6

Characterizations of Extreme Normalized Circulations Satisfying Linear Constraints*[†]

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ABSTRACT

This paper considers polytopes of circulations (flows) on a graph which satisfy given linear (homogenous) constraints. Algebraic characterizations of the extreme points of such polytopes are obtained. We also characterize subgraphs which support extreme points. Finally, a formula (geometric in nature) is obtained for the dimension of the flow space of a set of cycles.

1. INTRODUCTION

In our previous paper [2] optimal scaling problems for matrices led to the consideration of a polytope P of normalized circulations (flows) on a graph

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where each circulation satisfies a single constraint. In this paper we consider a polytope P of normalized circulations where there may be more than one linear (homogenous) constraint. After introducing some definitions in Section 2, we give algebraic characterizations of the extreme points of P. In particular, we define a space W of constrained circulations and show that, subject to an obvious normalization, a circulation u is an extreme point of P if and only if the intersection of the space W and the circulation space on the support of u is spanned by u. We draw various corollaries. These results may be found in Section 3. In Section 4 we obtain algebraic characterizations of subgraphs that support extreme points of P. Finally, in the Appendix we state a formula for the dimension of the flow space of a union of cycles. This formula is geometric in nature.

As remarked in [2], the characterization of the extreme points of the above polytope P is important, since it provides a characterization of the basic feasible solutions of linear programs whose corresponding feasible set is P. Although the explicit enumeration of the extreme points is usually not an efficient computational method, it might be possible to use our characterizations to develop variants of the simplex method which use the special structure of basic feasible solutions to accelerate computation. In particular, the geometric result in the Appendix concerning the dimension of the flow space of union of cycles might be useful. Our results have some application to the scalings of matrices which are optimal for certain measures of which the results in [2] are special cases.

2. DEFINITIONS AND NOTATION

Let R be the real field, and let R_+ be the set of nonnegative reals. By R^{nm} (respectively, R_+^{nm}) we denote the set of all $n \times m$ matrices with elements in R (respectively, R_+). As usual, R^n (R_+^n) will stand for R^{n1} (R_+^{n1}). Throughout, we use subscripts for coordinates. Also, the *rank* of a matrix M will be denoted rank M.

For a positive integer s let $\langle s \rangle \equiv \{1,\dots,s\}$. A (directed) *graph* G is an ordered pair (G_v, G_a). Elements of G_v are called *vertices*, and elements of G_a are called *arcs*. In this paper we shall have $G_v \subseteq \langle n \rangle$ for some positive integer n. With each arc a we associate two vertices, one called the *initial* and the other called the *final* vertex of a. Thus, we allow several arcs between two given vertices. We order G_a in some fixed manner, viz., $G_a = (a_1,\dots,a_m)$, where m denotes the number of elements of G_a. If G is a graph, the *vertex-arc incidence matrix*, denoted $\Gamma(G)$, is the $n \times m$ matrix defined in the following way: if arc a_q has initial vertex i and final vertex j, then

$$\Gamma(G)_{kq}=\begin{cases} 1 & \text{if } k=i\neq j, \\ -1 & \text{if } k=j\neq i, \\ 0 & \text{otherwise.}\end{cases}$$

[Observe that $\Gamma(G)_{iq}=0$ if $i=j$.]

We say that a graph $G'=(G'_v,G'_a)$ is a *subgraph* of $G=(G_v,G_a)$ if $G'_v=G_v$ and $G'_a\subseteq G_a$. (This definition is somewhat more restrictive than the usual one.) We write $G'\subseteq G$ when G' is a subgraph of G. We say that a subgraph H of a graph G is a *minimal* (*maximal*) subgraph of G satisfying a given property, if H satisfies that property and the only subgraph of H (the only subgraph of G of which H is a subgraph) satisfying that property is H itself.

Let G be a graph having n vertices and m arcs. Let $x^1,\ldots,x^s\in R^m$. The *support* $G(x^1,\ldots,x^s)$ of $\{x^1,\ldots,x^s\}$ is the subgraph G' of G having $G'_a=\{a_q: x^i_q\neq 0 \text{ for some } i=1,\ldots,s\}$ and $G'_v=\langle n\rangle$. A *circulation* of a *subgraph* G' of G is a vector $x\in R^m$ for which $\Gamma(G)x=0$ and $G(x)\subseteq G'$. The term *circulation* (when no subgraph is mentioned) will refer to a circulation for G itself, i.e., x is a circulation if and only if $\Gamma(G)x=0$. The set of all circulations for a given subgraph G' of G is clearly a subspace of R^m. This subspace will be called the *circulation space* of G' and will be denoted $C(G')$. We shall also use the abbreviated notation $C(x^1,\ldots,x^s)$ for $C(G(x^1,\ldots,x^s))$ where $x^i\in R^m$, $i=1,\ldots,s$. A *cycle* of a subgraph G' of G is a nonzero circulation z of G', with $z_q\in\{0,1,-1\}$ for all $q=1,\ldots,m$, such that $G(z)$ is a minimal subgraph of G which is a support of a nonzero circulation. The term *cycle* (when no subgraph of G is mentioned) will refer to a cycle of G itself. Of course, if z is a cycle, so is $-z$. Also, it is well known that the circulation space of any subgraph G' of G is spanned by the set of cycles of G' (cf. [1, p. 90]).

For $x\in R^m$, we define the norm $\|x\|=\Sigma_{i=1}^m|x_i|$. We say that the vectors $x^1,\ldots,x^s$ in R^m are *conforming* if for every $q=1,\ldots,m$ and $i,j=1,\ldots,s$, $x^i_q x^j_q\geqslant 0$. It is easily seen that $x^1,\ldots,x^s$ are conforming if and only if $\|\Sigma_{i=1}^s x^i\|=\Sigma_{i=1}^s\|x^i\|$. We say that $x^1,\ldots,x^s$ *conform with* a vector u in R^m if $x^1,\ldots,x^s$, u are conforming. Note that Tutte's definition in [4] is nonsymmetric and slightly different. We use the different (symmetric) definition because we wish to characterize extreme points of certain polytopes as nonnegative linear combinations of conforming cycles. Of course, each of these cycles conforms to the extreme point (in the nonsymmetric sense); but it seems to have value to use this conformality in the construction of the extreme point before this extreme point has been identified explicitly. Therefore the internal definition of conformality, which does not involve the extreme point itself, is useful to us.

Let $x^1,\dots,x^s$ be in R^m. We say that $(\alpha_1,\dots,\alpha_s)\in R^s$ is a *linear* relation on $(x^1,\dots,x^s)$ if $\sum_{i=1}^s \alpha_i x^i=0$. The *linear span* of $x^1,\dots,x^s$ will be denoted by $\operatorname{span}\{x^1,\dots,x^s\}$. The dimension of a subspace C of R^m will be denoted by $\dim C$.

We say that a point u is an *extreme point* of a convex set P if $u\in P$ and for any representation $u=\alpha x+(1-\alpha)y$ where $0<\alpha<1$ and $x,y\in P$ we have $x=y=u$.

Finally, we note that the symbol $\subset$ will be used for proper inclusion and the symbol $|\ |$ will be used to indicate the number of elements in a given set.

3. ALGEBRAIC CHARACTERIZATION OF EXTREME POINTS OF THE SET OF NORMALIZED CONSTRAINED CIRCULATIONS

Let G be a graph with incidence matrix $\Gamma\in R^{mn}$. Let $M\in R^{rm}$. The graph G and matrix M will be considered fixed throughout this paper. To avoid frequent explicit reference to G and M, we shall call a circulation u (for G) for which $Mu=0$ a *constrained circulation*. We let W be the subspace of constrained circulations, i.e., $W=\{u\in R^m\colon \Gamma u=0 \text{ and } Mu=0\}$. By P we denote the set of constrained circulations u such that $\|u\|\leqslant 1$. Since the unit ball of our norm is a polytope, we have that P is a polytope. In this section we obtain a number of algebraic characterizations of the extreme points of P. Our first result characterizes the case where $P=\{0\}$.

THEOREM 1. *The polytope $P\neq\{0\}$ if and only if there exist independent cycles $z^1,\dots,z^t$ such that $Mz^1,\dots,Mz^t$ are linearly dependent. In particular, $P\neq\{0\}$ if $\dim C(G)>\operatorname{rank} M$.*

Proof. If $P\neq\{0\}$, then there exists a circulation $0\neq x\in P$. Let $z^1,\dots,z^s$ be a basis of cycles for the circulation space of G. Then for some $\alpha_1,\dots,\alpha_s$, $x=\sum_{i=1}^s\alpha_i z^i$. Since $x\in P$, it follows that $0=Mx=\sum_{i=1}^s\alpha_i Mz^i$. Since $x\neq 0$, not all the α_i's are zero. Conversely, if $z^1,\dots,z^t$ are independent cycles and $Mz^1,\dots,Mz^t$ are linearly dependent, then there exist $\alpha_1,\dots,\alpha_t$, not all zero, such that $M(\sum_{i=1}^t\alpha_i z^i)=\sum_{i=1}^t\alpha_i Mz^i=0$. Since $z^1,\dots,z^t$ are linearly independent, $x=\sum_{i=1}^t\alpha_i z^i\neq 0$. It is easy to see that $0\neq\|x\|^{-1}x\in P$.

Next assume that $\dim C(G)>\operatorname{rank} M$. Let $z^1,\dots,z^s$ be a basis of cycles in $C(G)$. Then $Mz^1,\dots,Mz^s$ are linearly dependent, and this implies that $P\neq\{0\}$ by the first part of the theorem. ■

We need the following lemma that combines [4, (6.2)], which assures that every circulation u is a linear combination with positive coefficients of cycles which conform with u, with [2, Lemma 8], which states that for every circulation u, $C(u)$ has a basis of cycles which conform with u. Our lemma in [2] is closely related to the well-known result that the circulation space in a strongly connected graph has a basis of directed cycles. Our proof in [2] differs from that in Berge [1].

LEMMA 1. *Let $u \neq 0$ be a circulation. Then there exists a basis for $C(u)$ of conforming cycles $z^1,\ldots,z^s$ and $(\alpha_1,\ldots,\alpha_s) \in R^s_+$ such that $u = \sum_{i=1}^{s}\alpha_i z^i$.*

Proof. By [2, Lemma 8] there exists a basis for $C(u)$ of cycles which conform with u. We next demonstrate the existence of linearly independent cycles $z^1,\ldots,z^t$ which conform with u such that

$$u = \sum_{i=1}^{t} \alpha_i z^i \quad \text{and} \quad \alpha_i \geqslant 0, \quad i = 1,\ldots,t.$$

Then by the theorem of exchange, $z^1,\ldots,z^t$ can be completed to a basis of $C(u)$ all of whose elements conform with u. It is immediate to verify that these elements are conforming.

By [4, (6.2)] there exists an integer t and cycles $z^1,\ldots,z^t$ which conform with u such that u is a linear combination with positive coefficients of $z^1,\ldots,z^t$. Let t be the least integer for which there exist cycles, say $z^1,\ldots,z^t$, such that for some positive coefficients, say $\alpha_1,\ldots,\alpha_t$, we have $u = \sum_{i=1}^{t}\alpha_i z^i$. We shall show that $z^1,\ldots,z^t$ are linearly independent. Since $u \neq 0$, we must have that $t \geqslant 1$. Assume that $z^1,\ldots,z^t$ are not linearly independent. Then there exists $(\beta_1,\ldots,\beta_t) \neq 0$ such that $\sum_{i=1}^{t}\beta_i z^i = 0$. By possibly changing the sign of all β_i's, we may assume that at least one β_i is positive. Let $\theta = \min\{\alpha_i/\beta_i : \beta_i > 0\}$ and let $\gamma_i = \alpha_i - \theta\beta_i$, $i = 1,\ldots,t$. Then $\gamma_i \geqslant 0$, $i = 1,\ldots,t$, and at least one γ_i, say γ_1, is zero. Now, $\sum_{i=2}^{t}\gamma_i z^t = \sum_{i=1}^{t}\gamma_i z^i = \sum_{i=1}^{t}\alpha_i z^i - \theta\sum_{i=1}^{t}\beta_i z^i = u$. Consequently we get a contradiction to the minimality of t. ∎

The following example illustrates that it is possible to have a circulation u where there exists no basis for $C(u)$ of conforming cycles $z^1,\ldots,z^s$ such that $u = \sum_{i=1}^{s}\alpha_i z^i$ where $\alpha_i > 0$ for all $i = 1,\ldots,s$. The example also demonstrates that it is possible to have a circulation u and a basis for $C(u)$ of cycles conforming with u $z^1,\ldots,z^s$ such that $u = \sum_{i=1}^{s}\beta_i z^i$ and $\beta_i < 0$ for some $i = 1,\ldots,s$.

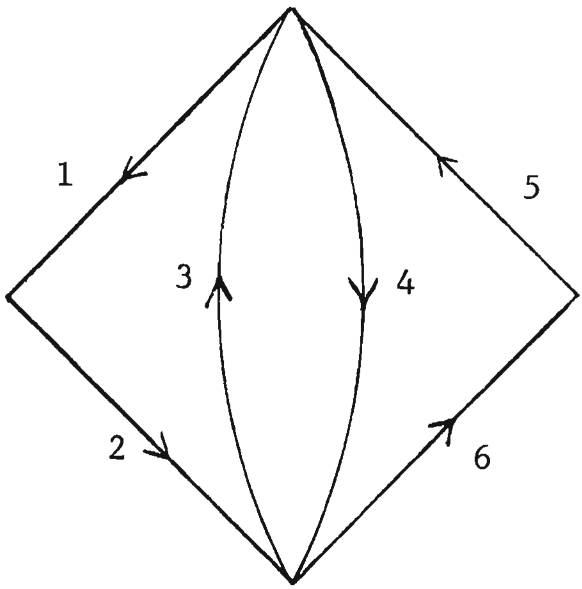

FIG. 1.

EXAMPLE 1. Let G be a graph whose planar representation is given by Figure 1. (The numbers in the above figure indicate the enumeration of the arcs.) The only cycles of this graph are: $z^1=(1,1,1,0,0,0)^T$, $z^2=(0,0,1,1,0,0)^T$, $z^3=(0,0,0,1,1,1)^T$, $z^4=(1,1,0,0,1,1)^T$, $z^5=(1,1,0,-1,0,0)^T$, $z^6=(0,0,-1,0,1,1)^T$, and $\{-z^i: i=1,\dots,6\}$. Let $u^1=z^2+z^4=(1,1,1,1,1,1)^T$ and $u^2=z^2+2z^4=(2,2,1,1,2,2)^T$. Now, if u^1 (u^2) has a representation as a linear combination with positive coefficients of conforming cycles, then these cycles must conform with u^1 (u^2). The only cycles conforming with u^1 and u^2 are z^1, z^2, z^3, and z^4, and any subset of three cycles out of these four form a basis for $C(u^1)=C(u^2)$. There are only two ways to express u^1 as a linear combination of z^1, z^2, z^3, and z^4: $u^1=z^2+z^4=z^1+z^3$. Consequently, there is no representation of u^1 as a linear combination with positive coefficients of elements in a basis of $C(u^1)$ consisting of conforming cycles. Also, the representation of u^2 in the basis $\{z^1,z^2,z^3\}$ of $C(u^2)$ is $u^2=2z^1+2z^3-z^2$, illustrating that there is a basis for $C(u^2)$ of conforming cycles where the representation of u^2 in the basis has a negative coefficients.

The following lemma characterizes the extreme points of P. It replaces the requirements of nonexistence of proper convex combinations by the requirement of nonexistence of certain conforming circulations.

LEMMA 2. *Suppose $P \neq \{0\}$. Then u is an extreme point of P if and only if*

(3.1) $u \in P$,
(3.2) $\|u\| = 1$, *and*
(3.3) *there exist no conforming linearly independent constrained circulations x^1 and x^2 for which $u = x^1 + x^2$.*

Proof. The proof follows exactly that of Lemma 1 of [2]. ■

We are now ready for the algebraic characterizations of the extreme points of P.

THEOREM 2. *Suppose $P \neq \{0\}$. Then u is an extreme point of P if and only if*:

(3.1) $u \in P$,
(3.2) $\|u\| = 1$,

and one, or equivalently each, of the following conditions holds:

(3.3a) $W \cap C(u) = \operatorname{span}\{u\}$,
(3.3b) $\dim C(u) = \dim\{Mx : x \in C(u)\} + 1$,
(3.3c) *there exists a basis for $C(u)$ of conforming cycles $z^1, \ldots, z^s$ and $(\alpha_1, \ldots, \alpha_s) \in R^s_+$ such that $u = \sum_{i=1}^s \alpha_i z^i$ and the space of linear relations on $Mz^1, \ldots, Mz^s$ is spanned by $(\alpha_1, \ldots, \alpha_s)$.*

Proof. In view of Lemma 2 it suffices to show the equivalence of (3.3), (3.3a), (3.3b), and (3.3c) for nonzero vectors u in P. Let $u \in P$, where $u \neq 0$.

(3.3) $\Rightarrow$ (3.3a): Suppose that (3.3a) is false. Clearly $W \cap C(u) \supseteq \operatorname{span}\{u\}$, and therefore there is a $v \in W \cap C(u)$ which is not a scalar multiple of u. There is $\varepsilon > 0$ (sufficiently small) such that $x^1 = \frac{1}{2}(u + \varepsilon v)$ and $x^2 = \frac{1}{2}(u - \varepsilon v)$ conform to each other (and to u). It is easily seen that x^1, x^2 are linearly independent. Since x^1, x^2 are constrained circulations and $u = x^1 + x^2$, it follows that (3.3) is false.

(3.3a) $\Rightarrow$ (3.3): Suppose that (3.3) is false, and $u = x^1 + x^2$, where x^1, x^2 are conforming linearly independent constrained circulations. Since x^1 and x^2 conform, it follows that $x^1 \in C(u)$. So $x^1 \in W \cap C(u)$. But x^1 and $u = x^1 + x^2$ are linearly independent. So $x^1 \notin \operatorname{span}\{u\}$, implying that $W \cap C(u) \neq \operatorname{span}\{u\}$. Hence (3.3a) is false.

(3.3a) $\Leftrightarrow$ (3.3b): For every $n \times m$ matrix A and subspace C of R^m, we have that

$$\dim C = \dim\{x \in C : Ax = 0\} + \dim\{Ax : x \in C\}.$$

Consequently, with $C=C(u)$ and $A=M$ we get that

$$\dim C(u)=\dim[C(u)\cap W]+\dim\{Mx: x\in C(u)\}.$$

If $u\in W$ and $u\neq 0$, we trivially have that $u\in W\cap C(u)$. But (3.3a) states that $\dim[C(u)\cap W]=1$, and its equivalence with (3.3b) is immediate.

(3.3a) $\Leftrightarrow$ (3.3c): By Lemma 1 there exists a basis for $C(u)$ of conforming cycles $z^1,\dots,z^s$ and $(\alpha_1,\dots,\alpha_s)\in R^s_+$ such that $u=\Sigma_{i=1}^s\beta_i z^i$. Assume (3.3a) holds, and let $(\beta_1,\dots,\beta_s)$ be a linear relation on $(Mz^1,\dots,Mz^s)$. Then $M(\Sigma_{i=1}^s\beta_i z^i)=\Sigma_{i=1}^s\beta_i Mz^i=0$. Consequently, $\Sigma_{i=1}^s\beta_i z^i\in W\cap C(u)=\operatorname{span}\{u\}$. So, for some $\gamma\in R$, $\Sigma_{i=1}^s\beta_i z^i=\gamma u=\Sigma_{i=1}^s\gamma\alpha_i z^i$. The independence of the z^i's assures that $(\beta_1,\dots,\beta_s)=\gamma(\alpha_1,\dots,\alpha_s)$, establishing (3.3c). Next assume that (3.3c) holds. For $v\in C(u)$ let $\beta_1,\dots,\beta_s$ be (the unique) coefficients such that $v=\Sigma_{i=1}^s\beta_i z^i$. Then $v\in C(u)\cap W$ implies that $0=Mv=\Sigma_{i=1}^s\beta_i Mz^i$, i.e., $(\beta_1,\dots,\beta_s)$ is a linear relation on $(Mz^1,\dots,Mz^s)$. Hence (3.3c) assures that for some γ we have $\beta_i=\gamma\alpha_i$, $i=1,\dots,s$, and therefore $v=\gamma\Sigma_{i=1}^s\alpha_i z^i=\gamma u$, establishing (3.3a). ■

The next example illustrates that one cannot require that (3.3c) be modified by requiring that the corresponding α_i's be all positive.

EXAMPLE 2. Consider the example described in Example 1. Let

$$M=\begin{bmatrix}0&0&1&-1&0&0\\1&1&1&1&-2&-2\end{bmatrix}.$$

The only circulations u satisfying $Mu=0$ are those proportional to $u^1=(1,1,1,1,1,1)=z^2+z^4=z^1+z^3$. Consequently, $\pm\frac{1}{6}u^1$ are the only extreme points of P. We have already seen in Example 1 that there exists no basis for $C(u)$ of conforming cycles $z^1,\dots,z^s$ such that $u^1=\Sigma_{i=1}^s\alpha_i z^i$ and all the α_i's are positive. Consequently (3.3c) does not hold with all α_i's positive for $u=u^1$.

COROLLARY 1. *If u is an extreme point of P, then* $\dim C(u)\leqslant \operatorname{rank} M+1$.

Proof. If $P=\{0\}$, then u is an extreme point if and only if $u=0$. In this case $C(u)=\{0\}$, and the conclusion is trivial. Next assume that $P\neq\{0\}$. Then by Theorem 2, $\dim C(u)=\dim\{Mx: x\in C(u)\}+1\leqslant\dim\{Mx: x\in R^m\}+1=\operatorname{rank} M+1$. ■

We next illustrate by examples that the bound in Corollary 1 is sometimes, but not always, obtained.

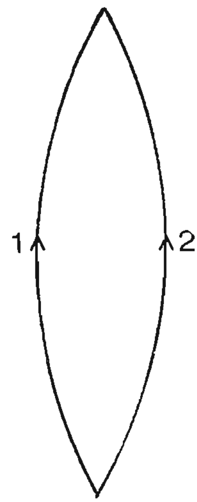

FIG. 2.

EXAMPLE 3. Let the incidence matrix Γ of our graph G (Figure 2) and the matrix M be given by

$$\Gamma=\begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}, \qquad M=(1,1).$$

Then $C(G)=W=\operatorname{span}\{(1,-1)^T\}$, $P=\{(\alpha,-\alpha): -0.5\leqslant\alpha\leqslant 0.5\}$, and the only two extreme points of P are $\pm(0.5,-0.5)$. For $u=(0.5,-0.5)$, $\dim C(u)=1<\operatorname{rank} M+1$.

EXAMPLE 4. Let the incidence matrix Γ of our graph and the matrix M be given by

$$\Gamma=\begin{pmatrix} 1 & 0 & -1 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & -1 & 1 \end{pmatrix}, \qquad M=(1,1,1,1,1).$$

Then $C(G)=\operatorname{span}\{(1,1,1,0,0)^T,(0,0,0,1,1)^T\}$, $W=\operatorname{span}\{(2,2,2,-3,-3)^T\}$, $P=\{\alpha(2,2,2,-3,-3)^T: -\frac{1}{12}\leqslant\alpha\leqslant\frac{1}{12}\}$, and the only two extreme points of P are $\pm\frac{1}{12}(2,2,2,-3,-3)^T$. For $u=\frac{1}{12}(2,2,2,-3,-3)^T$, $\dim C(u)=\dim C(G)=2=\operatorname{rank} M+1$.

4. ALGEBRAIC CHARACTERIZATION OF SUBGRAPHS THAT SUPPORT EXTREME POINTS OF P

THEOREM 3. *Suppose $P \neq \{0\}$. Let H be a subgraph of G. Then there exists an extreme point u of P such that $G(u) = H$ if and only if H is a minimal subgraph of G for which $W \cap C(H) \neq \{0\}$.*

Proof. Suppose that u is an extreme point of P for which $G(u) = H$. Then $C(u) = C(H)$. Since $P \neq \{0\}$, Theorem 1 implies that $u \neq 0$ and $W \cap C(H) = \operatorname{span}\{u\} \neq \{0\}$. Let $H' \neq H$ be a subgraph of H. Then $u \notin C(H')$. Since $u \in W \cap C(H)$, we conclude that $W \cap C(H') \subset W \cap C(H) = \operatorname{span}\{u\}$. As $W \cap C(H')$ is a subspace of R^m, it follows that $W \cap C(H') = \{0\}$.

Next assume that H is a minimal subgraph of G for which $W \cap C(H) \neq \{0\}$. Since $W \cap C(H)$ is a subspace of R^m, it follows that there exists $u \in W \cap C(H)$ such that $\|u\| = 1$. Observe that $G(u) \subseteq H$ and $0 \neq u \in W \cap C(u)$. So $W \cap C(u) \neq \{0\}$, and the minimality of H assures that $H = G(u)$. It follows from Theorem 1 that in order to prove that u is an extreme point of P it suffices to establish that $W \cap C(u) = \operatorname{span}\{u\}$. It is immediate that $W \cap C(u) \supseteq \operatorname{span}\{u\}$. Suppose that $v \in W \cap C(u)$. Then there exists $\alpha \in R$ such that $G(u - \alpha v) \subseteq G(u)$ and $G(u - \alpha v) \neq G(u)$. Since $u - \alpha v \in W \cap C(u - \alpha v)$, the minimality of H implies that $u - \alpha v = 0$. Since $u \neq 0$, this implies that $v \in \operatorname{span}\{u\}$. ■

The following is an immediate corollary of Theorem 3.

COROLLARY 2. *Suppose $P \neq \{0\}$. Let H be a subgraph of G. Then there exists an extreme point u of P such that $G(u) \subseteq H$ if and only if $W \cap C(H) \neq \{0\}$.*

COROLLARY 3. *If H is a subgraph of G for which* $\dim C(H) \geqslant \operatorname{rank} M + 1$, *then there exists an extreme point u of P such that $G(u) \subseteq H$.*

Proof. If $P = \{0\}$, then 0 is an extreme point of P, and clearly $G(0) \subseteq H$. Next assume that $P \neq \{0\}$. Then by Corollary 2, it suffices to show that $W \cap C(H) \neq \{0\}$. As in the proof of Theorem 1 and Corollary 1,

$$\begin{aligned}\dim C(H) &= \dim\{x \in C(H) : Mx = 0\} + \dim\{Mx : x \in C(H)\}\\ &\leqslant \dim[W \cap C(H)] + \dim\{Mx : x \in R^m\}\\ &= \dim[W \cap C(H)] + \operatorname{rank} M.\end{aligned}$$

So the condition $\dim C(H) \geqslant \operatorname{rank} M + 1$ implies that $\dim[W \cap C(H)] \geqslant 1$. We obtain $W \cap C(H) \neq \{0\}$. ■

We remark that Corollary 3 gives a sufficient but not a necessary condition for there to exist an extreme point u of P with $G(u) \subseteq H$, as is shown by Example 3 with $H = G$, where $\dim C(H) = 1 < 2 = \operatorname{rank} M + 1$.

5. APPENDIX

Let i and j be two vertices of a graph G. A *path* from i to $j \neq i$ is a vector $x \in R^m$ with $x_k \in \{-1, 0, 1\}$ for each $k \in \langle m \rangle$ and

$$[\Gamma(G)x]_q = \begin{cases} 1 & \text{if } \; q = i, \\ -1 & \text{if } \; q = j, \\ 0 & \text{otherwise.} \end{cases}$$

A graph is *connected* if there exists a path between any two distinct vertices. A *component* of a graph G is a maximal connected subgraph of G. Evidently, every graph may be uniquely partitioned into components.

Let $z^1, \dots, z^s$ be linearly independent cycles. Let $Q = G(z^1, \dots, z^s)$. We shall show that $\dim C(Q)$ depends only on the number of components of certain graphs.

For $\mu \subseteq \langle s \rangle$, let $Q_\mu = ((Q_\mu)_v, (Q_\mu)_a)$ be the graph whose set of arcs $(Q_\mu)_a = \bigcap_{i \in \mu} G(z^i)_a$ and whose set of vertices is $(Q_\mu)_v = \bigcap_{i \in \mu} G(z^i)_v$. We denote by m_μ, n_μ, p_μ, respectively, the number of arcs, vertices, components of Q_μ. Let $d_\mu = \dim C(Q_\mu)$. For the empty set $\varnothing$, we put $Q_\varnothing = Q$, and we write m for $m_\varnothing$, and similarly n, p, and d for $n_\varnothing$, $p_\varnothing$, and $d_\varnothing$.

Our next theorem generalizes [2, Lemma 6].

THEOREM 4. *Let $z^1, \dots, z^s$ be linearly independent cycles. Let $Q = G(z^1, \dots, z^s)$. Then*

$$\dim C(Q) = \sum_{\mu \subseteq \langle s \rangle, \, |\mu| \neq 1} p_\mu (-1)^{|\mu|}.$$

Proof. It is well known from the Euler formula (e.g., [1, p. 16]) that $\dim C(Q_\mu) = d_\mu = m_\mu - n_\mu + p_\mu$ for every $\mu \subseteq \langle s \rangle$. By the inclusion-exclusion principle (e.g. [3, p. 19])

$$m = \sum_{\varnothing \neq \mu \subseteq \langle s \rangle} m_\mu (-1)^{|\mu|-1},$$

$$n = \sum_{\varnothing \neq \mu \subseteq \langle s \rangle} n_\mu (-1)^{|\mu|-1}.$$

If $|\mu| = 1$, then Q_μ can be identified with a cycle and $d_\mu = p_\mu = 1$ and $m_\mu = n_\mu$. If $|\mu| > 1$, then $d_\mu = 0$, since $C(Q_\mu)$ consists only of the zero vector and hence $m_\mu - n_\mu = -p_\mu$. This yields

$$m - n + p = p + \sum_{\varnothing \neq \mu \subseteq \langle s \rangle,\ |\mu| \neq 1} p_\mu (-1)^{|\mu|},$$

and the theorem follows, as $p = p_\varnothing$ and by the Euler formula $\dim C(Q) = m - n + p$. ■

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