*

$$\Pi_w(A) = \Pi_w(C). \tag{5.4}$$

(c) *For each polygon w of $H(A)$ and for each minimal-support pseudodiagonal* (5.4) *holds.* □

It was shown in Saunders and Schneider (1978, Theorem 3.3 and 4.6) that if $H(A)$ has a pseudodiagonal, it is enough to suppose that (5.4) holds for each polygon and just one pseudodiagonal of $H(A)$ in order to conclude (a) of Corollary 5.6.

Corollary 5.7 (Saunders and Schneider (1979, Theorem 5.7 and Remark 5.6)). *Let $A, C \in \mathbb{R}_+^{pn}$ with $H(A)_a = H(C)_a$. Then the following are equivalent*:

(a) *There exist nonsingular diagonal matrices* $X \in \mathbb{R}_+^{pp}$ *and* $Y \in \mathbb{R}_+^{nn}$ *with* $\det X = \det Y$ *such that* $XAY^{-1} \leq C$.

(b) *For each nonnegative circulation or nonnegative pseudodiagonal w of* $H(A)$

$$\Pi_w(A) \leq \Pi_w(C), \tag{5.5}$$

for each nonpositive pseudodiagonal w of $H(A)$

$$\Pi_w(C) \leq \Pi_w(A). \tag{5.6}$$

(c) *For each nonnegative cycle or nonnegative minimal-support pseudodiagonal w of* $H(A)$ (5.6) *holds.* □

When $H(A)$ has no pseudodiagonals, conditions (b) and (c) of Corollary 5.7 reduce, respectively, to the corresponding conditions of Corollary 5.2. Formally, we obtain the following immediate corollary.

Corollary 5.8 (Saunders and Schneider (1979, Corollary 4.9)). *Let* $A, C \in \mathbb{R}^{pn}$ *with* $H(A)_a = H(C)_a$. *If* $H(A)$ *has no pseudodiagonals then there exist nonsingular diagonal matrices* $X \in \mathbb{R}^{pp}$ *and* $Y \in \mathbb{R}^{nn}$ *such that* $XAY^{-1} = C$ *if and only if there exist nonsingular diagonal matrices* $\tilde{X} \in \mathbb{R}^{pp}$ *and* $\tilde{Y} \in \mathbb{R}^{nn}$ *with* $\det \tilde{X} = \det \tilde{Y}$ *such that* $\tilde{X}A\tilde{Y}^{-1} = C$. □

Our methods can be used to study asymmetric and symmetric scalings of the form XAY^{-1} where $\{\log X_{ii}\}$ and $\{\log Y_{ii}\}$ satisfy a linear relation other than $\sum_i \log X_{ii} = \sum_i \log Y_{ii}$. We next state the results corresponding to the inequalities $\sum_i \log X_{ii} \leq \sum_i \log Y_{ii}$ and $\sum_i \log X_{ii} \geq \sum_i \log Y_{ii}$ (i.e. $\det X \leq \det Y$ and $\det Y \leq \det X$, respectively). We did not find other interesting examples and therefore do not include a formal presentation of the general result.

Theorem 5.9. *Let* $B, A, C \in \mathbb{R}_+^{pn}$ *with* $B \leq C$ *and* $H(A)_a \subseteq H(B)_a \subseteq H(C)_a$. *Then the following are equivalent*:

(a) *There exist nonsingular diagonal matrices* $X \in \mathbb{R}_+^{pp}$ *and* $Y \in \mathbb{R}_+^{nn}$ *with* $\det X \leq \det Y$ (resp. $\det X \geq \det Y$) *such that* $B \leq XAY^{-1} \leq C$.

(b) *For each circulation w of* $H(A)$ (5.1) *holds and for each pseudodiagonal w of* $H(A)$

$$\Pi_{w^+}(B)/\Pi_{w^-}(C) \leq \Pi_w(A) \tag{5.7}$$

(resp.

$$\Pi_w(A) \leq \Pi_{w^+}(C)/\Pi_{w^-}(B)). \tag{5.8}$$

(c) *For each cycle w of* $H(A)$, (5.1) *holds and for each minimal-support pseudodiagonal w of* $H(A)$, (5.7) (*resp.* (5.8)) *holds.* □

Appendix

In this Appendix we summarize some properties of pseudodiagonals.

Lemma A.1. *Let H be a (p, n)-bipartite graph for some positive integers p and n. Then*
(a) *Every minimal-support pseudodiagonal of H is an integral pseudodiagonal of H.*
(b) *Every diagonal of H is a minimal-support pseudodiagonal of H.*
(c) *Every nonnegative integral pseudodiagonal of H is a diagonal of H.*

Proof. Assertions (b) and (c) are straightforward. Also, assertion (a) is immediate from the (well-known) unimodularity of $\Gamma(H)$ and the (standard) observation that if w is a minimal-support pseudodiagonal of H then the columns of H corresponding to nonzero coordinates of w are independent. □

We next characterize bipartite graphs having pseudodiagonal. A simple necessary and sufficient condition was obtained by Saunders and Schneider (1978). We slightly generalize their results.

We need two additional definitions. A *component* of a graph $G = (G_v, G_a)$ is an equivalence class under the (equivalence) relation defined on G_v by saying that i and j are equivalent if $i = j$ or there exist $g_1, \ldots, g_t \in G_a$ and $i_1, \ldots, i_t \in G_v$ such that, with $i_0 = i$ and $i_t = k$, for $s = 1, \ldots, t$ either $g_s = (i_{s-1}, i_s)$ or $g_s = (i_s, i_s - 1)$. We say that G is *connected* if it has only one component.

Lemma A.2. *Let H be a (p, n)-bipartite graph for some positive integers p and n. Then the following are equivalent:*
(a) *H has a pseudodiagonal.*
(b) *H has an integral pseudodiagonal.*
(c) *H has a minimal-support pseudodiagonal.*
(d) *Each connected component of H has as many vertices in $\{1, \ldots, p\}$ as in $\{p+1, \ldots, p+n\}$.*

Moreover, a necessary condition for the above (equivalent) conditions to hold is that $p = n$. Also, a sufficient condition for the above (equivalent) conditions to hold is that $p = n$ and in addition H is connected.

Proof. The equivalence of (a), (b) and (c) follows by Lemma A.1. The equivalence of (b) and (d) was established in Saunders and Schneider (1978). The necessary condition and the sufficient condition for (a)–(d) to hold are immediate from (d). □

We remark that Saunders and Schneider (1978, p. 214) have illustrated that there exist bipartite graphs having no diagonals but having pseudodiagonals.

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