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ON THE CONTROLLABILITY OF MATRIX PAIRS (A, K) WITH K POSITIVE SEMIDEFINITE*

DAVID CARLSON†, B. N. DATTA‡ AND HANS SCHNEIDER§

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Abstract. The controllability of matrix pairs (A, K) is studied when K is positive semi-definite, and in particular when K is in the range of the Lyapunov map determined by A. This extends previous work of Chen, Wimmer, Carlson and Loewy, and Coppel.

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1. Introduction. This note is devoted to the study of the controllability of (A, K), where $A \in C^{n,n}$, $K \in H_n$ (the set of hermitian matrices in $C^{n,n}$), and K is positive semidefinite (which we shall write as $K \geqq 0$). A well-known result, proved independently by Chen [5] and Wimmer [11], states:

THEOREM 1. *Let $A \in C^{n,n}$, and suppose that $K = AH + HA^* \geqq 0$ for some H, $K \in H_n$. If (A, K) is controllable, then A has no eigenvalues on the imaginary axis and H is nonsingular (and, in fact, the numbers of eigenvalues of A with positive and negative real parts equal respectively the numbers of positive and negative eigenvalues of H).*

Using Theorem 1, Wimmer extended previous results in the damping of certain quadratic differential equations involved in linear vibration problems.

An example [3, p. 240] shows that the converse of Theorem 1 is false. However, working independently of Chen and Wimmer, Carlson and Loewy [3] established a converse under an additional hypothesis:

THEOREM 2. *Let $A \in C^{n,n}$, such that $\lambda + \bar{\mu} \neq 0$ for all eigenvalues λ, μ of A. Suppose that $K = AH + HA^* \geqq 0$ for some H, $K \in H_n$. Then the following are equivalent:*

(i) *(A, K) is controllable.*
(ii) *H is nonsingular.*

The question thus arose as to the role of the additional hypothesis of Theorem 2 in a more complete converse of Theorem 1. We answer this question by proving a result (Theorem 4) which will yield, under $K = AH + HA^* \geqq 0$, a condition equivalent to the controllability of (A, K) in terms of the spectrum of A and the nonsingularity of a matrix $\hat{H}$ determined by A and H. The matrix $\hat{H}$ is obtained from H via projections associated with an A-modal decomposition of C^n; see § 2 for definitions. Our proof of this result will use Theorem 1 and a result in [3] preliminary to Theorem 2; the result itself contains Theorem 2 as a special case.

As a consequence of Theorem 4 we will be able to discuss special cases (like that in Theorem 2) in which $\hat{H}$ may be replaced by H, that is, for which (i) and (ii) above are equivalent. This clarifies (see also [7]) Coppel's discussion in [6] of the relationship between dichotomies for linear differential equations and Lyapunov functions in the constant-coefficient case. Coppel's work, along with that of Chen, Wimmer, and Carlson and Loewy, has motivated our investigations.

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† Mathematical Sciences Department, San Diego State University, San Diego, California 92182. Permanent address, Mathematics Department, Oregon State University, Corvallis, Oregon 97331.

‡ Mathematical Sciences Department, Northern Illinois University, DeKalb, Illinois 60115.

§ Mathematics Department, University of Wisconsin, Madison, Wisconsin 53706. The research of this author was partly supported by the National Science Foundation under grant MCS 80-26132.

2. Definitions. So that our decompositions depend only on the spaces involved and not particular choices of bases for the spaces, we will set our results in an equivalent but seemingly more abstract setting. Let V be a finite-dimensional inner product space, and let $L(V)$, $H(V)$ be respectively the sets of linear operators and self-adjoint linear operators on V. $K \in H(V)$ is positive semi-definite iff $(x, Kx) \geqq 0$ for all $x \in V$.

Let $A \in L(V)$ have spectrum $\sigma(A) = \{\lambda_1, \cdots, \lambda_n\}$; let $\delta(A)$ be the number of eigenvalues λ_i which are imaginary, and let $\Delta(A) = \prod_{i,j=1}^{n} (\lambda_i + \bar{\lambda}_j)$. Evidently $\Delta(A) \neq 0$ is equivalent to $\sigma(A) \cap \sigma(-A^*) = 0$, where A^* is the adjoint of A, and $\delta(A) = 0$ is equivalent to $\sigma(A) \cap iR = 0$. Thus $\Delta(A) \neq 0$ implies that $\delta(A) = 0$; the converse is false. We denote the kernel and image of A by Ker A and Im A respectively, and the rank of A by $\rho(A)$.

Let $A, B \in L(V)$; the *control space* of (A, B) is $C(A, B) = \sum_{r=0}^{\infty} \operatorname{Im} A^r B$, the smallest A-invariant space containing Im B. Note that $C(A, B)$ depends only on A and Im B and that (because of the Cayley–Hamilton Theorem),

$$C(A, B) = \sum_{r=0}^{n-1} \operatorname{Im} A^r B.$$

The pair (A, B) is said to be *controllable* if $C(A, B) = V$.

For $A \in L(V)$, we may decompose V (generally in a number of ways) as $V = V_1 \oplus \cdots \oplus V_p$, so that each V_j is A-invariant, and so that the restrictions $A|V_j$ of A to distinct V_j have disjoint spectra. Following Wonham [12, p. 18], we call such decompositions *A-modal*. In the finest A-modal decomposition of V the V_j are the generalized eigenspaces of the distinct eigenvalues of A. We call this the *A-spectral decomposition* of V. Another natural A-modal decomposition is obtained by choosing $V_1 = V_+$, $V_2 = V_-$, and $V_3 = V_0$, the direct sums of the generalized eigenspaces of the eigenvalues of A with, respectively, positive, negative, and zero real parts. We call this the *A-inertial decomposition* of V.

If $V = V_1 \oplus \cdots \oplus V_p$ is an A-modal decomposition of V, for $j = 1, \cdots, p$ we let E_j denote the projection in $L(V)$ onto V_j which annihilates $\sum_{i \neq j} V_i$. It is well-known that $\sum_{j=1}^{p} E_j = I$, that $E_i E_j = 0$, $i \neq j$, and that each E_j is a polynomial in A, cf. [8, p. 221]. Also, $V = W_1 \oplus \cdots \oplus W_p$, where each W_j is the range in V of the corresponding projection E_j^* in $L(V)$. For $j = 1, \cdots, p$, as V_j is A-invariant, we may set

$$A_j = E_j A E_j = A E_j = E_j A, \tag{1}$$

so that

$$A = \sum_{j=1}^{p} A_j, \tag{2}$$

and for $K \in H(V)$, we set

$$K_{jj} = E_j K E_j^*, \tag{3}$$

$$\hat{K} = \sum_{j=1}^{p} K_{jj}. \tag{4}$$

Note that $\rho(\hat{K}) = \sum_{j=1}^{p} \rho(K_{jj})$.

The restrictions of the linear operations A and A_j to V_j are equal, and the restrictions to W_j of the Hermitian forms induced by K and K_{jj} are equal: if $x, y \in W_j$,

$$(y, K_{jj} x) = (y, E_j K E_j^* x) = (E_j^* y, K E_j^* x) = (y, Kx).$$

3. Results.

LEMMA 1. *Let* $A \in L(V)$, *and suppose that* $V = V_1 \oplus \cdots \oplus V_p$ *is an* A-*modal decomposition. If* $K \in H(V)$, *with* $K \geqq 0$, *then* $C(A, \hat{K}) = C(A, K)$.

Proof. We observe that

$$(5) \qquad C(A, \hat{K}) = C(A_1, K_{11}) \oplus \cdots \oplus C(A_p, K_{pp}) \qquad j = 1, \cdots, p,$$

since $A^r\hat{K} = \sum_{j=1}^p A_j^r K_{jj}$, and $\operatorname{Im} A^r K_{jj} \subseteq V_j$, $r = 0, \cdots, n-1$; it follows that $\operatorname{Im} A^r \hat{K} = \sum_{j=1}^r \operatorname{Im} A_j^r K_{jj}$. Thus to prove $C(A, K) \supseteq C(A, \hat{K})$, it is sufficient to show that $C(A, K) \supseteq C(A_j, K_{jj})$, $j = 1, \cdots, p$. Since E_j is a polynomial in A, we have

$$\operatorname{Im}(A_j^r K_{jj}) = \operatorname{Im}(E_j A^r E_j K E_j^*) \subseteq \operatorname{Im}(E_j A^r E_j K) \subseteq C(A, K)$$

and the inclusion follows. (We have not used $K \geqq 0$ here.)

To prove $C(A, K) \subseteq C(A, \hat{K})$, we first note the easily-proved result that $K \geqq 0$ implies that $\operatorname{Ker} \hat{K} \subseteq \operatorname{Ker} K$. It follows that $\operatorname{Im} K \subseteq \operatorname{Im} \hat{K}$ and hence $\operatorname{Im} A^r K \subseteq \operatorname{Im} A^r \hat{K}$, $r = 1, \cdots, n-1$. The result follows. □

THEOREM 3. *Let* $A \in L(V)$, *and suppose that* $V = V_1 \oplus \cdots \oplus V_p$ *is an* A-*modal decomposition of* V. *Suppose that* $K \in H(V)$, *with* $K \geqq 0$. *Then the following are equivalent*:

(i) (A, K) *is controllable.*
(ii) $C(A_j, K_{jj}) = V_j$, $j = 1, \cdots, p$.
(iii) $(A\hat{K})$ *is controllable.*
(iv) $(x, Kx) > 0$ *for every eigenvector* x *of* A^*.

Proof. The equivalence of (i), (ii), and (iii) follows immediately from (5) and Lemma 1. The equivalence of (i) and (iv) is a special case of [2, Lemma 3]. □

We remark that a related result which holds for all $K \in L(V)$ is known, cf. [12, p. 45, Exercise 1.5].

The equivalence of (i) and (iv) was noted in [3] (under the unnecessary assumption that $\Delta A \neq 0$); it is in fact merely a rephrasing of Hautus' criterion for controllability (cf. [11]) in the case that $K \geqq 0$. We cannot drop the condition $K \geqq 0$ from either Lemma 1 or Theorem 3: let $V = C^2 = V_1 \oplus V_2$, where V_1, V_2 are the coordinate subspaces, and let

$$A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad K = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix};$$

then (A, K) is controllable, but $C(A, K) = \{0\}$ since $\hat{K} = 0$.

In Theorem 3 we considered the controllability of pairs (A, K) where the only restriction on K is $K \geqq 0$. We shall now assume that $K = AH + HA^* \geqq 0$, where $H \in H(V)$. We note that the mapping $H \to AH + HA^*$ of $H(V)$ into itself is onto $H(V)$ if and only if $\Delta(A) \neq 0$.

Before stating Lemma 2, we must take care of a trivial but awkward technicality which we require to relate our results to Theorems 1 and 2. If $K = AH + HA^*$, note that $K_{jj} = A_j H_{jj} + H_{jj} A_j^*$ and indeed, for any $c \in C$, $K_{jj} = B_j H_{jj} + H_{jj} B_j^*$, where $B_j = A_j + c(\sum_{i \neq j} E_i)$. Then $\sigma(B_j) = \sigma(A|V_j) \cup \{c\}$. Hence, if $\Delta(A|V_j) \neq 0$, $j = 1, \cdots, p$, we may choose $c \in R$ so that $\Delta(B_j) \neq 0$, $j = 1, \cdots, p$.

LEMMA 2. *Let* $A \in L(V)$, *and let* $V = V_1 \oplus \cdots \oplus V_p$ *be an* A-*modal decomposition with* $\Delta(A|V_j) \neq 0$, $j = 1, \cdots, p$. *Let* H, $K \in H_n$ *with* $K = AH + HA^* \geqq 0$. *Then* $C(A, K) = \operatorname{Im} \hat{H}$.

Proof. For $j = 1, \cdots, p$, since $\Delta(A|V_j) \neq 0$, we choose $c \in R$ so that $B_j = A_j + c\sum_{i \neq j} E_i$ has $\Delta(B_j) \neq 0$. Also $C(B_j, K_{jj}) = C(A_j, K_{jj})$ and

$$B_j H_{jj} + H_{jj} B_j^* = A_j H_{jj} + H_{jj} A_j^* = K_{jj} \geqq 0.$$

By Corollary 2 of [3], then,

$$\operatorname{Im} H_{jj} = C(B_j, K_{jj}) = C(A_j, K_{jj}),$$

and the lemma follows by Lemma 1 and (5). □

THEOREM 4. *Let $A \in L(V)$ with $\delta(A)=0$. Suppose that $V = V_1 \oplus \cdots \oplus V_p$ is an A-modal decomposition and that, for each $j=1,\cdots,p$, $\Delta(A|V_j) \neq 0$. Let $K = AH + HA^* \geqq 0$ for $H, K \in H(V)$. The following are equivalent*:

(i) *(AK) is controllable.*
(ii) *$\hat{H}$ is nonsingular.*
(iii) *$(x, Hx) \neq 0$ for each eigenvector x of A^*.*
(iv) *$(x, \hat{H}x) \neq 0$ for each eigenvector x of A^*.*
(v) *H is nonsingular and $(A^*, H^{-1}K)$ is controllable.*

Proof. We note first that $\delta(A)=0$ guarantees that there exists an A-modal decomposition $V = V_1 \oplus \cdots \oplus V_p$ for which $\Delta(A|V_j) \neq 0$, $j=1,\cdots,p$.

The equivalence of (i) and (ii) follows immediately from Lemma 2.

To show that (i) and (iii) are equivalent, note that for any $x \in V$ for which $A^*x = \lambda x$,

$$(x, Kx) = (x, (AH + HA^*)x) = (A^*x, Hx) + (x, H(A^*x)) = (\bar{\lambda} + \lambda)(x, Hx),$$

and use condition (iv) from Theorem 3. To show that (iii) and (iv) are equivalent, we observe that if $A^*x = \lambda x$, then $x \in W_j$ for some j, $1 \leqq j \leqq p$, where W_j is defined in § 2. Hence $(x, Hx) = (x, H_{jj}x) = (x, \hat{H}x)$.

To show that (i) and (v) are equivalent, suppose that H is nonsingular. Then

$$A^*H^{-1} + H^{-1}A = H^{-1}KH^{-1},$$

and the equivalence follows easily from [1, Thm. 4]. □

We state the special case of A-inertial decomposition as a

COROLLARY 1. *Let $A \in L(V)$ and let $V = V_+ \oplus V_- \oplus V_0$ be the A-inertial decomposition of V. Suppose $K = AH + HA^* \geqq 0$ for some $H, K \in H(V)$. Then (A, K) is controllable if and only if $\delta(A)=0$ and $\hat{H}$ is nonsingular.*

Proof. If $\delta(A)=0$, then $V = V_+ \oplus V_-$ is an A-modal decomposition with $\Delta(A|V_+) \neq 0$, $\Delta(A|V_-) \neq 0$, and Theorem 4 applies.

If $\delta(A) \neq 0$, then by Theorem 1, (A, K) is not controllable. □

As an example, let $V = C^2$ and let

$$A = \begin{pmatrix} 1 & 0 \\ 2 & -1 \end{pmatrix}, \quad H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad K = AH + HA^* = \begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix} \geqq 0.$$

Here $\delta(A) = \delta(H) = 0$, yet $(A, \hat{K})$ is not controllable. We have

$$V_+ = \left\langle \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\rangle, \quad V_- = \left\langle \begin{pmatrix} 0 \\ -1 \end{pmatrix} \right\rangle, \quad E_1 = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}, \quad E_2 = \begin{pmatrix} 0 & 0 \\ -1 & 1 \end{pmatrix},$$

and

$$\hat{H} = E_1 H E_1^* + E_2 H E_2^* = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix},$$

which is singular.

Finally, we observe that if $V = V_1 \oplus \cdots \oplus V_p$ is any A-modal decomposition and each V_j is also H-invariant (in particular, this is true if $p=1$ or if A and H commute), then H commutes with all E_j (cf. [8, p. 221]) and

$$\hat{H}=\sum_{j=1}^{p} E_j H E_j^* = H\left(\sum_{j=1}^{p} E_j E_j^*\right).$$

It is easily shown that $\sum_{j=1}^{p} E_j E_j^*$ is nonsingular, so that H and $\hat{H}$ are singular or nonsingular together, and we may replace $\hat{H}$ by H in (ii) of Theorem 4. If, for example, all eigenvalues of A are known to have negative real part, then (A, K) is controllable if and only if H is nonsingular. This result is stated in [7].

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